BULL. SHIMANE UNIV. (NATURAL SCI.)
No. 10, MARCH, 1961

Quasi-regular Bands

By Miyuki YAMADA

(*Received Nov. 16, 1960*)

山田深雪：擬正則巾等準群

A band is synonymous with an idempontent semigroup. Let S be a band.

Then there exist, up to isomorphism, a unique semilattice Γ, and a disjoint family of rectangular subbands of S indexed by Γ, $\{S_\gamma : \gamma \in \Gamma\}$, such that

$$(1) \quad S = \cup \{S_\gamma : \gamma \in \Gamma\}$$

and

$$(2) \quad S_\alpha S_\beta \subset S_{\alpha\beta} \quad \text{for all } \alpha, \beta \in \Gamma.$$

(See McLean [3]).

Following Kimura [1], Γ is called the *strcture semilattice* of S, and S_γ the γ-*kernel*.

And this decomposition is called the *structure decomposition* of S, and denoted by $S \sim \sum \{S_\gamma : \gamma \in \Gamma\}$.

For each subset $\varDelta$ of Γ, we first define the relation $\Re_\Delta$ on S as follows:

$$a \, \Re_\Delta \, b \text{ if and only if } \begin{cases} ab = a \text{ and both } a \text{ and } b \text{ are contained in a common } S_\gamma, \ \gamma \in \varDelta, \\ \text{or} \\ ab = b \text{ and both } a \text{ and } b \text{ are contained in a common } S_\gamma, \ \gamma \notin \varDelta. \end{cases}$$

Then, it is easily seen that $\Re_\Delta$ is an equivalence relation and especially $\Re_\phi$ and $\Re_\Gamma$ (where ϕ is the empty subset of Γ) coincide with $\mathfrak{P}$ and $\mathfrak{Q}$ in Kimura [2] respectively.

The following two theorems have been proved by [2]:

THEOREM I. *$\Re_\phi(\Re_\Gamma)$ is a congruence on S if and only if S is left (right) semiregular.*

Further, in this case the quotient semigroup $S/\Re_\phi$ $(S/\Re_\Gamma)$ is left (right) regular.

THEOREM II. *Both $\Re_\phi$ and $\Re_\Gamma$ are congruences on S if and only if S is regular.*

Further, in this case S is isomorphic to the spined product of $S/\Re_\phi$ and $S/\Re_\Gamma$ with respect to Γ.

In this paper, we shall present a necessary and sufficient condition for $\Re_\Delta$ to be a congruence on S, and make some generalizations of Theorems I and II. Further, this paper contains an example which shows the existence of a band $B \equiv \{B_\omega : \Omega\}$[2] on which for some subset $\varLambda$ of Ω both $\Re_\Lambda$ and $\Re_{\Omega/\Lambda}$ (where $\Omega \backslash \varLambda$ is the complement of $\varLambda$ in Ω) are congruences although neither $\Re_\phi$ nor $\Re_\Omega$ is a congruence, that is, the existence of a band which is quasi-regular but neither left semiregular nor right semiregular.

Notations and terminologies. If M and N are sets such that $M \supset N$, then $M \setminus N$ will denote the complement of N in M. The notation ϕ will denote always the empty set. Throughout the paragraphs 1 and 2, S will denote a band unless otherwise mentioned. The structure semilattice of S and the γ-kernel, for each γ of the structure semilattice, will be denoted by Γ and S_γ res-

1) Yamada [5] is an abstract of this paper.

2) The notation 'a band $B \equiv \{B_\omega : \Omega\}$ 'stands for' a band B whose structure decomposition is $B \sim \sum \{B_\omega : \omega \in \Omega\}$'

pectively. And the structure decomposition of S will be denoted naturally by $S\sim\sum\{S_\gamma:\gamma\in\Gamma\}$. Further, an element of S_γ will be denoted by a small letter with the sufflix γ such as a_γ, b_γ, e_γ etc. Any other notation or terminology without definition should be referred to [1].

1. Quasi-separativity.

Let Δ be a subest of the structure semilattice Γ of S, and put $\cup\{S_\gamma:\gamma\in\Delta\}=S(\Delta)$.

Then, we have immediately

LEMMA 1. *For* $a, b\in S(\Delta)$, $a\Re_\Delta b$ *if and only if* $ab=a$ *and* $ba=b$.

