A Remark on Periodic Semigroups

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(*Received Nov. 10, 1958*)

山田深雪：周期準群についての一注意

1. Following Š. Schwarz [3], a semigroup S is called a *periodic semigroup*, if every element of S has a finite order, that is, if, for every element a of S, the subsemigroup $(a)=\{a \mid a, a^2, \ldots, a^n, \ldots\}$ generated by a contains a finite number of different elements. Any strongly reversible periodic semigroup is the class sum of mutually disjoint unipotent semigroups[1]. This theorem has been proved by K. Iséki [2]. The purpose of this note is to present a necessary and sufficient condition for a periodic semigroup to be a band of unipotent homogroups[2], and some relevant matters. Throughout the paper, S will denote a periodic semigroup.

2. In the first place, we introduce the following three conditions, which will play principal roles in this note.

Condition A. For any elements a and b, if $a^n=b^m$ for some integers n and m then there exist three positive integers r, s and t such that $(ab)^r=a^sb^t=b^ta^s$.

Condition B. For any elements a and b, and for any positive integers n and m, there exist two positive integers r and s such that $(ab)^r=(a^nb^m)^s$.

Condition C. For any elements a and b, and for any positive integers n and m, there exist three positive integers r, s and t such that $(ab)^r=(a^nb^m)^s=(b^ma^n)^t$.

As to integers, the present paper deals exclusively with positive integers, and hereafter the modifer "positive" will be omitted.

LEMMA 1. *For periodic semigroups, Condition A is a consequence of Condition B.*

Proof. Let S satisfy Condition B. Suppose that a and b are elements of S such that $a^n=b^m$. Then there exists an integer j and an idempotent $e \in S$, satisfying $a^{nj}=b^{mj}=e$. By Condition B, there exist integers r and s such that $(ab)^r=(a^{nj}b^{mj})^s=e$. Hence, we

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1) A semigroup S is called "unipotent" if it contains exactly one idempotent [T. Tamura: Note on unipotent inversible semigroups, Kōdai Math. Sem. Rep., 1954, 93-95 (1954)].

2) An element u of a semigroup S is called a zeroid element of S if, for each element a of S, there exist x and y in S such that $ax=ya=u$ [A. H. Clifford and D. D. Miller: Semigroups having zeroid elements, Amer. Jour. Math., 70, 117-125 (1948)]. A homogroup is a synonym of a semigroup having zeroid elements [G. Thierrin: Sur quelques classes de demi-groupes, C. R. Paris, 236, 33-35 (1953)].

have $(ab)^r = a^{nj}b^{mj} = b^{mj}a^{nj}$.

LEMMA 2. *For semigroups, Condition B is a consequence of Condition C.*

Proof. Immediate from the definitions of Conditions B and C.

G. Thierrien [4] defined strong reversibility of semigroups as follows: A semigroup is said to be *strongly reversible*, if, for any two elements a and b, there exist three integers r, s and t such that $(ab)^r = a^s b^t = b^t a^s$.

We obtain immediately

LEMMA 3. *For periodic semigroups, Condition C is a cosequence of strong reversibility.*

Proof. Let n and m be any two integers, and a and b any two elements of S. Then there exist two integers i, j and two idempotents e_1, e_2 such that $a^i = e_1$ and $b^j = e_2$. By strong reversibility, there exist also integers r, r_1, r_2, s, s_1, s_2, t, t_1 and t_2 such that

$$(ab)^r = a^s b^t = b^t a^s,$$

$$(a^n b^m)^{r_1} = a^{ns_1} b^{mt_1} = b^{mt_1} a^{ns_1},$$

$$\text{and} \quad (b^m a^n)^{r_2} = b^{mt_2} a^{ns_2} = a^{ns_2} b^{mt_2}.$$

Hence, we have

$$(ab)^{rij} = a^{sij} b^{tij} = e_1 e_2,$$

$$(a^n b^m)^{r_1 ij} = a^{ns_1 ij} b^{mt_1 ij} = e_1 e_2$$

$$\text{and} \quad (b^m a^n)^{r_2 ij} = a^{ns_2 ij} b^{mt_2 ij} = e_1 e_2.$$

Thus

$$(ab)^{rij} = (a^n b^m)^{r_1 ij} = (b^m a^n)^{r_2 ij}.$$

Let e be an idempotent of S, and $K^{(e)}$ the set of all elements a such that the subsemigroup generated by a contains the idempotent e, *i. e.* $K^{(e)} = \{a \mid a^n = e$ for some integer $n\}$. It is well known that S is the class sum of mutually disjoint sets $K^{(e)}$. Further, K. Iséki [2] proved that if S is strongly reversible, then each $K^{(e)}$ is a unipotent semigroup, and consequently S is decomposed into the class sum of mutually disjoint unipotent semigroups $K^{(e)}$.

