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TYPE (A) HYPERSURFACES IN A COMPLEX PROJECTIVE SPACE

SADAHIRO MAEDA

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ABSTRACT. In a complex $n(\geqq 2)$-dimensional complex projective space $\mathbb{C}P^n(c)$ of constant holomorphic sectional curvature $c(>0)$, a type (A) hypersurface is one of fundamental examples in the theory of real hypersurfaces isometrically immersed into this ambient space. The purpose of this paper is to make a survey of fundamental properties of type (A) hypersurfaces in $\mathbb{C}P^n(c)$.

1. INTRODUCTION

We first recall the classification theorem of homogeneous real hypersurfaces M^{2n-1} in $\mathbb{C}P^n(c), n \geqq 2$, that is they are orbits of some subgroups of the full isometry group $\mathrm{I}(\mathbb{C}P^n(c))(=\mathrm{SU}(n+1))$. By virtue of the results in [32, 25, 12] we obtain the following.

In $\mathbb{C}P^n(c)$ $(n \geqq 2)$, a homogeneous real hypersurface is locally congruent to one of the following Hopf hypersurfaces all of whose principal curvatures are constant:

(A_1) A geodesic sphere of radius r, where $0 < r < \pi/\sqrt{c}$;
(A_2) A tube of radius r around a totally geodesic $\mathbb{C}P^\ell(c)$ $(1 \leqq \ell \leqq n-2)$, where $0 < r < \pi/\sqrt{c}$;
(B) A tube of radius r around a complex hyperquadric $\mathbb{C}Q^{n-1}$, where $0 < r < \pi/(2\sqrt{c}\,)$;
(C) A tube of radius r around the Segre embedding of $\mathbb{C}P^1(c) \times \mathbb{C}P^{(n-1)/2}(c)$, where $0 < r < \pi/(2\sqrt{c}\,)$ and n $(\geqq 5)$ is odd;
(D) A tube of radius r around the Plüker embedding of a complex Grassmannian $\mathbb{C}G_{2,5}$, where $0 < r < \pi/(2\sqrt{c}\,)$ and $n = 9$;
(E) A tube of radius r around a Hermitian symmetric space $\mathrm{SO}(10)/\mathrm{U}(5)$, where $0 < r < \pi/(2\sqrt{c}\,)$ and $n = 15$.

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These real hypersurfaces are said to be of types (A_1), (A_2), (B), (C), (D) and (E). Unifying real hypersurfaces of types (A_1) and (A_2), we call them *type* (A) *hypersurfaces.* The numbers of distinct principal curvatures of these real hypersurfaces are $2, 3, 3, 5, 5, 5$, respectively (for details, see [33]).

We next recall the following fact (see [18]): Let M^{2n-1} be a real hypersurface in $\mathbb{C}P^n(c), n \geqq 2$. Then the length of the derivative of the shape operator A of M satisfies $\|\nabla A\|^2 \geqq (c^2/4)(n-1)$ at its each point. In particular, $\|\nabla A\|^2 = (c^2/4)(n-1)$ holds on M if and only if M is locally congruent to a type (A) hypersurface. So it is natural to pay attention to type (A) hypersurfaces in the class of all real hypersurfaces in $\mathbb{C}P^n(c)$.

On the other hand, $\mathbb{C}P^n(c)$ does not admit totally umbilic real hypersurfaces M^{2n-1}. Hence, there exist no real hypersurfaces all of whose geodesics are mapped to circles in this space. So, in some sense the geometry of real hypersurfaces in $\mathbb{C}P^n(c)$ is a bit complicated.

The purpose of this paper is to survey geometric properties of type (A) hypersurfaces M by observing the extrinsic shape of geodesics on M from the ambient space $\mathbb{C}P^n(c)$. In particular, we investigate hypersurfaces of type (A_1) in detail.

Needless to say, there do exist similarly real hypersurfaces isometrically immersed into a complex $n(\geqq 2)$-dimensional complex hyperbolic space $\mathbb{C}H^n(c)$ of constant holomorphic sectional curvature $c(< 0)$, which are called type (A) hypersurfaces (for details, see [25]). There are some analogous results to our Theorem 1, Theorem 2, Theorem 3, Theorem 5, Theorem 6, Theorem 7 and Theorem 10. However we emphasize that analogous results to our Theorem 4, Theorem 8 and Theorem 9 do *not* hold.

2. Terminologies and fundamental results on real hypersurfaces

Let M^{2n-1} be a real hypersurface with unit normal vector field $\mathcal{N}$ of an $n(\geqq 2)$-dimensional complex projective space $\mathbb{C}P^n(c)$ of constant holomorphic sectional curvature $c(> 0)$. The Riemannian connections $\widetilde{\nabla}$ of $\mathbb{C}P^n(c)$ and ∇ of M are related by the following:

$$\widetilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)\mathcal{N} \quad \text{and} \quad \widetilde{\nabla}_X \mathcal{N} = -AX \tag{2.1}$$

for all vector fields X and Y on M, where g denotes the metric induced from the standard Riemannian metric of $\mathbb{C}P^n(c)$ and A is the shape operator of M in $\mathbb{C}P^n(c)$ associated with $\mathcal{N}$. On M an almost contact metric structure (ϕ, ξ, η, g) associated with $\mathcal{N}$ is canonically induced from the Kähler structure (g, J) of the ambient space $\mathbb{C}P^n(c)$. They are defined by

$$g(\phi X, Y) = g(JX, Y),\ \xi = -J\mathcal{N} \quad \text{and} \quad \eta(X) = g(\xi, X) = g(JX, \mathcal{N}).$$

Note that by changing $\mathcal{N}$ for $-\mathcal{N}$ we have two almost contact metric structures (ϕ, ξ, η, g) and $(\phi, -\xi, -\eta, g)$ on M. It follows from (2.1) and the property $\widetilde{\nabla} J = 0$ that

(2.2)
$$\nabla_X \xi = \phi AX \quad \text{and} \quad (\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi \quad \text{for each } X, Y \in TM.$$

The above equations do not depend on the choice of the unit normal vector $\mathcal{N}$. We denote by R the curvature tensor of M. Then R is given by

$$
\begin{aligned}
(2.3) \qquad g((R(X,Y)Z,W) = (c/4)\{&g(Y,Z)g(X,W) - g(X,Z)g(Y,W) \\
&+ g(\phi Y,Z)g(\phi X,W) - g(\phi X,Z)g(\phi Y,W) - 2g(\phi X,Y)g(\phi Z,W)\} \\
&+ g(AY,Z)g(AX,W) - g(AX,Z)g(AY,W).
\end{aligned}
$$

The following is called the equation of Codazzi.

$$(\nabla_X A)Y - (\nabla_Y A)X = (c/4)(\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X,Y)\xi).$$

Let K be the sectional curvature of M. That is, K is defined by $K(X,Y) = g(R(X,Y)Y,X)$, where X and Y are orthonormal vectors on M. Then it follows from (2.3) that

$$K(X,Y) = (c/4)\big(1 + 3g(\phi X,Y)^2\big) + g(AX,X)g(AY,Y) - g(AX,Y)^2.$$

We call eigenvalues and eigenvectors of the shape operator A *principal curvatures* and *principal curvature vectors* of M in $\mathbb{C}P^n(c)$, respectively. Here and in the following, we set $V_\lambda := \{X \in TM | AX = \lambda X\}$. We usually call M a *Hopf hypersurface* if the characteristic vector ξ of M is a principal curvature vector at each point of M.

3. Circles in Riemannian geometry

The notion of circles is a key in this paper. A smooth real curve $\gamma = \gamma(s)$ parametrized by its arclength s on a Riemannian manifold M with Riemannian connection ∇ is called a *circle* of curvature k if it satisfies ordinary equations

$$(3.1) \qquad \nabla_{\dot\gamma}\dot\gamma = kY_s \quad \text{and} \quad \nabla_{\dot\gamma}Y_s = -k\dot\gamma$$

along the curve γ, where k is a positive constant and Y_s is the principal normal unit vector perpendicular to $\dot\gamma(s)$. A geodesic is regarded as a circle of null curvature. By virtue of the unique existence theorem on ordinary equations for given each positive k and each pair of orthonormal vectors X and Y at an arbitrary point p of M there exists locally the unique circle $\gamma = \gamma(s)$ of curvature k satisfying the initial condition that $\gamma(0) = p, \dot\gamma(0) = X$ and $Y_0 = Y$. It is known that in a complete Riemannian manifold every circle can be defined for $-\infty < s < \infty$ (see [27]).