Proof. McLean [3] has proved that for elements x, y, satisfying $xy=x$ and $yx=y$, of a band $B\equiv\{B_\omega:\Omega\}$ there exists B_ω which contains both x and y. The 'only if' part follows from the definition of $\Re_\Delta$ and the fact that in any S_γ $xy=x$ implies $yx=y$.

The 'if' part is proved as follows. Let a and b be elements of $S(\Delta)$. If $ab=a$ and $ba=b$, then it follows from the McLean's result that there exists S_γ which contains both a and b. Since $S(\Delta)\ni a$, γ is an element of Δ. Hence $a\ \Re_\Delta\ b$.

Similarly, we obtain

LEMMA 2. *For* $a, b\notin S(\Delta)$, $a\ \Re_\Delta\ b$ *if and only if* $ab=b$ *and* $ba=a$.

LEMMA 3. *If* $\Re_\Delta$ *is a congruence on* S, *then for an element* $\alpha\in\Delta$ *the following (1) and (2) are equivalent*:

(1) $a_\alpha b_\alpha=a_\alpha$.

(2) *For any* c_β, $\begin{cases} c_\beta a_\alpha c_\beta b_\alpha=c_\beta a_\alpha & \text{if } \alpha\beta\in\Delta,\\ a_\alpha c_\beta=b_\alpha c_\beta & \text{if } \alpha\beta\notin\Delta\end{cases}$

Proof. By the definition of $\Re_\Delta$, the relation (1) implies $a_\alpha\ \Re_\Delta\ b_\alpha$. Since $\Re_\Delta$ is a congruence on S, two relations $c_\beta a_\alpha\Re_\Delta c_\beta b_\alpha$ and $a_\alpha c_\beta\Re_\Delta b_\alpha c_\beta$ hold for any $c_\beta\in S$.

We have then

$$\begin{cases} c_\beta a_\alpha c_\beta b_\alpha=c_\beta a_\alpha & \text{if } \alpha\beta\in\Delta,\\ a_\alpha c_\beta=b_\alpha c_\beta & \text{if } \alpha\beta\notin\Delta\end{cases}$$

because $a_\alpha c_\beta=a_\alpha b_\alpha c_\beta=a_\alpha b_\alpha c_\beta b_\alpha c_\beta=a_\alpha c_\beta b_\alpha c_\beta=b_\alpha c_\beta$ if $\alpha\beta\notin\Delta$. Thus, (1) implies (2). Conversely, assume the relation (2). Putting $c_\beta=a_\alpha$ in (2), we have $a_\alpha b_\alpha=a_\alpha$ since $\alpha\alpha\in\Delta$. Accordingly, (2) implies (1).

Similarly, we obtain

LEMMA 4. *If* $\Re_\Delta$ *is a congruence on* S, *then for an element* $\alpha\notin\Delta$ *the following (1) and (2) are equivalent*:

(1) $a_\alpha b_\alpha=b_\alpha$.

(2) *For any* c_β, $\begin{cases} a_\alpha c_\beta b_\alpha c_\beta=b_\alpha c_\beta & \text{if } \alpha\beta\notin\Delta,\\ c_\beta b_\alpha=c_\beta a_\alpha & \text{if } \alpha\beta\in\Delta.\end{cases}$

Using Lemmas 1-4,, we have

THEOREM 1. $\Re_\Delta$ *is a congruence on* S *if and only if* S *satisfies the condition*

$$(C)\quad\begin{cases} cabacba=caba & \text{if } ab\in S(\Delta) \text{ and } abc\in S(\Delta),\\ abac=bac & \text{if } ab\in S(\Delta) \text{ and } abc\notin S(\Delta),\\ caba=cab & \text{if } ab\notin S(\Delta) \text{ and } abc\in S(\Delta),\\ abcabac=abac & \text{if } ab\notin S(\Delta) \text{ and } abc\notin S(\Delta).\end{cases}$$

Proof. Necessity. Suppose that $\Re_\Delta$ is a congruence on S. We shall prove first that in the case $ab\in S(\Delta)$ and $abc\in S(\Delta)$ the relation $cabacba=caba$ holds. First of all, there exist S_α and S_β such that $ab\in S_\alpha$ and $abc\in S_\beta$. Since $ab\in S(\Delta)$ and $abc\in S(\Delta)$, both α and β are elements of Δ. Further, it is clear that the elements aba and ba are contained in S_α, and the elements $caba$ and cba are contained in S. Since $(aba)(ba)=aba$, the relation $cabacba=caba$ follows from Lemma 3. The proofs of the other cases are obtained by similar methods.

