REGULARLY TOTALLY ORDERED SEMIGROUPS II

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1. A semigroup $S^{(1)}$ is said to be (*partially*) *ordered* if for some pairs of elements x, y an *ordering relation* $x \leqq y$ exists (also denoted by $y \geqq x$) which satisfies

(1) $a \leqq a$ for every $a \in S$,

(2) $a \leqq b$, $b \leqq a$ imply $a = b$,

(3) $a \leqq b$, $b \leqq c$ imply $a \leqq c$,

and (4) $a \leqq b$ implies both $ac \leqq bc$ and $ca \leqq cb$ for every $c \in S$.

We write usually $a < b$ if $a \leqq b$ but $a \neq b$. Especially, we shall call S *totally* (*or 'simply'*) *ordered* if all pairs x, y are ordered. In a totally ordered semigroup S (with an ordering relation $\leqq$), we can consider the *Archimedean property*: S is *archimedean* if it satisfies the following

(A) For any non-zero elements a, b of S, there exist positive integers n, m such that $a^n > b$, $b^m > a$.

Now, let S be a commutative semigroup. Then we shall call S *regularly totally ordered* (*r. t. o.*) if S satisfies the following conditions;

(1) for any different $a, b \in S$, either $aS \subset bS^{(2)}$ or $bS \subset aS$ holds,

(2) if $aS \subset bS$, then there exists a positive integer n such that $a^n \in b^n S$,

where the symbol $\subset$ means 'is a proper subset of.'

It is easy to see that if S is a r. t. o. semigroup then S becomes a totally ordered semigroup if an ordering relation in S is defined as follows; $a \leqq b$ means $aS \supseteqq bS^{(3)}$.

Next, we define the *locally nilpotency of semigroups*: A commutative semigroup S is said to be *locally nilpotent* if it satisfies the following conditions; for any element a of S,

$$\bigcap_n a^n S^{(4)} \begin{cases} = \phi & \text{if } S \text{ has no zero element,} \\ = \{o\} & \text{if } S \text{ has a zero element } o, \end{cases}$$

(1) By the term *semigroup* we shall mean a system consisting of a class Σ of elements, $a, b, c, \ldots$ in which there is defined an associative binary operation: $a(bc) = (ab)c$.

(2) Let A, B be subsets of a semigroup S. Then, AB denotes the subset $\{ab \mid a \in A, b \in B\}$ of S. Especially, if A consists of only one element a then we use aB as a substitute for $\{a\}B$.

(3) The symbol $\subseteqq$ means 'is a subset of'.

(4) $\bigcap_n a^n S$ will denote the intersection of all $a^n S$ ($n = 1, 2, 3, \ldots$).

where ϕ and $\{o\}$ denote the empty set and the set consisting of o alone, respectively.

In a r. t. o. semigroup, however, it is easily seen that the locally nilpotency is equivalent to the Archimedean property. In fact: Let S be a r. t. o. semigroup. Suppose that S is locally nilpotent. Take up any non-zero elements a, b from S. If there exists no integer n such that $a^n > b$, then we have $a^n \leqq b$ for every integer n, whence $bS \subseteqq \bigcap_n a^n S = \phi$ or $\{o\}$. Since $bS \neq \phi$ we have $b=o$, contrary to $b \neq o$.

Conversely, suppose S to be archimedean. Let x, y be any non-zero elements of S. Then there exist integers n, m such that $x^n > y$, $y^m > x$. Hence we obtain $\bigcap_i x^i S = \bigcap_i (x^n)^i S \subseteqq \bigcap_i y^i S = \bigcap_i (y^m)^i S \subseteqq \bigcap_i x^i S$, i. e., $\bigcap_i x^i S = \bigcap_i y^i S$. Assume now that there exists an element a such that $\bigcap_i a^i S$ contains at least one non-zero element, say, an element z. Since $a \neq o$, $\bigcap_i a^i S = \bigcap_i z^i S \ni z$. This implies $z^2 S \supseteqq zS$, whence $z^2 = z$. Since S is archimedean, from $z^2 = z$ we conclude $z = o$ (otherwise, there exists a positive integer j such that $z^j > z$), contrary to $z \neq o$.

In a previous paper [2] the author gave some results concerning the structure of archimedean r. t. o. semigroups:

(S. 1) A semigroup is $\left\{\begin{array}{l}\text{a discrete, archimedean r. t. o. semigroup} \\ \text{a non-discrete, archimedean r. t. o. semigroup}\end{array}\right\}$ without zero if and only if it is isomorphic with $\left\{\begin{array}{l}\text{an I-subgroup} \\ \text{a } \beta\text{-dense, I-subgroup}\end{array}\right\}$ of $\left\{\begin{array}{l}\text{a closed half line } L[1] \\ \text{an open half line } L(\beta)\end{array}\right\}$.

(S. 2) If an archimedean r. t. o. semigroup S contains a zero element and if S satisfies the *cancellation law* (in the sense mentioned below[5]), then the problem of determining the structure of S is reduced to the problem of determining the structure of either an archimedean r. t. o. semigroup without zero or a dense-in-itself (or simply 'dense') segment.