In this paper, we consider circles in $\mathbb{C}P^n(c), n \geqq 2$. Given a circle $\gamma = \gamma(s)$ satisfying (3.1) we call $\tau(s) := g(\dot\gamma(s), JY_s)$ the *holomorphic torsion* of γ, where J is the standard complex structure of $\mathbb{C}P^n(c)$. The function $-1 \leqq \tau \leqq 1$ is constant along every circle γ in $\mathbb{C}P^n(c)$. In fact,

$$
\begin{aligned}
\nabla_{\dot\gamma}(g(\dot\gamma, JY_s)) &= g(\nabla_{\dot\gamma}\dot\gamma, JY_s) + g(\dot\gamma, J\nabla_{\dot\gamma}Y_s) \\
&= k\cdot g(Y_s, JY_s) - k\cdot g(\dot\gamma, J\dot\gamma) = 0.
\end{aligned}
$$

It is well-known that all geodesics on $\mathbb{C}P^n(c)$ are congruent to each other by some $\varphi \in \mathrm{I}(\mathbb{C}P^n(c))$. The congruence theorem on circles of positive curvature in $\mathbb{C}P^n(c)$ is expressed as:

Lemma 1 ([6]). *Let $\gamma_i = \gamma_i(s)\,(i = 1, 2)$ be circles of positive curvature k_i and holomorphic torsion τ_i in $\mathbb{C}P^n(c), n \geqq 2$. Then these two circles γ_1 and γ_2 are congruent to each other if and only if $k_1 = k_2$ and $|\tau_1| = |\tau_2|$. Precisely, when $\tau_1 = \tau_2$ (resp. $\tau_1 = -\tau_2$), they are congruent to each other by some holomorphic isometry (resp. anti-holomorphic isometry) in this space.*

A circle γ of the holomorphic torsion τ with $\tau = 1$ or $\tau = -1$ (resp. $\tau = 0$) is called a *Kähler circle* (resp. *totally real circle*). Note in $\mathbb{C}P^n(c)$ that a circle γ is totally real if and only if γ lies locally on a totally real totally geodesic surface $\mathbb{R}P^2(c/4)$ of constant sectional curvature $c/4$ and a circle γ is Kähler if and only if γ lies locally on a totally geodesic holomorphic line $\mathbb{C}P^1(c)$. They are closed curves. But, in general a circle with holomorphic torsion $\tau \neq 0, \pm 1$ is not necessarily closed (see [6, 3]).

4. Frenet curves

In general, M^n denotes a real n-dimensional Riemannian manifold. In this paper, M_n denotes a complex n-dimensional Kähler manifold.

A smooth curve $\gamma = \gamma(s)$ parametrized by its arclength s on a complex n-dimensional Kähler manifold M_n furnished with Riemannian metric g and Riemannian connection ∇ is said to be a *Frenet curve of proper order* $d\,(2 \leqq d \leqq 2n)$ if there exist an orthonormal system $\{V_1 = \dot{\gamma}, V_2, V_3, \ldots, V_d\}$ of vector fields along γ and positive smooth functions $\kappa_1(s), \ldots, \kappa_{d-1}(s)$ satisfying the following system of ordinary differential equations:

$$\nabla_{\dot{\gamma}} V_j(s) = -\kappa_{j-1}(s)V_{j-1}(s) + \kappa_j(s)V_{j+1}(s),\ 1 \leqq j \leqq d.$$

Here, $\kappa_0 V_0$ and $\kappa_d V_{d+1}$ are null vector fields along γ. The functions $\kappa_1, \ldots, \kappa_{d-1}$ and the orthonormal frames $\{V_1, V_2, \ldots, V_d\}$ are called the *curvatures* and the *Frenet frame* of the curve γ, respectively. Roughly speaking, a Frenet curve is a smooth curve having no inflection points of higher order. For the Frenet frame $\{V_1, V_2, \ldots, V_d\}$ of γ, we set $\tau_{ij}(s) := g(V_i(s), JV_j(s))$ with $1 \leqq i < j \leqq d$ and call them the *holomorphic torsions* along γ.

A Frenet curve is said to be a *helix* when all of its curvatures $\kappa_1, \ldots, \kappa_{d-1}$ are constant functions. A helix of proper order 2 is a circle of curvature $k(= \kappa_1 > 0)$.

A real curve $\gamma = \gamma(s)$ in M_n is said to be *homogeneous* if it is an orbit of one-parameter subgroup of $\mathrm{I}(M_n)$.

In the following, we adopt $\mathbb{C}P^n(c), n \geqq 2$ as a Kähler manifold M_n. In the study of Frenet curves in $\mathbb{C}P^n(c)$, the notion of holomorphic torsions plays an important role. We can give the necessary and sufficient condition for a Frenet curve to be homogeneous in $\mathbb{C}P^n(c)$ by using the notion of curvatures and holomorphic torsions. In fact, the following is known.

Proposition 1 ([21]). *A real curve $\gamma = \gamma(s)$ is a homogeneous curve in $\mathbb{C}P^n(c), n \geqq 2$ if and only if it is a helix and all of its holomorphic torsions are constant functions.*

For a circle γ of positive curvature k in $\mathbb{C}P^n(c)$, we have just one holomorphic torsion $\tau(s) := \tau_{12}(s) = g(V_1(s), JV_2(s))$. By easy computation in the previous section we find that the holomorphic torsion τ of a circle γ of positive curvature is automatically a constant function. Hence, by Proposition 1 we see that every circle of positive curvature is homogeneous in $\mathbb{C}P^n(c)$. This, together with a fact that all geodesics are homogeneous curves in $\mathbb{C}P^n(c)$, implies that all circles are homogeneous in this space. Hence every circle in $\mathbb{C}P^n(c)$ is an integral curve of some (Killing) vector field, which implies that it is a simple curve in this space.

5. Congruence theorem on geodesics of type (A) hypersurfaces

We first review the following:

Lemma 2 ([29, 18])**.** *Let M be a real hypersurface isometrically immersed into $\mathbb{C}P^n(c), n \geqq 2$. Then the following four conditions are mutually equivalent*:
(1) *M is locally congruent to a type* (A) *hypersurface*;
(2) *The shape operator A of M satisfies*

$$(\nabla_X A)Y = -(c/4)(g(\phi X, Y)\xi + \eta(Y)\phi X) \quad \textit{for } X, Y \in TM;$$

(3) *The shape operator A' of a hypersurface $\pi^{-1}(M)$ in a Euclidean sphere $S^{2n+1}(c/4)$ of constant sectional curvature $c/4$ is parallel, where $\pi : S^{2n+1}(c/4) \to \mathbb{C}P^n(c)$ is the Hopf fibration*;
(4) *The structure tensor ϕ and the shape operator A of M satisfy $\phi A = A\phi$ on M.*

Remark 1. We explain principal curvatures of type (A) hypersurfaces M in $\mathbb{C}P^n(c)$ with $n \geqq 2$. It is well-known that if M is of type (A_1), M has two distinct constant principal curvatures $\delta = \sqrt{c}\,\cot(\sqrt{c}\,r)$ with multiplicity 1 and $\lambda = (\sqrt{c}/2)\cdot \cot(\sqrt{c}\,r/2)$ with multiplicity $2n-2$, and that if M is of type (A_2), M has three distinct constant principal curvatures $\delta = \sqrt{c}\,\cot(\sqrt{c}\,r)$ with multiplicity 1, $\lambda_1 = (\sqrt{c}/2)\cot(\sqrt{c}\,r/2)$ with multiplicity $2n-2\ell-2$ and $\lambda_2 = -(\sqrt{c}/2)\tan(\sqrt{c}\,r/2)$ with multiplicity 2ℓ, where $A\xi = \sqrt{c}\,\cot(\sqrt{c}\,r)\xi$ and δ can be expressed as: $\delta = \lambda_1 + \lambda_2$ (cf. [25]).