Sufficiency. Let S satisfy the condition (C). Since $\mathfrak{R}_\Delta$ is an equivalence relation, we need only to show that $a\ \mathfrak{R}_\Delta\ b$ implies both $ac\ \mathfrak{R}_\Delta\ bc$ and $ca\ \mathfrak{R}_\Delta\ cb$ for every element c of S. To show this, for an arbitrary element c and elements a, b such that $a\ \mathfrak{R}_\Delta\ b$, we divide into four cases as follows; (i) $a, b\in S(\Delta)$ and $ac\in S(\Delta)$, (ii) $a, b\in S(\Delta)$ and $ac\notin S(\Delta)$, (iii) $a, b\notin S(\Delta)$ and $ac\in S(\Delta)$, (iv) a, $b\notin S(\Delta)$ and $ac\notin S(\Delta)$. In the case (i), both $ab=a$ and $ba=b$ follows from Lemma 1 and elements bc, ca, ba, cb, ab and abc are all contained in $S(\Delta)$ since $a, b\in S(\Delta)$ and $ac\in S(\Delta)$. From these and the condition (C), we have

$$acbc=abcbc=abc=ac,$$
$$bcac=bacac=bac=bc,$$
$$cacb=cabacba=caba=ca$$
$$\text{and}\quad cbca=cbabcab=cbab=cb.$$

Accordingly, we conclude $ac\ \mathfrak{R}_\Delta\ bc$ and $ca\ \mathfrak{R}_\Delta\ cb$ by using Lemma 1. By an analogous argument we can easily prove $ac\ \mathfrak{R}_\Delta\ bc$ and $ca\ \mathfrak{R}_\Delta\ cb$ also in the case (ii), (iii) or (iv). So we omit the proofs in these cases.

COROLLARY. *Both $\mathfrak{R}_\Delta$ and $\mathfrak{R}_{\Gamma\setminus\Delta}$ are congruences on S if and only if S satisfies the condition*

$$(C^*)\quad \begin{cases} \left.\begin{array}{l} cabacba = caba \\ abcabac = abac \end{array}\right\} & \text{if } ab\in S(\Delta) \text{ and } abc\in S(\Delta), \text{ or if } ab\in S(\Gamma\setminus\Delta) \text{ and } abc\in S(\Gamma\setminus\Delta), \\ \left.\begin{array}{l} caba = cab \\ abac = bac \end{array}\right\} & \text{if } ab\in S(\Delta) \text{ and } abc\in S(\Gamma\setminus\Delta), \text{ or if } ab\in S(\Gamma\setminus\Delta) \text{ and } abc\in S(\Delta). \end{cases}$$

Now, we shall define here (Γ,Δ)-semiregularity, $\Gamma(\Delta)$-regularity, quasi-separativity and quasi-regularity. S is called *(Γ, Δ)-semiregular* (or *(Γ, Δ)-separative*) if it satisfies the condition (C) in Theorem 1. Further, S is called *quasi-separative* if it is (Γ,Δ)-semiregular for some subset Δ of Γ. Moreover, S is is called *$\Gamma(\Delta)$-regular* (or *$\Gamma(\Delta)$-separative*) if it satisfies the condition (C^*) in the foregoing corollary. And S is called *quasi-regular* (or *separative*) if it is $\Gamma(\Delta)$-regular for some subset of Γ. Of course, it is obvious from the definition that $\Gamma(\Delta)$-regularity is equivalent to $\Gamma(\Gamma\setminus\Delta)$-regularity.

Under these definitions, Theorem 1 and its corollary can be paraphrased as follows.

