On a Large Deflection of Clamped Parallelogrammic Isotropic Flat Plate
I. Derivation of the Expression of the Total Potential Energy of the Plate

By Misao HASEGAWA

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長谷川　節：固定平行四辺形板の大きな撓みについて

1. Introduction

Recently the study of the bending of elastic plates with large deflection became important specially in aeroelasticity. The laterally loaded clamped square plate with large deflection was studied by S. Way[(1)] and the author[(2)]. The problem in the case of the rectangular plates was discussed by S. Levy [(3)].

The problm to be treated here is that of a parallelogrammic, laterally loaded, clamped plate having large deflections of the same order of magnitude as the thickness.

2. Basic Equations

Let the plate be situated with respect to the two coordinate systems shown in Fig 1. O-xyz is the cartesian coordinate system, O-$x'y'z'$ the oblique coordinate system, and the axes of x and z coincide with those of x' and z'. The z-axis is normal to the plate.

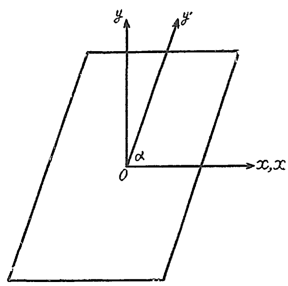

Fig. 1

Let the coordinates of a point of the plate be expressed by (x, y, z) in cartesian coordinates, and by (x', y', z') in oblique ones. Let α be expressed the angle between x'- and y'- axes. Let the components of displacements of points in the middle surface be expressed by (u, v, w) in cartesian system, and by (u', v', w') in oblique ones.

Let the thickness and the lengths of sides of the plate be h, $2a$ and $2b$. Let Young's and shear moduli, and Poissons ratio of the plate be E, G and μ. Then the flexural rigidity D of the plate is expressed by

$$D=\frac{Eh^3}{12(1-\mu^2)}$$

Let the extensions, shear and stresses in the middle surface of the plate be e_x, e_y, γ_{xy},

σ_x, σ_y and τ_{xy} in cartesian coordinates. Then e_x, e_y and γ_{xy} are related to the displacements (u, v, w) by the following equations

$$\left.\begin{aligned} e_x&=\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2,\\ e_y&=\frac{\partial v}{\partial y}+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2,\\ \gamma_{xy}&=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}. \end{aligned}\right\} \tag{1}$$

The stresses σ_x, σ_y and τ_{xy} are related to the strain e_x, e_y and γ_{xy} by the following equations

$$\left.\begin{aligned} \sigma_x&=\frac{E}{1-\mu^2}(e_x+\mu e_y),\\ \sigma_y&=\frac{E}{1-\mu^2}(e_y+\mu e_x),\\ \tau_{xy}&=G\gamma_{xy}. \end{aligned}\right\} \tag{2}$$

If we introduce the stress function F such that

$$\sigma_x=\frac{\partial^2 F}{\partial x^2},\quad \sigma_y=\frac{\partial^2 F}{\partial y^2},\quad \tau_{xy}=-\frac{\partial^2 F}{\partial x\partial y}, \tag{3}$$

the equilibrium equations of the plate are expressed in terms of the displacements and the stress function by the following equations

$$D\Delta\Delta w-p-h\left\{\frac{\partial^2 F}{\partial y^2}\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 F}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-2\frac{\partial^2 F}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y}\right\}=0, \tag{4}$$

$$\Delta\Delta F=E\left\{\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2-\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right\} \tag{5}$$

in which p is the lateral load per unit area of the plate and $\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$.

As the equations (4) and (5) are nonlinear in w and F, it is very difficult to solve eqs. (4) and (5). Therefore we shall apply the Ritz method to this problem.

In applying the Ritz method, we make use of the expression for the total potential energy U of the plate and the loads upon it. The expression of U is

$$U=\frac{D}{2}\iint(\Delta w)^2\,dxdy-\iint pw\,dxdy+\frac{h}{2}\iint(\sigma_x e_x+\sigma_y e_y+\tau_{xy}\gamma_{xy})dxdy \tag{6}$$

for a clamped plate. The integrals are taken over the entire plate.