Remark 2. We review another expression of type (A) hypersurfaces M in $\mathbb{C}P^n(c)$ with $n \geqq 2$. We set $M' = \pi^{-1}M$. By Lemma 2(3) we know that M' is a Clifford hypersurface $M_{2p+1,2\ell+1}(c_1, c_2) := S^{2p+1}(c_1) \times S^{2\ell+1}(c_2)$ in the ambient sphere $S^{2n+1}(c/4)$, where p, ℓ are integers with $p + \ell = n-1$ and $p \geqq \ell \geqq 0$ except $p = \ell = 0$, and c_1, c_2 are positive constants with $1/c_1 + 1/c_2 = 4/c$. As a matter of course c_1, c_2 and $c/4$ are sectional curvatures of these spheres. This hypersurface $M_{2p+1,2\ell+1}(c_1, c_2)$ has two distinct constant principal curvatures $c_1/\sqrt{c_1+c_2}$ with multiplicity $2p+1$ and $-c_2/\sqrt{c_1+c_2}$ with multiplicity $2\ell+1$. Then the real hypersurface $M^{\mathbb{C}}_{p,\ell} := \pi(M_{2p+1,2\ell+1}(c_1, c_2))$ with $p\ell \neq 0$ in $\mathbb{C}P^n(c)$ has three constant principal curvatures $(c_1 - c_2)/\sqrt{c_1+c_2}$ with multiplicity 1 which is the principal curvature of the characteristic vector ξ on $M^{\mathbb{C}}_{p,\ell}$, $c_1/\sqrt{c_1+c_2}$ with multiplicity $2p$ and $-c_2/\sqrt{c_1+c_2}$ with multiplicity 2ℓ (for details, see [18]). Note that the hypersurface $M^{\mathbb{C}}_{p,\ell} = \pi(M_{2p+1,2\ell+1}(c_1, c_2))$ is either a hypersurface of type (A_1) or a hypersurface of type (A_2) in the introduction when $\ell = 0$ or $\ell > 0$, respectively. Hence, the radius r of a type (A) hypersurface must satisfy $\cot(\sqrt{c}\,r/2) = \sqrt{c_1/c_2}$.

Next, let $\gamma = \gamma(s)$ be a geodesic parametrized by its arclength s on a type (A) hypersurface M^{2n-1} in $\mathbb{C}P^n(c), n \geqq 2$. We consider two functions $\rho_\gamma = \rho_\gamma(s)$ and $\kappa_\gamma = \kappa_\gamma(s)$ along the curve γ defined by $\rho_\gamma(s) := g(\dot\gamma(s), \xi_{\gamma(s)})$ and $\kappa_\gamma(s) := g(A\dot\gamma(s), \dot\gamma(s))$. Then by the first equality in (2.2) and the skew-symmetry of ϕ : $g(\phi X, Y) = -g(X, \phi Y)$ and Lemma 2 we see that these functions ρ_γ and κ_γ are constant along each geodesic γ on every type (A) hypersurface. We call ρ_γ and κ_γ of a geodesic γ on a type (A) hypersurface M the *strcture torsion* and the *normal curvature* of the curve γ, respectively.

Using these two invariants ρ_γ and κ_γ of a geodesic γ, we can describe the following congruence theorem on geodesics:

Lemma 3 ([4]). *Let $\gamma_i\,(i = 1, 2)$ be geodesics on a type (A) hypersurface M in $\mathbb{C}P^n(c), n \geqq 2$. Then the following hold*:
(A_1) *When M is of type* (A_1)*, the structure torsions of these curves satisfy $|\rho_{\gamma_1}| = |\rho_{\gamma_2}|$ if and only if they are congruent to each other, i.e., there exists an isometry on M with $\gamma_2(s + s_0) = (\varphi \circ \gamma_1)(s)$ for each s and some s_0.*
(A_2) *When M is of type* (A_2)*, the structure torsions and normal curvatures of these curves satisfy $|\rho_{\gamma_1}| = |\rho_{\gamma_2}|$ and $\kappa_{\gamma_1} = \kappa_{\gamma_2}$ if and only if they are congruent to each other.*

Given a submanifold M^n isometrically immersed through f into a Riemannian manifold $\widetilde{M}^{n+p}$ we call a smooth curve $\gamma = \gamma(s)$ an *extrinsic geodesic* on M^n if the curve $f \circ \gamma$ is a geodesic in $\widetilde{M}^{n+p}$. Needless to say, every extrinsic geodesic is also a geodesic on the submanifold.

Here, using Lemma 3, we count the number of congruent classes of extrinsic geodesics on every type (A) hypersurface M with respect to the full isometry group $\mathrm{I}(M)$ of M.

Proposition 2. *Let M be a type* (A) *hypersurface of radius r $(0 < r < \pi/\sqrt{c}\,)$ of $\mathbb{C}P^n(c), n \geqq 2$. Then the following hold*:
(1) *When M is of type* (A_1) *of radius r $(0 < r < \pi/(2\sqrt{c}\,))$, M has no extrinsic geodesics.*
(2) *When M is of type* (A_1) *of radius r $(\pi/(2\sqrt{c}\,) \leqq r < \pi/\sqrt{c}\,)$, M has just one congruent class of extrinsic geodesics.*
(3) *When M is of type* (A_2) *of radius r $(0 < r < \pi/\sqrt{c}\,)$, M has uncountably infinite congruent classes of extrinsic geodesics.*

Proof. The proofs of Statements (1) and (2) are given in the proof of Theorem 1 in [30]. It remains to show Statement (3). For an extrinsic geodesic $\gamma = \gamma(s)$ on M of type (A_2) we set $\dot\gamma(0) = \rho_\gamma \xi_{\gamma(0)} + aX + bY$, where ρ_γ, a, b are nonnegative with $\rho_\gamma^2 + a^2 + b^2 = 1$ and X, Y are unit vectors with $X \in V_{(\sqrt{c}/2)\cot(\sqrt{c}\,r/2)}$ and $Y \in V_{-(\sqrt{c}/2)\tan(\sqrt{c}\,r/2)}$. By the constancy of the normal curvature κ_γ of the geodesic γ we see that γ is an extrinsic geodesic if and only if $g(A\dot\gamma(0), \dot\gamma(0)) = 0$. Then by simple computation we have $\rho_\gamma^2 = \tan^2(\sqrt{c}\,r/2) - a^2\sec^2(\sqrt{c}\,r/2)$. This, together with $\rho_\gamma^2 \leqq 1$, yields inequalities $\sin^2(\sqrt{c}\,r/2) - \cos^2(\sqrt{c}\,r/2) \leqq a^2 \leqq \sin^2(\sqrt{c}\,r/2)$. So we get a one-parameter family $\gamma_a = \gamma_a(s)$ of extrinsic geodesics on our type (A) hypersurface. Hence we obtain the desirable conclusion (see Lemma 3((A_2))). □

As an immediate consequence of the proof in Proposition 2(3) we have

Remark 3 ([30]). When M is of type (A_2) of radius $\pi/(2\sqrt{c}\,)$, M has a one-parameter family of extrinsic geodesics $\gamma_a = \gamma_a(s)$ with $0 \leqq a \leqq 1/\sqrt{2}$. The initial vector $\dot\gamma(0)$ is written as: $\dot\gamma(0) = \sqrt{1-2a^2}\,\xi_{\gamma(0)} + aX + aY$, where X and Y are unit vectors with $X \in V_{\sqrt{c}/2}$ and $Y \in V_{-\sqrt{c}/2}$. Note that two curves γ_a and γ_b are congruent to each other with respect to $\mathrm{I}(M)$ if and only if $a = b \in [0, 1/\sqrt{2}\,]$. On the contrary these curves are congruent to each other with respect to the isometry group $\mathrm{SU}(n+1)$ of the ambient space $\mathbb{C}P^n(c)$ for any $a, b \in [0, 1/\sqrt{2}\,]$.

6. Extrinsic shape of geodesics on type A hypersurfaces

It is known that every geodesic $\gamma = \gamma(s)$ on a type (A) hypersurface M is a homogeneous curve (see [24]). This, together with a fact that our real hypersurface M is homogeneous in $\mathbb{C}P^n(c)$, the curve γ is also mapped to a homogeneous curve in this ambient space. Hence the curve γ is a helix in $\mathbb{C}P^n(c)$, so that it has all constant curvatures $k_1, \ldots, k_{d-1}$. Motivated by this fact, we give the following characterization of type (A) hypersurfaces.