THEOREM 1′. *$\mathfrak{R}_\Delta$ is a congruence on S if and only if S is (Γ,Δ)-semiregular.*

COROLLARY. *Both $\mathfrak{R}_\Delta$ and $\mathfrak{R}_{\Gamma\setminus\Delta}$ are congruences on S if and only if S is $\Gamma(\Delta)$-regular.*

REMARK. (Γ, ϕ)-semiregularity ((Γ, Γ)-semiregularity, $\Gamma(\phi)-(\equiv\Gamma(\Gamma)-)$ regularity) coincides with left semiregularity (right semiregularity, regularity). Accordingly, it is particularly noted from Theorem 1′ and its corollary that

$$\left\{\begin{array}{l} \mathfrak{R}_\phi \text{ is a congruence} \\ \mathfrak{R}_\Gamma \text{ is a congruence} \\ \text{both } \mathfrak{R}_\phi \text{ and } \mathfrak{R}_\Gamma \text{ are congruences} \end{array}\right\} \text{ on } S \text{ if and only if S is } \left\{\begin{array}{l} \text{left semiregular} \\ \text{right semiregular} \\ \text{regular} \end{array}\right\}.$$

Therefore, Theorem 1′ or its corollary can be considered as a generalization of the first half of Theorem I or II respectively.

The next theorem gives a necessary and sufficient condition for S to be quasi- separative:

THEOREM 2. *S is quasi-separative if and only if it is the class sum of mutually disjoint subsets A, B having the properties*

(1) $A\ni a$, $axa=a$ and $xax=x$ imply $x\in A$,

(2) *$B \ni b$, $byb=b$ and $yby=y$ imply $y \in B$,*

(3) *$A \ni ab$ and $B \ni abc$ imply $abac=bac$,*

(4) *$A \ni abc$ and $B \ni ab$ imply $caba=cab$,*

(5) *$B \ni ab$ and $B \ni abc$ imply $abcabac=abac$,*

(6) *$A \ni ab$ and $A \ni abc$ imply $cabacba=caba$.*

Proof. Necessity. By the definition of quasi-separativity, for some $\Delta \subset \Gamma$, $\mathfrak{R}_\Delta$ is a congruence on S. Let $A=S(\Delta)$ and $B=S(\Gamma \backslash \Delta)$. Then, it is easily seen from Theorem 1 that A and B have the properties (1)-(6). Since $S=A \cup B$, the proof of the 'only if' part is complete.

Sufficiency. Suppose that S is partitioned into disjoint subsets A, B having the properties (1)-(6). Let $\Delta=\{\alpha : A \cap S_\alpha \neq \phi,\ \alpha \in \Gamma\}$. We shall first prove the relation $A=S(\Delta)$. Take up an arbitrary element x from $S(\Delta)$. Then, there exists S_β such that $x \in S_\beta \subset S(\Delta)$. Since the set $A \cap S_\beta$ is non-empty, there exists b_β such that $b_\beta \in A \cap S_\beta$.

By the rectangularity of S_β, we have $b_\beta x b_\beta = b_\beta$ and $x b_\beta x = x$, which implies $x \in A$ by the property (1). Thus we have $S(\Delta) \subset A$. Since the converse relation $A \subset S(\Delta)$ is clear, the relation $A= S(\Delta)$ holds. Now, it follows from the relation $A=S(\Delta)$, Theorem 1 and the properties (3)-(6) that $\mathfrak{R}_\Delta$ is a congruence on S. Therefore, the 'if' part is valid.

COROLLARY. *S is quasi-regular if and only if it is the class sum of mutually disjoint subsets A, B having the properties*

(1) *$A \ni a$, $axa=a$ and $xax=x$ imply $x \in A$,*

(2) *$B \ni b$, $byb=b$ and $yby=y$ imply $y \in B$,*

$$(3)\quad \left.\begin{array}{l} A \ni ab \text{ and } A \ni abc \\ \text{or} \\ B \ni ab \text{ and } B \ni abc \end{array}\right\} \text{imply } cabacba=caba \text{ and } abcabac=abac,$$

$$(4)\quad \left.\begin{array}{l} A \ni ab \text{ and } B \ni abc \\ \text{or} \\ A \ni abc \text{ and } B \ni ab \end{array}\right\} \text{imply } abac=bac \text{ and } caba=cab.$$

2. The structure of quasi-regular bands.

A band is called *bi-regular* if for any given elements a, b it satisfies at least one of the relations $aba=ba$ and $aba=ab$.

LEMMA 5. *S is left singular or right singular if and only if it is rectangular and bi-regular.*

Proof. Let S be rectangular and bi-regular. Then S may be considered as the direct product $L \times R$ of a left singular band L and a right singular band R, since S is rectangular. (See Kimura [1]).