3. The total potential energy

The expression (6) of the total potential energy U is expressed in x, y, z, u, v and w. We shall express it in x', y', z', u', v' and w'. It will be easily found that the relations between (x, y, z, u, v, w) and (x', y', z', u', v', w') are

$$\left.\begin{aligned} &x=x'+y'\cos\alpha,\ y=y'\sin\alpha\\ &z=z' \end{aligned}\right\} \tag{7}$$

$$\left.\begin{aligned}&u=u'+v'\cos\alpha,\quad v=v'\sin\alpha\\&w=w'.\end{aligned}\right\}\tag{8}$$

By using the above relations, Δw is transformed into

$$\Delta w=\operatorname{cosec}^2\alpha\left\{\frac{\partial^2 w'}{\partial x'^2}-2\cos\alpha\frac{\partial^2 w'}{\partial x'\partial y'}+\frac{\partial^2 w'}{\partial y'^2}\right\}\tag{9}$$

By the same transformation, $\sigma_x e_x+\sigma_y e_y+\tau_{xy}\gamma_{xy}$ is transformed into the following expression

$$\begin{aligned}\sigma_x e_x+\sigma_y e_y+\tau_{xy}\gamma_{xy}=&\frac{E}{1-\mu^2}\Bigg[\frac{2+(1-\mu)\cot^2\alpha}{2}\left(\frac{\partial u'}{\partial x'}\right)^2\\
&-(1-\mu)\cos\alpha\operatorname{cosec}^2\alpha\frac{\partial u'}{\partial x'}\frac{\partial u'}{\partial y'}+\frac{1}{2}(1-\mu)\operatorname{cosec}^2\alpha\left(\frac{\partial u'}{\partial y'}\right)^2\\
&+(1-\mu)\cos\alpha(2+\cos 2\alpha\operatorname{cosec}^2\alpha)\frac{\partial u'}{\partial x'}\frac{\partial v'}{\partial x'}\\
&+\{\mu(2+\cot^2\alpha)-\cot^2\alpha\}\frac{\partial u'}{\partial x'}\frac{\partial v'}{\partial y'}\\
&-(1-\mu)\cos 2\alpha\operatorname{cosec}^2\alpha\frac{\partial u'}{\partial y'}\frac{\partial v'}{\partial x'}+(1-\mu)\cos\alpha\operatorname{cosec}^2\alpha\frac{\partial u'}{\partial y'}\frac{\partial v'}{\partial y'}\\
&+(1-\mu)(\operatorname{cosec}^2\alpha-2\cos^2\alpha)\left(\frac{\partial v'}{\partial x'}\right)^2\\
&+(\mu-1)\cos\alpha\,(2+\cos 2\alpha\operatorname{cosec}^2\alpha)\frac{\partial v'}{\partial x'}\frac{\partial v'}{\partial y'}\\
&+\frac{1}{2}\left(\frac{\partial w'}{\partial x'}\right)^2\Bigg\{4\cot^2\alpha\frac{\partial u'}{\partial x'}-2(1-\mu)\cot\alpha\operatorname{cosec}\alpha\frac{\partial u'}{\partial y'}\\
&+2\cos\alpha\,(1-\cot^2\alpha)\frac{\partial v'}{\partial x'}+2\mu\operatorname{cosec}^2\alpha\frac{\partial v'}{\partial y'}\Bigg\}\\
&+\frac{\partial w'}{\partial x'}\frac{\partial w'}{\partial y'}\Bigg\{-(1+\mu)\operatorname{cosec}\alpha\cot\alpha\frac{\partial u'}{\partial x'}\\
&+(1-\mu)\operatorname{cosec}^2\alpha\frac{\partial u'}{\partial y'}+(1-\mu)\operatorname{cosec}^2\alpha\frac{\partial v'}{\partial x'}-(1+\mu)\operatorname{cosec}\alpha\cot\alpha\frac{\partial v'}{\partial y'}\Bigg\}\\
&+\frac{\operatorname{cosec}^4\alpha}{4}\left(\frac{\partial w'}{\partial x'}\right)^4-\operatorname{cosec}^3\alpha\cot\alpha\left(\frac{\partial w'}{\partial x'}\right)^3\left(\frac{\partial w'}{\partial y'}\right)\\
&+\frac{\operatorname{cosec}^2\alpha}{4}(3-\mu+6\cot^2\alpha)\left(\frac{\partial w'}{\partial x'}\right)^2\left(\frac{\partial w'}{\partial y'}\right)^2\\
&-\cot\alpha\operatorname{cosec}^2\alpha\frac{\partial w'}{\partial x'}\left(\frac{\partial w'}{\partial y'}\right)^3+\frac{\operatorname{cosec}^4\alpha}{4}\left(\frac{\partial w'}{\partial y'}\right)^4\\
&+\left(\frac{\partial w'}{\partial y'}\right)^2\Bigg\{2\mu\operatorname{cosec}^2\alpha\frac{\partial u'}{\partial x'}+2(\mu-1)\operatorname{cosec}^2\alpha\cos\alpha\frac{\partial v'}{\partial x'}\\
&+2\operatorname{cosec}^2\alpha\frac{\partial v'}{\partial y'}\Bigg\}\Bigg].\end{aligned}\tag{10}$$