Theorem 1 ([17]). *Let M^{2n-1} be a connected real hypersurface isometrically immersed into $\mathbb{C}P^n(c), n \geqq 2$. Then M is locally congruent to a type* (A) *hypersurface if and only if every geodesic γ of M, considered as a curve in the ambient space $\mathbb{C}P^n(c)$, has constant first curvature $k_1 (:= \|\widetilde\nabla_{\dot\gamma}\dot\gamma\|)$ along γ, where $\widetilde\nabla$ is the Riemannian connection on $\mathbb{C}P^n(c)$.*

The following is fundamental.

Lemma 4. *Let M^n be a connected hypersurface isometrically immersed into a Riemannian manifold $\widetilde{M}^{n+1}$. Then the following are equivalent:*
(1) *The shape operator A of M^n in $\widetilde{M}^{n+1}$ is expressed as $A = \lambda \mathrm{I}_n$, where λ is a constant function on M^n and I_n is a unit matrix;*
(2) *Every geodesic γ on M^n, considered as a curve in the ambient space $\widetilde{M}^{n+1}$, is a circle;*
(3) *Every geodesic γ on M^n, considered as a curve in the ambient space $\widetilde{M}^{n+1}$, is a circle of the same curvature $|\lambda|$, where λ is given in Condition* (1).

By virtue of Lemma 4 and a fact that there exist no totally umbilic real hypersurfaces in $\mathbb{C}P^n(c)$ we find that $\mathbb{C}P^n(c)$ admits no real hypersurfaces M^{2n-1} all of whose geodesics are mapped to circles in this ambient space. We now pose the following problem:

Problem 1. Does there exist real hypersurfaces M^{2n-1} some of whose geodesics are mapped to circles of the same curvature in $\mathbb{C}P^n(c), n \geqq 2$ through an isometric immersion?

The following is a partial answer to Problem 1.

Fact 1 ([20]). (1) *Every geodesic $\gamma = \gamma(s)$ on M^{2n-1} of type* (A_1) *of radius r $(0 < r < \pi/\sqrt{c}\,)$ with initial vector $\dot\gamma(0)$ perpendicular to the characteristic vector $\xi_{\gamma(0)}$*

is mapped to a circle of the same positive curvature $(\sqrt{c}/2)\cot(\sqrt{c}\,r/2)$ *in* $\mathbb{C}P^n(c)$.
(2) *For a geodesic* $\gamma = \gamma(s)$ *on* M^{2n-1} *of type* (A_2) *the following hold*:
$(2i)$ *When the initial vector* $\dot\gamma(0)$ *is a principal curvature vector of* M^{2n-1} *in* $\mathbb{C}P^n(c)$ *with principal curvature* $(\sqrt{c}/2)\cot(\sqrt{c}\,r/2)$, *the curve* γ *is mapped to a circle of the same positive curvature* $(\sqrt{c}/2)\cot(\sqrt{c}\,r/2)$ *in the ambient space*;
$(2ii)$ *When the initial vector* $\dot\gamma(0)$ *is a principal curvature vector of* M^{2n-1} *in* $\mathbb{C}P^n(c)$ *with principal curvature* $-(\sqrt{c}/2)\tan(\sqrt{c}\,r/2)$, *the curve* γ *is mapped to a circle of the same positive curvature* $(\sqrt{c}/2)\tan(\sqrt{c}\,r/2)$ *in the ambient space.*

Considering the converse of Fact 1, we obtain the following (see Proposition 2):

Theorem 2 ([20, 30])**.** *Let* M^{2n-1} *be a connected real hypersurface isometrically immersed into* $\mathbb{C}P^n(c), n \geqq 2$. *Suppose that there exist orthonormal vectors* $v_1, v_2, \dots, v_{2n-2}$ *orthogonal to the characteristic vector* ξ_p *at each fixed point* p *of* M *such that all geodesics* $\gamma_i = \gamma_i(s)$ *on* M *with* $\gamma_i(0) = p$ *and* $\dot\gamma_i(0) = v_i$ $(1 \leqq i \leqq 2n-2)$ *are mapped to circles of the same positive curvature* $k(p)$ *in* $\mathbb{C}P^n(c)$. *Then the function* $k = k(p)$ *is locally constant on* M *and* M *is locally congruent to either a geodesic sphere* $G(r)$ *of radius* r $(0 < r < \pi/\sqrt{c}\,)$ *or a hypersurface of type* (A_2) *of radius* $\pi/(2\sqrt{c}\,)$. *Moreover, the following hold*:

(1) *When* M *has no extrinsic geodesics,* M *is locally congruent to a geodesic sphere* $G(r)$ *of radius* $r = (2/\sqrt{c}\,)\cot^{-1}(2k/\sqrt{c}\,)$ $(0 < r < \pi/(2\sqrt{c}\,))$ *in* $\mathbb{C}P^n(c)$ *with* $k > \sqrt{c}/2$;
(2) *When* M *has just one congruent class of extrinsic geodesics with respect to isometry group* $\mathrm{I}(M)$ *of* M, M *is locally congruent to a geodesic sphere* $G(r)$ *of radius* $r = (2/\sqrt{c}\,)\cot^{-1}(2k/\sqrt{c}\,)$ $(\pi/(2\sqrt{c}\,) \leqq r < \pi/\sqrt{c}\,)$ *in* $\mathbb{C}P^n(c)$ *with* $k \leqq \sqrt{c}/2$;
(3) *When* M *has at least two congruent classes of extrinsic geodesics with respect to* $\mathrm{I}(M)$ *of* M, M *is locally congruent to a hypersurface of type* (A_2) *of radius* $\pi/(2\sqrt{c}\,)$ *in* $\mathbb{C}P^n(c)$ *with* $k = \sqrt{c}/2$.

In the following, we shall characterize all homogeneous real hypersurfaces M^{2n-1} in $\mathbb{C}P^n(c), n \geqq 2$ by observing the extrinsic shape of geodesics from this ambient space. For this purpose we prepare the following two lemmas:

Lemma 5. *Let* M^n *be a connected hypersurface isometrically immersed into a Riemannian manifold* $\widetilde{M}^{n+1}$. *If a geodesic* $\gamma = \gamma(s)$ $(s \in \mathrm{I} \subset \mathbb{R})$ *on* M *is mapped to a circle of positive curvature* k, *then the shape operator* A *of* M^n *in* $\widetilde{M}^{n+1}$ *satisfies either* $A\dot\gamma(s) = k\dot\gamma(s)$ *for all* $s \in \mathrm{I}$ *or* $A\dot\gamma(s) = -k\dot\gamma(s)$ *for all* $s \in \mathrm{I}$.

Lemma 6 ([2])**.** *Let* $\gamma = \gamma(s)$ *be a geodesic on each homogeneous real hypersurface* M^{2n-1} *in* $\mathbb{C}P^n(c), n \geqq 2$. *If the initial vector* $\dot\gamma(0)$ *is a principal curvature unit vector with principal curvature (, say)* λ, *then the curve* γ *is mapped to a circle of curvature* $|\lambda|$ *in* $\mathbb{C}P^n(c)$. *In particular, when the initial vector* $\dot\gamma(0)$ *is perpendicular to* $\xi_{\gamma(0)}$, *the curvature* $|\lambda|$ *is positive.*

In view of Lemmas 5, 6 and the classification theorem of all homogeneous real hypersurfaces in $\mathbb{C}P^n(c)$ we obtain the following:

Theorem 3 ([2]). *Let M^{2n-1} be a connected real hypersurface isometrically immersed into $\mathbb{C}P^n(c), n \geqq 2$. Then M is locally congruent to a homogeneous real hypersurface if and only if there exist orthonormal vectors $v_1, \ldots, v_{2n-2}$ orthogonal to the characteristic vector ξ_p at each point p of M such that all geodesics $\gamma_i = \gamma_i(s)$ $(1 \leqq i \leqq 2n-2)$ on M with $\gamma_i(0) = p$ and $\dot\gamma_i(0) = v_i$ are mapped to circles of positive curvature in $\mathbb{C}P^n(c)$.*

Note that all circles in Theorem 3 are totally real circles in the ambient space $\mathbb{C}P^n(c)$.