Pick up a_1 and a_2 from L, and b_1 and b_2 from R.

Then,

$$(a_1,b_1)\,(a_2,b_2)=(a_1,b_2) \text{ and } (a_2,b_2)\,(a_1,b_1)=(a_2,b_1).$$

On the other hand,

$$(a_1,b_1) = (a_1,b_1)\,(a_2,b_2)\,(a_1,b_1) = \begin{cases} (a_1,b_1)\,(a_2,b_2) \\ \text{or} \\ (a_2,b_2)\,(a_1,b_1) \end{cases}$$

by bi-regularity.

Hence we have either $(a_1,b_1)=(a_1,b_2)$ or $(a_1,b_1)=(a_2,b_1)$, whence $b_1=b_2$ or $a_1=a_2$.

Accordingly, at least one of L and R consists of a single element. This means that S is left singular or right singular. Thus, the proof of the 'if' part is complete. The 'only if' part is clear.

The global structure of bi-regular bands is given by

THEOREM 3. *S is bi-regular if and only if each γ-kernel S_γ is left singular or right singular.*

Proof. Necessity. Let S be bi-regular. It is easily seen that any subband of a bi-regular band is bi-regular. Hence, each γ-kernel S_γ is rectangular and bi-regular. According to Lemma 5, S_γ is then left singular or right singular.

Sufficiency. Assume that each γ-kernel S_γ of S is left singular or right singular. Let a_α and b_β be arbitrary elements of S. Then, both $a_\alpha b_\beta$ and $b_\beta a_\alpha$ are contained in $S_{\alpha\beta}$.

Now, we have

$$a_\alpha b_\beta a_\alpha = (a_\alpha b_\beta)(b_\beta a_\alpha) = \begin{cases} a_\alpha b_\beta & \text{if } S_{\alpha\beta} \text{ is left singular,} \\ b_\beta a_\alpha & \text{if } S_{\alpha\beta} \text{ is right singular.} \end{cases}$$

Thus, S is bi-regular.

Let $B \equiv \{B_\omega : \Omega\}$ be a bi-regular band. From Theorem 3, each ω-kernel B_ω is then left singular or right singular. Let Λ be a subset of Ω.

B is said to be *(Ω,Λ)-regular* if it satisfies the following (P):

$$(P) \quad \begin{cases} \text{For } \omega \in \Lambda,\ B_\omega \text{ is left singular.} \\ \text{For } \omega \notin \Lambda,\ B_\omega \text{ is right singular.} \end{cases}$$

It should be noted that (Ω,Ω)-regularity and (Ω,ϕ)-regularity coincide with left regularity and right regularity respectively. Further, it is sometimes possible that a band $B \equiv \{B_\omega : \Omega\}$ is both (Ω,Λ_1)-and (Ω,Λ_2)-regular for some different subsets Λ_1 and Λ_2 of Ω.

For example, take up a commutative band (i.e. semilattice) T. Then, the structure decomposition of T is $T \sim \sum\{\{t\} : t \in T\}$. Since every t-kernel consists of the single element t, it is left and right singular. Hence, T is (T,T_1)-regular for an arbitrary subet T_1 of T.

THEOREM 4. *Let S be (Γ,Δ)-semiregular. Then the quotient semigroup $S/\Re_\Delta$ is a $(\Gamma,\Gamma\backslash\Delta)$-regular band, and its structure decomposition is $S/\Re_\Delta \sim \sum\{S_\gamma/\Re_\Delta : \gamma \in \Gamma\}$.*

Proof. We shall first prove that $S/\Re_\Delta$ is bi-regular. Denote by $\bar{x}$ the congruence class conthining x mod $\Re_\Delta$. Pick up two congruence classes $\bar{a}_\alpha$, $\bar{b}_\beta$ from $S/\Re_\Delta$. If $\alpha\beta \in \Delta$ then $a_\alpha b_\beta a_\alpha b_\beta a_\alpha = a_\alpha b_\beta a_\alpha$, whence $\bar{a}_\alpha \bar{b}_\beta \bar{a}_\alpha = \bar{b}_\beta \bar{a}_\alpha$. If conversely $\alpha\beta \notin \Delta$ then $a_\alpha b_\beta a_\alpha a_\alpha b_\beta = a_\alpha b_\beta$, whence $\bar{a}_\alpha \bar{b}_{\beta\alpha\alpha} = \bar{a}_\alpha \bar{b}_\beta$.