The author was, however, not able to know whether any archimedean r. t. o. semigroup satisfies necessarily the cancellation law or not. In the present paper he will give a solution for this problem, and show that we may as well eliminate the 'if S satisfies the cancellation law' metioned in (S. 2).

2. In this section, to complete our previous paper [2] we shall prove that any archimedean r. t. o. semigroup satisfies the cancellation law. Let G be an archimedean r. t. o. semigroup with zero o. By a *zero divisor* we shall mean a non-zero element x such that $xy = o$ for some non-zero element y of G. Moreover, by a *nil-element* we shall mean an

(5) Cancellation law in a semigroup with zero o: If $ab = ac \neq 0$, then $b = c$.

element x which satisfies $x^n=o$ for some positive integer n. It was already shown by [2] that if an archimedean r. t. o. semigroup S has no zero divisor (accoringly, as a matter of course, if S has no zero element) then S satisfies the cancellation law. Therefore, we shall restrict our attention to an archimedean r. t. o. semigroup which has at least one zero divisor. Henceforth, S will denote an archimedean r. t. o. semigroup which has at least one zero divisor, and o will denote the zero element of S.

Lemma 1. $xS=\{o\}$ *implies* $x=o$.

The proof of this lemma is apparent.

Lemma 2. Every element of S *is a nil-element.*

Proof. Let a be a zero divisor of S. Then, there exists a non-zero element b such that $ab=o$. Take up any element x of S. Were $x^n \leqq b$ for every positive integer n, we would have $bS \subseteqq \bigcap_i x^i S=\{o\}$, contrary to $b \neq o$. Thus, there exists an integer i such that $x^i > b$. Similarly there exists an integer j such that $x^j > a$. From $ab=o$ we have $o=ab \leqq x^{i+j}$, whence $x^{i+j}=o$. (It is obvious that o is the greatest element of S).

Lemma 3. $yx=x$ *implies* $x=o$.

Proof. Since $y^n x=x$ for every positive integer n, we have $x \in \bigcap_n y^n S=\{o\}$. This implies $x=o$.

Lemma 4. For any element x *of* S, *there exist elememts* y, z *such that* $yz<x$.

Proof. Assume that there exists an element x which satisfies $yz \geqq x$ for any elements y, z of S. Then, for any positive integer n and for any $2n$ elements $x_1, x_2, \ldots\ldots, x_{2n}$ of S, we have

$$x_1 x_2 \ldots\ldots x_{2n} \geqq x^n.$$

Since every element of S is a nil-element, $x^m=o$ for some positive integer m. Thus we have

$$x_1 x_2 \ldots\ldots x_{2m}=o$$

for any $2m$ elements $x_1, x_2, \ldots\ldots, x_{2m}$ of S. This implies $x_1 x_2 \ldots\ldots x_{2m-1} S=\{o\}$, that is, $x_1 x_2 \ldots\ldots x_{2m-1}=o$. Repeating such a process successively $2m-1$ times, we have consequently $x_1=o$. Since x_1 is any element of S this implies $S=\{o\}$, contrary to $S \neq \{o\}$.

Lemma 5. If $x<y$, *then there exists* z *such that* $xz<y$.

Proof. Assume that for every element t of S, $xt \geqq y$ is satisfied. Then $xtS \subseteqq yS$ for every element t of S, and this implies $xS^2 \subseteqq yS$. From $x<y$ we have $xw \notin yS$ for some $w \in S$. By Lemma 4, there exist elements w_1, w_2 of S such that $w_1 w_2 < w$, i. e., $w_1 w_2 S \supsetneqq wS$. Therefore $w_1 w_2 v=wx$ for some $v \in S$. From $w_1 w_2 v=wx$ and $w_1 w_2<w$, we have $w_1 w_2 x \leqq wx=w_1 w_2 v$. This is however impossible by the reasons as follows. Were $w_1 w_2 x=wx$, we would have $wx \in xS^2 \subseteqq yS$, contrary to $wx \notin yS$. Were $w_1 w_2 x<wx=w_1 w_2 v$, since $w_2 x<w_2 v$ we would have $w_2 w_1 v=w_2 x u$ for some $u \in S$, whence $wx=w_1 w_2 v=x w_2 u \in$

$xS^2 \subseteqq yS$, contrary to $wx \notin yS$. Hence, there exists an element z of S such that $xz < y$.

Lemma 6. S satisfies the cancellation law. That is: If a, x, y *are elements of S such that* $ax = ay \neq o$, *then* $x = y$.

Proof. To prove this lemma, we assume that $ax = ay$. If, say, $x < y$, then $xt < y$ for some $t \in S$.. From $ax = ay \geqq axt$, $ax \leqq axt$ and Lemma 3, we conclude $ax = o$.

Using this lemma, we obtain immediately a complete form of Theorem 5 of our previous paper [2]:

Theorem. If an archimedean r. t. o. semigroup has a zero divisor, then it is a dense-in-itself segment (in the sense of Clifford [1]).

References

[1] A. H. Clifford, 'Naturally totally ordered commutative semigroups,' American Journal of Mathematics, vol. 76 (1954), pp. 631–646.

[2] M. Yamada, 'Regularly totally ordered semigroups 1,' Bulletin of the Shimane University (Natural Sciences), No. 7 (1957), pp. 14–23.