7. Real hypersurfaces having ϕ-invariant shape operator

We first recall the following:

Proposition 3 ([11]). *Let (M_n, g, J) be a complex n-dimensional Kähler manifold immersed into a $(2n+p)$-dimensional sphere $S^{2n+p}(c)$ of constant sectional curvature c through an isometric immersion f. Then f has parallel second fundamental form σ if and only if σ is J-invariant, namely $\sigma(JX, JY) = \sigma(X, Y)$ holds for all vectors X, Y on M_n.*

Remark 4. The "if" part in Proposition 3 is obtained by tensor calculus. The "only if" part in this proposition is based on a fact that under the only if part our Kähler manifold M_n is locally congruent to a compact Hermitian symmetric space and moreover this isometric parallel immersion f of the compact Hermitian symmetric space into the ambient sphere $S^{2n+p}(c)$ is locally realized as a part of the embedding as the symmetric R-space (cf. [11, 15, 26]).

Inspired by Proposition 3, for each real hypersurface M^{2n-1} furnished with almost contact metric structure (ϕ, ξ, η, g) isometrically immersed into $\mathbb{C}P^n(c), n \geqq 2$ we introduce the following conditions concerning ϕ-invariances of the shape operator A of M in the ambient space $\mathbb{C}P^n(c)$.

The shape operator A of M is called *strongly ϕ-invariant* if A satisfies

$$g(A\phi X, \phi Y) = g(AX, Y), \text{ i.e., } \sigma(\phi\mathrm{X}, \phi\mathrm{Y}) = \sigma(\mathrm{X}, \mathrm{Y}) \tag{7.1}$$

for all vectors X and Y on M. Also, it is called weakly ϕ-invariant if A satisfies

$$g(A\phi X, \phi Y) = g(AX, Y), \text{ i.e., } \sigma(\phi\mathrm{X}, \phi\mathrm{Y}) = \sigma(\mathrm{X}, \mathrm{Y}) \tag{7.2}$$

for all vectors X and Y *orthogonal to the characteristic vector* ξ on M.

The following is a classification theorem of real hypersurfaces in $\mathbb{C}P^n(c)$ having strongly ϕ-invariant shape operator.

Theorem 4 ([19]). *Let M^{2n-1} be a connected real hypersurface isometrically immersed into $\mathbb{C}P^n(c), n \geqq 2$. Then the following conditions* (1), (2) *and* (3) *are mutually equivalent.*

(1) *M is locally congruent to a type* (A) *hypersurface with radius $\pi/(2\sqrt{c}\,)$.*

(2) *The shape operator A of M in $\mathbb{C}P^n(c)$ is strongly ϕ-invariant.*

(3) *M satisfies the following two conditions*:

(3i) *At each fixed point* $p \in M$*, there exist orthonormal vectors* $v_1, v_2, \ldots, v_{2n-2}$ *orthogonal to the characteristic vector* ξ_p *of* M *such that all geodesics* $\gamma_i = \gamma_i(s)$ *on* M *with* $\gamma_i(0) = p$ *and* $\dot{\gamma}_i(0) = v_i$ $(1 \leqq i \leqq 2n-2)$ *are mapped to circles of the same positive curvature in* $\mathbb{C}P^n(c)$*;*

(3ii) *There exists at least one integral curve of the characteristic vector field* ξ *of* M *which is an extrinsic geodesic on* M*.*

Remark 5. Theorem 4(3) gives a geometric meaning of a condition that M has strongly ϕ-invariant shape operator in the ambient space $\mathbb{C}P^n(c)$.

Next, we classify Hopf hypersurfaces having weakly ϕ-invariant shape operator in $\mathbb{C}P^n(c)$.

Proposition 4 ([19])**.** *For a real hypersurface* M^{2n-1} *isometrically immersed into* $\mathbb{C}P^n(c), n \geqq 2$ *the following two conditions are mutually equivalent.*
(1) M *is a Hopf hypersurface having weakly* ϕ*-invariant shape operator.*
(2) M *is locally congruent to a type* (*A*) *hypersurface.*

Remark 6. There exist many non-Hopf hypersurfaces M^{2n-1} having weakly ϕ-invariant shape operator in $\mathbb{C}P^n(c)$ (for details, see [19]).

8. Length spectrum of hypersurfaces of type (A_1) in $\mathbb{C}P^n(c)$

Having respect for [8], we call a compact simply connected Riemannian manifold a *Berger sphere* if its sectional curvatures lie in the interval $[\delta K, K]$ with some positive constants K and $\delta \in (0, 1/9)$ and has a closed geodesic of length shorter than $2\pi/\sqrt{K}$. When M is even dimensional, Klingenberg ([13]) showed that on a compact simply connected Riemannian manifold whose sectional curvatures lie in $(0, K]$ with some constant K every geodesic on M has length not shorter than $2\pi/\sqrt{K}$. Thus, we have only odd dimensional Berger spheres. Weinstein ([36]) gave examples of Berger spheres by observing geodesic spheres $G(r)$ of the radius r $(0 < r < \pi/\sqrt{c}\,)$ with $\tan^2(\sqrt{c}\, r/2) > 2$ in $\mathbb{C}P^n(c)$. We find easily the following:

Lemma 7. *Let* $G(r)$ *be a geodesic sphere of radius* r $(0 < r < \pi/\sqrt{c}\,)$ *in* $\mathbb{C}P^n(c), n \geqq 2$*. Then the following three conditions are mutually equivalent*:
(1) *The radius* r *satisfies an inequality* $\tan^2(\sqrt{c}\, r/2) > 2$.
(2) *The sectional curvature* K *of* $G(r)$ *satisfies sharp inequalities* $\delta L \leqq K \leqq L$ *for some* $\delta \in (0, 1/9)$ *at its each point, where* $L = c + (c/4)\cot^2(\sqrt{c}\, r/2)$*.*
(3) *The length of every integral curve of the characteristic vector field* ξ *on* $G(r)$ *is shorter than* $2\pi/\sqrt{L}$*, where* L *is the maximal sectional curvature of* $G(r)$ *which is given by Condition* (2).

Remark 7. For every geodesic sphere $G(r)$ $(0 < r < \pi/\sqrt{c}\,)$ in $\mathbb{C}P^n(c)$ every integral curve of the characteristic vector field ξ on $G(r)$ is a geodesic.

In the following, we call a geodesic sphere $G(r)$ $(0 < r < \pi/\sqrt{c}\,)$ with $\tan^2(\sqrt{c}\, r/2) > 2$ in $\mathbb{C}P^n(c)$ a *Berger sphere.* Motivated by their works, we shall investigate the distribution of lengths of closed geodesics on every hypersurface of type (A_1) in $\mathbb{C}P^n(c), n \geqq 2$. We first review fundamental notion on geodesics.

A smooth curve $\gamma : \mathbb{R} \to M$ on a Riemannian manifold M parameterized by its arclength s is said to be closed if there is a positive s_c satisfying $\gamma(s+s_c) = \gamma(s)$ for all s. When γ is closed, the minimum positive such s_c is called its length and is denoted by length(γ). When γ is open, that is, it is not closed, we set length$(\gamma) = \infty$. We here review the congruency for smooth real curves. We say two smooth curves γ_1, γ_2 on M parameterized by its arclength s to be congruent to each other if there exist an isometry φ of M and a constant s_0 satisfying $\gamma_2(s) = \varphi \circ \gamma_1(s+s_0)$ for all s. When we can take $s_0 = 0$, we say that they are congruent to each other in strong sense. We denote by $\mathcal{G}(M)$ the moduli space of geodesics of unit speed on M, which is the set of all congruent classes of geodesics. We can then define a map $\mathcal{L} : \mathcal{G}(M) \to (0, \infty]$ by $\mathcal{L}([\gamma]) = \text{length}(\gamma)$, where $[\gamma]$ denotes the congruent class containing a geodesic γ. This map or sometimes its image $\mathcal{L}(\mathcal{G}) \cap (0, \infty)$ on the real line is called the length spectrum of M.