Therefore

$$\bar{a}_\alpha \bar{b}_\beta \bar{a}_\alpha = \begin{cases} \bar{a}_\alpha \bar{b}_\beta \\ \text{or} \\ \bar{b}_\beta \bar{a}_\alpha. \end{cases}$$

This shows $S/\Re_\Delta$ to be bi-regular. Further, it is easily seen that the structure decomposition of $S/\Re_\Delta$ is $S/\Re_\Delta \sim \sum\{S_\gamma/\Re_\Delta : \gamma \in \Gamma\}$. Thus, to complete the proof it is sufficient to prove that S_γ/R_Δ is left singular if $\gamma \notin \Delta$ and $S_\gamma/\Re_\Delta$ is right singular if $\gamma \in \Delta$.

(i) The case $\gamma \in \Delta$. Pick up two different elements $\bar{a}, \bar{b}$ ($a \in S_\gamma$, $b \in S_\gamma$) from $S_\gamma/\Re_\Delta$.

Then, $\bar{a} = \overline{aba} = \bar{a}\,\bar{b}\,\bar{a} = \bar{a}\,\bar{b}$ or $= \bar{b}\,\bar{a}$ by the bi-regularity of S. If $\bar{a} = \bar{a}\,\bar{b}$, then we have $\bar{a} = \bar{b}$ which is contrary to $\bar{a} \neq \bar{b}$. Hence, $\bar{a} = \bar{b}\,\bar{a}$. Thus, $S_\gamma/\Re_\Delta$ is right singular.

(ii) The case $\gamma \notin \Delta$. The left singularity of $S_\gamma/\Re_\Delta$ is proved by an analogous argument to that in (i).

THEOREM 5. *If both $\Re_\Delta$ and $\Re_{\Gamma\setminus\Delta}$ are congruences on S, then S is isomorphic to the spined poduct of $S/\Re_\Delta$ and $S/\Re_{\Gamma\setminus\Delta}$ with respect to Γ.*

Proof. Define the relation $\mathfrak{D}$ on S as follows:

$a \mathfrak{D} b$ if and only if a and b are contained in a common S_γ.

Then,

(i) $\mathfrak{R}_\Delta$, $\mathfrak{R}_{\Gamma\setminus\Delta} \leqq \mathfrak{D}$

(ii) $\mathfrak{R}_\Delta \cap \mathfrak{R}_{\Gamma\setminus\Delta} = O$

and (iii) $\mathfrak{R}_\Delta \cup \mathfrak{R}_{\Gamma\setminus\Delta} = \mathfrak{D}$

are obvious.

Next, we shall prove

(iv) $\mathfrak{R}_\Delta$, $\mathfrak{R}_{\Gamma\setminus\Delta}$ are permutable.

Let $a\, \mathfrak{R}_\Delta\, x$ and $x \mathfrak{R}_{\Gamma\setminus\Delta} b$. Then, there exists S_γ containing a, x, b. If $\gamma \in \Delta$, then $b \mathfrak{R}_\Delta ab$ and $ba \mathfrak{R}_{\Gamma\setminus\Delta}\, a$ follow from $bab=b$ and $aab=ab$. Conversely if $\gamma \notin \Delta$, then $b\, \mathfrak{R}_\Delta\, ba$ and $ba\, \mathfrak{R}_{\Gamma\setminus\Delta}\, a$ follow from $bba=ba$ and $aba=a$. Thus, it was proved that there exists an element y such that $b\, \mathfrak{R}_\Delta y$ and $y\, \mathfrak{R}_{\Gamma\setminus\Delta}\, a$. Therefore, $\mathfrak{R}_\Delta$, $\mathfrak{R}_{\Gamma\setminus\Delta}$ are permutable.

Since $\mathfrak{R}_\Delta$, $\mathfrak{R}_{\Gamma\setminus\Delta}$ satisfy (i)-(iv), S is isomorphic to the spined product of $S/\mathfrak{R}_\Delta$ and $S/\mathfrak{R}_{\Gamma\setminus\Delta}$ with respect to Γ. (See Yamada [5]).