Then the expression of U takes the following form

$$\begin{aligned}U=&\frac{D\operatorname{cosec}^3\alpha}{2}\iint\left\{\frac{\partial^2 w'}{\partial x'^2}-2\cos\alpha\frac{\partial^2 w'}{\partial x'\partial y'}+\frac{\partial^2 w'}{\partial y'^2}\right\}^2dx'dy'-\sin\alpha\iint pw'dx'dy'\\
&+\frac{6D\sin\alpha}{h^2}\iint\Bigg[\frac{2+(1-\mu)\cot^2\alpha}{2}\left(\frac{\partial u'}{\partial x'}\right)^2-(1-\mu)\cos\alpha\operatorname{cosec}^2\alpha\frac{\partial u'}{\partial x'}\frac{\partial u'}{\partial y'}\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}(1-\mu)\mathrm{cosec}^2\alpha\left(\frac{\partial u'}{\partial y'}\right)^2+(1-\mu)\cos\alpha\,(2+\cos 2\alpha\ \ \mathrm{cosec}^2\alpha)\frac{\partial u'}{\partial x'}\frac{\partial v'}{\partial x'}\\
&+\{\mu(2+\cot^2\alpha)-\cot^2\alpha\}\frac{\partial u'}{\partial x'}\frac{\partial v'}{\partial y'}-(1-\mu)\cos 2\alpha\ \ \mathrm{cosec}^2\alpha\frac{\partial u'}{\partial y'}\frac{\partial v'}{\partial x'}\\
&+(1-\mu)\cos\alpha\ \ \mathrm{cosec}^2\alpha\frac{\partial u'}{\partial y'}\frac{\partial v'}{\partial y'}+(1-\mu)(\mathrm{cosec}^2\alpha-2\cos^2\alpha)\left(\frac{\partial v'}{\partial x'}\right)^2\\
&+(\mu-1)\cos\alpha(2+\cos 2\alpha\ \ \mathrm{cosec}^2\alpha)\frac{\partial v'}{\partial x'}\frac{\partial v'}{\partial y'}+\frac{1}{2}(2+\cot^2\alpha-\mu\cot^2\alpha)\left(\frac{\partial v'}{\partial y'}\right)^2\\
&+\frac{1}{2}\left(\frac{\partial w'}{\partial x'}\right)^2\Big\{4\cot^2\alpha\frac{\partial u'}{\partial x'}-2(1-\mu)\cot\alpha\ \ \mathrm{cosec}\,\alpha\frac{\partial u'}{\partial y'}+2\cos\alpha(1-\cot^2\alpha)\frac{\partial v'}{\partial x'}\\
&+2\,\mu\,\mathrm{cosec}^2\alpha\frac{\partial v'}{\partial y'}\Big\}+\frac{\partial w'}{\partial x'}\frac{\partial w'}{\partial y'}\Big\{-(1+\mu)\mathrm{cosec}\,\alpha\ \ \cot\alpha\frac{\partial u'}{\partial x'}\\
&+(1-\mu)\mathrm{cosec}^2\alpha\frac{\partial u'}{\partial y'}+(1-\mu)\mathrm{cosec}^2\alpha\frac{\partial v'}{\partial x'}-(1+\mu)\mathrm{cosec}\,\alpha\ \ \cot\alpha\frac{\partial v'}{\partial y'}\Big\}\\
&+\frac{\mathrm{cosec}^4\alpha}{4}\left(\frac{\partial w'}{\partial x'}\right)^4-\mathrm{cosec}^3\alpha\ \ \cot\alpha\left(\frac{\partial w'}{\partial x'}\right)^3\left(\frac{\partial w'}{\partial y'}\right)\\
&+\frac{\mathrm{cosec}^2\alpha}{4}(3-\mu+6\cot^2\alpha)\left(\frac{\partial w'}{\partial x'}\right)^2\left(\frac{\partial w'}{\partial y'}\right)^2-\cot\alpha\ \ \mathrm{cosec}^3\alpha\frac{\partial w'}{\partial x'}\left(\frac{\partial w'}{\partial y'}\right)^3\\
&+\frac{\mathrm{cosec}^4\alpha}{4}\left(\frac{\partial w'}{\partial y'}\right)^4+\left(\frac{\partial w'}{\partial y'}\right)^2\Big\{2\mu\,\mathrm{cosec}^2\alpha\frac{\partial u'}{\partial x'}\\
&+2(\mu-1)\mathrm{cosec}^2\alpha\ \ \cos\alpha\frac{\partial v'}{\partial x'}+2\,\mathrm{cosec}^2\,\alpha\frac{\partial v'}{\partial y'}\Big\}\Big]dx'dy' \qquad (11)
\end{aligned}
$$

In the following paper we shall apply the Ritz method to treat the problem of clamped parallelogrammic plate with large deflection.