The aim of this section is to study how lengths of closed geodesics on geodesic spheres $G(r)$ $(0 < r < \pi/\sqrt{c}\,)$ of $\mathbb{C}P^n(c)$ are distributed on the real line. To do this we define functions $m_{G(r)}, n_{G(r)} : (0, \infty) \to \mathbb{N} \cup \{\infty\}$, where $\mathbb{N}$ is the set of positive integers, as follows: For a positive number λ, we denote by $m_{G(r)}(\lambda)$ the number of congruent classes of geodesics of length λ, and denote by $n_{G(r)}(\lambda)$ the number of congruent classes of geodesics of length not longer than λ. We call $m_{G(r)}(\lambda)$ the multiplicity of lengths at λ. We first have

Theorem 5 ([7])**.** *Given a geodesic γ on a geodesic sphere $G(r)$ of radius r $(0 < r < \pi/\sqrt{c}\,)$ in $\mathbb{C}P^n(c), n \geqq 2$ through a natural isometric embedding ι we find the following*:
(1) *The curve $\iota \circ \gamma$ is either a geodesic, a Kähler circle, a totally real circle or a homogeneous curve of proper order 4 in the ambient space $\mathbb{C}P^n(c)$;*
(2) *If the structure torsion of γ is ± 1, then γ is closed and its length is* $(2\pi/\sqrt{c}\,)\sin(\sqrt{c}\, r)$;
(3) *If γ has null structure torsion, then γ is also closed and its length is* $(4\pi/\sqrt{c}\,)\sin(\sqrt{c}\, r/2)$;
(4) *When the structure torsion of γ is of the form* $\sin\theta$ $(0 < |\theta| < \pi/2)$, *it is closed if and only if*

$$\sin\theta = \frac{\pm q}{\sin(\sqrt{c}\, r/2)\sqrt{p^2 \tan^2(\sqrt{c}\, r/2) + q^2}}$$

with some relatively prime positive integers p and q with $q < p\tan^2(\sqrt{c}\, r/2)$. In this case, its length is

$$\text{length}(\gamma) = \begin{cases} (4\pi/\sqrt{c}\,)\sqrt{p^2\sin^2(\sqrt{c}\, r/2) + q^2\cos^2(\sqrt{c}\, r/2)} \\ \qquad \textit{if } pq \textit{ is even}, \\ (2\pi/\sqrt{c}\,)\sqrt{p^2\sin^2(\sqrt{c}\, r/2) + q^2\cos^2(\sqrt{c}\, r/2)} \\ \qquad \textit{if } pq \textit{ is odd}. \end{cases}$$

As a direct consequence of Theorem 5, for a geodesic sphere $G(r)$ of radius r in $\mathbb{C}P^n(4)$, we can see that

$$\begin{aligned}\mathrm{Lspec}(G(r)) &= \{\pi\sin 2r\}\cup\{2\pi\sin r\}\\ &\cup\left\{2\pi\sqrt{p^2\sin^2 r+q^2\cos^2 r}\;\middle|\;\begin{array}{c}p \text{ and } q \text{ are relatively prime}\\ \text{positive integers which satisfy}\\ pq \text{ is even and } q<p\tan^2 r\end{array}\right\}\\ &\cup\left\{\pi\sqrt{p^2\sin^2 r+q^2\cos^2 r}\;\middle|\;\begin{array}{c}p \text{ and } q \text{ are relatively prime}\\ \text{positive integers which satisfy}\\ pq \text{ is odd and } q<p\tan^2 r\end{array}\right\}.\end{aligned}$$

Therefore we obtain the following

Theorem 6 ([7])**.** *On a geodesic sphere $G(r)$ $(0<r<\pi/\sqrt{c}\,)$ in $\mathbb{C}P^n(c)$, there exist countably infinite congruent classes of closed geodesics. Moreover the length spectrum* $\mathrm{Lspec}(G(r))$ *of $G(r)$ is a discrete unbounded subset in the real line $\mathbb{R}$.*

Moreover, we establish the following

Theorem 7 ([7])**.** *Geodesics on a geodesic sphere $G(r)$ of radius r $(0<r<\pi/\sqrt{c}\,)$ in $\mathbb{C}P^n(c)$ with $n\geqq 2$ satisfy the following properties.*
(1) *When $\tan^2(\sqrt{c}\,r/2)$ is irrational, two closed geodesics on $G(r)$ are congruent to each other in strong sense if and only if they have a common length.*
(2) *When $\tan^2(\sqrt{c}\,r/2)$ is rational, the multiplicity $m_{G(r)}(\lambda)$ at λ is finite at each positive λ, but is not uniformly bounded. It satisfies $\limsup_{\lambda\to\infty} m(\lambda)=\infty$ and $\lim_{\lambda\to\infty}\lambda^{-\delta}m(\lambda)=0$ for each $\delta>0$.*
(3) *We have*

$$\lim_{\lambda\to\infty}\frac{n_{G(r)}(\lambda)}{\lambda^2}=\frac{3c\sqrt{c}\,r}{8\pi^4\sin(\sqrt{c}\,r)}.$$

In particular, we have countably infinitely many congruent classes of closed geodesics.

In the rest of this section we have the following characterizations of Berger spheres from the viewpoint of submanifold theory.

Theorem 8 ([14])**.** *Let M^{2n-1} be a connected real hypersurface of $\mathbb{C}P^n(c)$, $n\geqq 2$ through an isometric immersion. Then M is locally congruent to a Berger sphere, namely a geodesic sphere $G(r)$ of radius r with $\tan^2(\sqrt{c}\,r/2)>2$, with respect to the full isometry group $\mathrm{SU}(n+1)$ of the ambient space $\mathbb{C}P^n(c)$ if and only if at each point p of M there exists an orthonormal basis $v_1,\dots,v_{2n-2},\xi_p$ of T_pM such that all geodesics $\gamma_i=\gamma_i(s)$ $(1\leqq i\leqq 2n-2)$ with initial condition that $\gamma_i(0)=p$ and $\dot\gamma_i(0)=v_i$ are mapped to circles of the same positive curvature $k(p)$ with $k(p)<\sqrt{c}/(2\sqrt{2}\,)$ in the ambient space $\mathbb{C}P^n(c)$, where ξ_p is the characteristic vector of M at $p\in M$. In this case, the function $k=k(p)$ on M is automatically constant with $k=(\sqrt{c}/2)\cot(\sqrt{c}\,r/2)$.*

Theorem 9 ([14])**.** *Let M^{2n-1} be a connected real hypersurface of $\mathbb{C}P^n(c)$, $n\geqq 2$ through an isometric immersion. Then M is locally congruent to a Berger sphere if and only if M satisfies the following two conditions.*

(1) *There exists a positive constant k with $k < \sqrt{c}/(2\sqrt{2}\,)$ such that the exterior derivative $d\eta$ of the contact form η on M satisfies either $d\eta(X,Y) = kg(\phi X, Y)$ for all $X, Y \in TM$ or $d\eta(X,Y) = -kg(\phi X, Y)$ for all $X, Y \in TM$, where g and ϕ are the Riemannian metric and the structure tensor on M, respectively.*
(2) *There exists a point x of M satisfying that every sectional curvature of M at x is positive.*

The definition of $d\eta$ on a real hypersurface M is given by $d\eta(X,Y) = (1/2)\cdot \{X(\eta(Y)) - Y(\eta(X)) - \eta([X,Y])\}$ for all $X, Y \in TM$.

We note for comparison the recent paper of Li, Vrancken and Wang ([16]), which gives a characterization of 3-dimensional Bereger spheres as Lagrangian submanifolds of $\mathbb{C}P^3$. They showed the following (for details, see Theorem 1.2 in [16]): Let ϕ be a Lagrangian isometric immersion of a homogeneous 3-manifold M^3 into a complex space form $M_3(c)(= \mathbb{C}P^3(c), \mathbb{C}H^3(c)$ or $\mathbb{C}^3)$. Then $c > 0$ and ϕ is minimal and M^3 is locally congruent to the Berger sphere.

At the end of this section we give some comments:

Remark 8. (1) *Except* geodesics with structure torsion ± 1, every geodesic γ on $G(r)$ $(0 < r < \pi/\sqrt{c}\,)$ in $\mathbb{C}P^n(c)$ satisfies $\text{length}(\gamma) > 4\pi/\sqrt{c\big(4+\cot^2(\sqrt{c}\,r/2)\big)}$, i.e., it satisfies an inequality of Klingenberg's type.
(2) In the class of all *homogeneous* real hypersurfaces M^{2n-1} in $\mathbb{C}P^n(c)$, M is of type (A_1) if and only if every sectional curvature of M is positive, and M is of type (A) if and only if every sectional curvature of M is nonnegative (see [22]).
(3) In the statement of Theorem 9, if we remove Condition (2), this theorem does not hold. The Berger sphere and a certain homogeneous real hypersurface of type (B) satisfy Theorem 9(1). We here review a fact that $\mathbb{C}P^n(c)$ admits *no* real hypersurfaces with $d\eta = 0$ (see [25]). On the other hand, a complex Euclidean space $\mathbb{C}^n$ has real hypersurfaces M^{2n-1} with $d\eta = 0$ (for example, the totally geodesic real hypersurface $\mathbb{R}^{2n-1}$ satisfies this condition). So, in some sense the geometry of real hypersurfaces of $\mathbb{C}P^n(c)$ is more complicated than that of $\mathbb{C}^n$. Motivated by them, we establish Theorem 9.