Combining Theorems 4 and 5, we have

COROLLARY. *If S is $\Gamma(\Delta)$-regular, then S is isomorphic to the spined product of a $(\Gamma,\Gamma/\Delta)$-regular band and a (Γ,Δ)-regular band with respect to Γ.*

REMARK. It is noted that Theorem 4 (5) is a generalization of the latter half of Theorem I (II).

Let $B_1 \equiv \{B_\omega^{(1)}: \Omega\}$ and $B_2 \equiv \{B_\omega^{(2)}: \Omega\}$ be bi-regular bands with the same structure semilattice Ω. Then, B_1 and B_2 are called *mutually associated bands* if the following (A) is satisfied:

$$\text{(A)} \quad \text{For any given } \omega \in \Omega, \quad \begin{cases} B_\omega^{(1)} \text{ is left singular and } B_\omega^{(2)} \text{ is right singular.} \\ \text{or} \\ B_\omega^{(1)} \text{ is right singular and } B_\omega^{(2)} \text{ is left singular.} \end{cases}$$

From the definition of quasi-regularity and the foregoing corollary, we have immediately.

THEOREM 6. *A quasi-regular band $B \equiv \{B_\omega: \Omega\}$ is isomorphic to the spined product of mutually associated bi-regular bands with respect to Ω.*

COROLLARY. *S is isomorphic to the spined product of mutually associated bi-regular bands with respect to Γ if it is the class sum of mutually disjoint subsets A, B having the properties (1)-(4) in Corollary to Theorem 2.*

3. Example.

Let Ω be the semilattice consisting of 0 and 1 with respect to the ordinary multiplication.

And let

$$B_1 = \{(a_i, a_j^*): i, j=1, 2\},$$

$$B_0 = \{(b_m, b_n^*): m, n=1, 2, 3\}$$

and $B=B_1 \cup B_0$.

Then, B becomes a band with respect to the multiplication defined by

(I) $(a_i,a_j^*)\,(a_k,a_s^*)=(a_k,a_j^*)$ for all $i, j, k, s,$

(II) $(b_m,b_n^*)\,(b_t,b_u^*)=(b_m,b_u^*)$ for all $m, n, t, u,$

$$\text{(III)}\ (a_i,a_j^*)\,(b_m,b_n^*)=\begin{cases} (b_m,b_n^*) & \text{for } m=2,3 \text{ and for all } i, j, n, \\ (b_3,b_n^*) & \text{for } i=1,\ m=1 \text{ and for all } j, n, \\ (b_2,b_n^*) & \text{for } i=2,\ m=1 \text{ and for all } j, n, \end{cases}$$

(IV) $(b_m, b_n^*)(a_i, a_j^*) = \begin{cases} (b_m, b_n^*) & \text{for } n=2,3 \text{ and for all } i, j, m, \\ (b_m, b_3^*) & \text{for } n=1, j=1 \text{ and for all } i, m, \\ (b_m, b_2^*) & \text{for } n=1, j=2 \text{ and for all } i, m. \end{cases}$

The structure decomposition of B is $B \sim \sum\{B_\omega : \omega \in \Omega\}$. Further, it is easily seen from simple consideration that both $\mathfrak{R}_{\{0\}}$ and $\mathfrak{R}_{\{1\}}$ are congruences on B although neither $\mathfrak{R}_\phi$ nor $\mathfrak{R}_\Omega$ is a congruence. That is, B is a $\Omega(\{0\})$-regular band and isomorphic to the spined product of the mutually associated bands $B/\mathfrak{R}_{\{0\}}$ and $B/\mathfrak{R}_{\{1\}}$ with respect to Ω, but B is neither left semiregular nor right semiregular. Consequently, it has been proved that there exists a quasi-regular band which is neither left semiregular nor right semiregular.

REFERENCES

1. Naoki Kimura: *The structure of idempotent semigroups (1)*, Pacific Journal of Math., 8(1958), 257-275.
2. Naoki Kimura: *Note on idempotent semigroups III*, Proc. Japan Acad., 34 (1958), 113-114.
3. David McLean: *Idempotent semigroups*, Amer. Math. Monthly, 61 (1954), 110-113.
4. Miyuki Yamada: *Gertain congruences and the structure of some special bands*, Proc. Japan Acad., 36 (1960), 408-410.
5. Miyuki Yamada: *A note on subdirect decompositions of idempotent semigroups*, Proc. Japan Acad., 36 (1960), 411-414.