9. Almost contact structures on real hypersurfaces in $\mathbb{C}P^n(c)$

We first clarify the meaning of the condition that a real hypersurface M in $\mathbb{C}P^n(c), n \geqq 2$ is a Sasakian manifold with respect to almost contact metric structures (ϕ, ξ, η, g) and $(\phi, -\xi, -\eta, g)$. We call a real hypersurface M *Sasakian* if M satisfies either $(\nabla_X \phi)Y = g(X,Y)\xi - \eta(Y)X$ for all tangent vectors $X, Y \in T_pM$ at an arbitrary point $p \in M$ or $(\nabla_X \phi)Y = -g(X,Y)\xi + \eta(Y)X$ for all vectors $X, Y \in T_pM$ at an arbitrary point $p \in M$, where ∇ denotes the Riemannian connection of M (see Theorem 6.3 in [9]). For a tangent vector X of a Sasakian manifold M which is orthogonal to ξ, the sectional curvature of the plane spanned by X and ϕX is called ϕ-sectional curvature of the ϕ-section determined by X. We say a Sasakian manifold to be a *Sasakian space form* of constant ϕ-sectional curvature k if ϕ-sectional curvatures of all tangent vectors orthogonal to ξ are k.

By easy computation we find the following

Lemma 8 ([1]). *For a connected real hypersurface M^{2n-1} isometrically immersed into $\mathbb{C}P^n(c), n \geqq 2$ the following four conditions are mutually equivalent.*
(1) *M is locally congruent to a geodesic sphere $G(r)$ of radius r $(0 < r < \pi/\sqrt{c}\,)$ with $(\sqrt{c}/2)\cot(\sqrt{c}\,r/2) = 1$.*
(2) *The shape operator A of M in the ambient space $\mathbb{C}P^n(c)$ is expressed as either $AX = -X + (c/4)\eta(X)\xi$ for each $X \in TM$ or $AX = X - (c/4)\eta(X)\xi$ for each $X \in TM$. That is, M is a member of totally η-umbilic real hypersurfaces in $\mathbb{C}P^n(c)$.*
(3) *M is a Sasakian manifold with respect to the almost contact metric structure (ϕ, ξ, η, g) induced from the standard Kähler structure (g, J) on $\mathbb{C}P^n(c)$.*
(4) *M is a Sasakian space form with respect to the almost contact metric structure (ϕ, ξ, η, g) induced from the standard Kähler structure (g, J) on $\mathbb{C}P^n(c)$. In this case, M has constant ϕ-sectional curvature $c+1$.*

Remark 9. (1) It follows from a fact that every geodesic sphere $G(r)$ $(0 < r < \pi/\sqrt{c}\,)$ in $\mathbb{C}P^n(c)$ is diffeomorphic to a Euclidean sphere and Lemma 8 that our geodesic sphere $G(r)$ $(0 < r < \pi/\sqrt{c}\,)$ of radius r with $(\sqrt{c}/2)\cot(\sqrt{c}\,r/2) = 1$ is a complete simply connected Sasakian space form $N(k)(:= N^{2n-1}(k))$ of constant ϕ-sectional curvature $k = c+1(> 1)$. Hence by virtue of the unique existence theorem on complete simply connected Sasakian space forms in [35] our geodesic sphere $G(r)$ with $(\sqrt{c}/2)\cot(\sqrt{c}\,r/2) = 1$ is congruent to a standard example of Sasakian space forms having constant ϕ-sectional curvature greater than 1 constructed in page 114 of [9].
(2) In view of (2.3) and Lemma 8(2) we can write the curvature tensor R of a Sasakian space form $N(k)$ of constant ϕ-sectional curvature $k(> 1)$ (cf. [28]).

Using the results in [7] and an equality $(\sqrt{c}/2)\cot(\sqrt{c}\,r/2) = 1$, we give the following list of the length spectrum $\mathrm{LSpec}(N(k)) = \mathcal{L}\big(\mathcal{G}(N(k))\big) \cap \mathbb{R}$ of a complete simply connected Sasakian space form $N(k)$ with $k > 1$

$$\begin{aligned}\mathrm{LSpec}(N(k)) = &\left\{\frac{8\pi}{k+3},\ \frac{4\pi}{\sqrt{k+3}}\right\} \\ &\bigcup \left\{ 4\pi\sqrt{\frac{(k-1)p^2+4q^2}{(k-1)(k+3)}} \;\middle|\; \begin{array}{c} p \text{ and } q \text{ are relatively prime} \\ \text{positive integers which satisfy} \\ pq \text{ is even and } 4p < (k-1)q \end{array} \right\} \\ &\bigcup \left\{ 2\pi\sqrt{\frac{(k-1)p^2+4q^2}{(k-1)(k+3)}} \;\middle|\; \begin{array}{c} p \text{ and } q \text{ are relatively prime} \\ \text{positive integers which satisfy} \\ pq \text{ is odd and } 4p < (k-1)q \end{array} \right\}.\end{aligned}$$

Geodesics on a complete simply connected Sasakian space form $N(k)$ with $k > 1$ have the following properties.

Proposition 5. *Every geodesic on $N(k)$ $(k > 1)$ is homogeneous, that is, it is an orbit of some one-parameter subgroup of the isometry group $\mathrm{I}(N(k))$. Hence it is a simple curve, i.e., it does not have self-intersections.*

Proposition 6 ([5]). *We have countably infinite congruent classes of closed geodesics on $N(k)$ $(k > 1)$. Moreover, we have the following.*
(1) *Every $N(k)$ $(k > 9)$ is a Berger sphere.*

(2) *When k is irrational, two closed geodesics on $N(k)$ are congruent to each other if and only if they have a common length.*
(3) *When k is rational, we have $m_{N(k)}(\lambda)$ is finite for each positive λ, but is not uniformly bounded*; $\limsup_{\lambda\to\infty} m_{N(k)}(\lambda) = \infty$. *The growth order of the function $m_{N(k)}$ is less than polynomial order. More precisely, we have $\lim_{\lambda\to\infty} \lambda^{-\delta} m_{N(k)}(\lambda) = 0$ for each positive δ.*

Though the feature of the function $m_{N(k)}$ of multiplicities depends whether k is rational or irrational, the functions $n_{N(k)}$ of numbers of congruent classes of closed geodesics have a common property.

Theorem 10 ([5])**.** *The function $n_{N(k)}$ with $k > 1$ satisfies*

$$\lim_{\lambda\to\infty} \frac{n_{N(k)}(\lambda)}{\lambda^2} = \frac{3(k+3)\sqrt{k-1}}{16\pi^4} \tan^{-1}(\sqrt{k-1}/2).$$

For the definitions of functions $m_{N(k)}$ and $n_{N(k)}$ see Section 8.

10. A certain homogeneous submanifold in a sphere

In this section we show that every sufficiently high dimensional Euclidean sphere admits an odd dimensional Riemannian submanifold M having the following properties:
(1) M is diffeomorphic but not isometric to a Euclidean sphere.
(2) M is a homogeneous submanifold with nonzero parallel mean curvature vector in the ambient sphere.
(3) M is a Berger sphere.
(4) M is a Sasakian space form of constant ϕ-sectional curvature.
For this purpose we establish the following theorem.

Theorem 11 ([23])**.** (I) *For each of $c > 0, n \geqq 2, N > n(n+2)-1$ and $\tilde{c} \leqq (n+1)c/(2n)$, there exists a $(2n-1)$-dimensional submanifold M^{2n-1} isometrically immersed into an N-dimensional sphere $S^N(\tilde{c})$ of constant sectional curvature $\tilde{c}$, which has the following properties*:
(1) *M is diffeomorphic but not isometric to a Euclidean sphere.*
(2) *M is a homogeneous submanifold which has nonzero parallel mean curvature vector with respect to the normal connection in $S^N(\tilde{c})$.*
(3) *M is a Berger sphere.*
(4) *When $c = 8n+4$, M is a Sasakian space form of constant ϕ-sectional curvature $8n+5$.*
(II) *For each of $c > 0$ and $n \geqq 2$, when $N = n(n+2)-1$, there exists also a $(2n-1)$-dimensional submanifold M^{2n-1} in an N-dimensional sphere $S^N(\tilde{c})$ of constant sectional curvature $\tilde{c} = (n+1)c/(2n)$, which has the above properties* $(1), (2), (3), (4)$.

We explain an idea in the proof of Theorem 11. We denote by (M, ι_M) a real hypersurface M^{2n-1} of $\mathbb{C}P^n(c)$ through an isometric immersion $\iota_M : M \to \mathbb{C}P^n(c)$. In the following, we regard a real hypersurface M in $\mathbb{C}P^n(c)$ as a submanifold of the sphere $S^{n(n+2)-1}((n+1)c/(2n))$ of constant sectional curvature $(n+1)c/(2n)$

through an isometric immersion $f_1 \circ \iota_M$, where f_1 is the parallel equivariant minimal embedding of $\mathbb{C}P^n(c)$ into $S^{n(n+2)-1}((n+1)c/(2n))$.

We here recall the definition and fundamental properties of f_1. The embedding f_1 is defined by eigenfunctions of the first eigenvalue of the Laplacian Δ on $\mathbb{C}P^n(c)$ (for details, see [10, 34]). In submanifold theory, this embedding f_1 is well-known as the only example of a full minimal *parallel immersion*, i.e., the second fundamental form σ_1 of f_1 is parallel, of a complex projective space endowed with Fubini-Study metric into a Euclidean sphere. The inner product of the first normal space of f_1 is given by

$$
\begin{aligned}
(10.1) \qquad \langle \sigma_1(X,Y), \sigma_1(Z,W)\rangle &= -(c/(2n))\langle X,Y\rangle\langle Z,W\rangle + (c/4)(\langle X,W\rangle\langle Y,Z\rangle \\
&\quad + \langle X,Z\rangle\langle Y,W\rangle + \langle JX,W\rangle\langle JY,Z\rangle + \langle JX,Z\rangle\langle JY,W\rangle)
\end{aligned}
$$

for all vectors X, Y, Z, W on $\mathbb{C}P^n(c)$, where J is the complex structure on $\mathbb{C}P^n(c)$. Equation (10.1) shows the following properties of f_1:
(i) The embedding f_1 is minimal.
(ii) It holds that $\sigma_1(JX, JY) = \sigma_1(X, Y)$ for all vectors X, Y on $\mathbb{C}P^n(c)$, i.e., σ_1 is J-invariant. Hence the second fundamental form σ_1 of $\mathbb{C}P^n(c)$ in $S^{n(n+2)-1}((n+1)c/(2n))$ is parallel (see Proposition 3).
(iii) The length of a normal vector $\sigma_1(X, X)$ is written as $\|\sigma_1(X,X)\| = \sqrt{(n-1)c/(2n)}$ for each unit vector X on $\mathbb{C}P^n(c)$, namely our embedding f_1 is $\sqrt{(n-1)c/(2n)}$-isotropic (cf. [31]).

We next explain the embeddings $f_1 \circ \iota_M : M \to S^{n(n+2)-1}((n+1)c/(2n))$. This class contains some homogeneous submanifolds of $S^{n(n+2)-1}((n+1)c/(2n))$, that is they are expressed as orbits of some subgroups of the isometry group $\mathrm{SO}(n(n+2))$ of the ambient sphere. In fact, if we take a homogeneous real hypersurface M of $\mathbb{C}P^n(c)$, the immersion $f_1 \circ \iota_M$ gives a homogeneous submanifold M of the sphere. As a matter of course these homogeneous submanifolds have constant mean curvature in the sphere. On the other hand, the second fundamental form of the immersion $f_1 \circ \iota_M : M \to S^{n(n+2)-1}((n+1)c/(2n))$ is *not* parallel for each real hypersurface M of $\mathbb{C}P^n(c)$ because $\mathbb{C}P^n(c)$ admits no real hypersurfaces which are locally symmetric (for example, see [25]). Hence it is natural to pose the following problem:

Problem 2. Classify submanifolds $(M^{2n-1}, f_1 \circ \iota_M)$ of $S^{n(n+2)-1}((n+1)c/(2n))$ satisfying that the isometric immersion $f_1 \circ \iota_M$ has parallel mean curvature vector with respect to the normal connection.

The following answer to Problem 2 is a key lemma.

Lemma 9 ([23]). *Let M^{2n-1} be a connected real hypersurface of $\mathbb{C}P^n(c), n \geqq 2$ through an isometric immersion ι_M and $f_1 : \mathbb{C}P^n(c) \to S^{n(n+2)-1}((n+1)c/(2n))$ the first standard minimal embedding. Then M is locally congruent to the geodesic sphere $G(r)$ $(0 < r < \pi/\sqrt{c}\,)$ with $\tan^2(\sqrt{c}\, r/2) = 2n+1$ in $\mathbb{C}P^n(c)$ if and only if the isometric immersion $f_1 \circ \iota_M : M \to S^{n(n+2)-1}((n+1)c/(2n))$ has parallel mean curature vector with respect to the normal connection. Moreover, this submanifold $(M, f_1 \circ \iota_M)$ is homogeneous in the ambient sphere.*

We find easily the following lemma.

Lemma 10 ([23]). *The geodesic sphere $G(r)$ of radius r $(0 < r < \pi/\sqrt{c}\,)$ with $\tan^2(\sqrt{c}\,r/2) = 2n+1$ in $\mathbb{C}P^n(c)$ is a Sasakian manifold with respect to the almost contact metric structure (ϕ, ξ, η, g) induced from the Kähler structure (g, J) on $\mathbb{C}P^n(c), n \geqq 2$ if and only if $c = 8n+4$. Furthermore, this geodesic sphere is a Sasakian space form of constant ϕ-sectional curvature $8n+5$.*

Moreover, we need the following lemma in order to show Statement (2) in Theorem 11.

Lemma 11 ([23]). *We consider the following isometric embedding $\tilde{f}$ of the geodesic sphere $G(r)$ of radius r $(0 < r < \pi/\sqrt{c}\,)$ with $\tan^2(\sqrt{c}\,r/2) = 2n+1$ in $\mathbb{C}P^n(c)$ into an $N(\geqq n(n+2)-1)$-dimensional sphere $S^N(\tilde{c})$ of constant sectional curature $\tilde{c}(\leqq (n+1)c/(2n))$.*
(1) *When $N > n(n+2)-1$, $\tilde{f}$ is given by*

$$\tilde{f} = \iota \circ (f_1 \circ \iota_{G(r)}) : G(r) \stackrel{f_1 \circ \iota_{G(r)}}{\longrightarrow} S^{n(n+2)-1}((n+1)c/(2n)) \stackrel{\iota}{\longrightarrow} S^N(\tilde{c}),$$

where ι is a totally umbilic embedding, so that $(n+1)c/(2n) \geqq \tilde{c}$.
(2) *When $N = n(n+2)-1$, $\tilde{f}$ is nothing but $f_1 \circ \iota_{G(r)}$, so that $(n+1)c/(2n) = \tilde{c}$. Then our geodesic sphere is homogeneous in $S^N(\tilde{c})$ and it has nonzero parallel mean curvature vector with respect to the normal connection in this sphere.*

Thus, in view of Lemmas 9, 10 and 11 we establish Theorem 11. At the end of this paper we pose the following open problem.

Problem 3. Let f_1 be a minimal parallel full immersion of a complex n-dimensional compact Hermitian symmetric space $\widetilde{M}_n$ into a Euclidean sphere $S^{2n+p}(\tilde{c})$. If there exists a real hypersurface (M^{2n-1}, ι_M) of $\widetilde{M}_n$ satisfying that the corresponding submanifold $(M^{2n-1}, f_1 \circ \iota_M)$ has parallel mean curvature vector with respect to the normal connection in the ambient sphere $S^{2n+p}(\tilde{c})$, is our Hermitian symmetric space $\widetilde{M}_n$ holomorphically isometric to a complex projective space $\mathbb{C}P^n(c)$ of constant holomorphic sectional curvature $c = 2n\tilde{c}/(n+1)$ and $p = n^2-1$?

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S. Maeda: Department of Mathematics, Saga University, Saga 840-8502, Japan
Email address: sadahiromaeda0801@gmail.com