Since Condition A is clearly a consequence of strong reversibility, the following is a generalization of K. Iséki's result.

THEOREM 1. *A periodic semigroup is decomposable into the class sum of mutually disjoint unipotent homogroups if and only if it satisfies Condition A. Further, in this case such a decomposition is uniquely determined.*

Proof. Let S satisfy Condition A. To prove the "if" part of this theorem, we need to show only that each $K^{(e)}$ is a homogroup.

If a and b are two elements of $K^{(e)}$, then there exist integers n and m such that a^n

$=b^m=e$. Since S satisfies Condition A, $(ab)^r=a^sb^t=b^ta^s$ for some integers r, s and t. Hence, $(ab)^{rnm}=a^{snm}b^{tnm}=e$. This implies that $K^{(e)}$ is a semigroup. Since e is clearly a zeroid element of $K^{(e)}$, the semigroup $K^{(e)}$ is a homogroup.

Conversely, assume that S is decomposed into the class sum of mutually disjoint unipotent homogroups H_α, and suppose that $a^n=b^m$. Then both a and b are contained in the same homogroup, say H_α, since there exists an idempotent e and an integer s such that $a^{ns}=b^{ms}=e$. Therefore, $ab\in H_\alpha$. Hence, we have

$$a^{ns}b^{ms}=b^{ms}a^{ns}=e=(ab)^r$$

for some integer r. Thus, the proof of the first half of this theorem is complete. The latter half of this theorem is clear.

A *band* is a synonym of an idempotent semigroup. Let J be a band. A semigroup G is said to be a band J of semigroups of type T (A. H. Clifford [1]), if G is the class sum of a set $\{G_\alpha | \alpha\in J\}$ of mutually disjoint subsemigroups G_α, each of type T, such that for any α, $\beta\in J$, $G_\alpha G_\beta\subset G_{\alpha\beta}$, where $\alpha\beta$ is the product of α and β in J. If J is a commutative band, that is, if J is a semilattice, then S is called a semilattice of semigroups of type T.

Since Conditions B and C are consequences of strong reversibility, the next theorem and Theorem 3 below are also generalizations of K. Iséki's result.

THEOREM 2. *A periodic semigroup is decomposable into a band of unipotent homogroups if and only if it satisfies Condition B.*

Further, in this case such a decomposition is uniquely determined.

Proof. Assume that S is a band of unipotent homogroups H_α. Let a and b be any elements of S. Let n and m be any integers. Then, there exist H_α and H_β which contain a and b respectively. Since, by the assumption on S, both ab and a^nb^m are contained in $H_{\alpha\beta}$, there exist integers r and s such that $(ab)^r=e_{\alpha\beta}$ and $(a^nb^m)^s=e_{\alpha\beta}$, where $e_{\alpha\beta}$ is the idempotent of $H_{\alpha\beta}$.

Conversely, let S satisfy Condition B. By Theorem 1, S is then the class sum of mutually disjoint unipotent homogroups H_α, since S satisfies also Condition A. Pick up any a_1, $a_2\in H_\alpha$ and b_1, $b_2\in H_\beta$ respectively. There exist integers n_1, n_2, m_1 and m_2 such that $a_1^{n_1}=e_\alpha$, $a_2^{n_2}=e_\alpha$, $b_1^{m_1}=e_\beta$ and $b_2^{m_2}=e_\beta$, where e_α and e_β are idempotents of H_α and H_β respectively.

By Condition B,

$$(a_1b_1)^{r_1}=(e_\alpha e_\beta)^{s_1}$$

$$\text{and}\quad (a_2b_2)^{r_2}=(e_\alpha e_\beta)^{s_2}$$

for some integers r_1, s_1, r_2 and s_2.

If $e_\alpha e_\beta$ is contained in H_γ, there exists an integer n such that $(e_\alpha e_\beta)^n = e_\gamma$, where e_γ is the idempotent of H_γ. Therefore, we have

$$(a_1 b_1)^{r_1 n} = (e_\alpha e_\beta)^{s_1 n} = e_\gamma = (e_\alpha e_\beta)^{s_2 n} = (a_2 b_2)^{r_2 n}.$$

This implies that both $a_1 b_1$ and $a_2 b_2$ are contained in H_γ, that is $H_\alpha H_\beta \subseteq H_\gamma$. Thus, the proof of the first half of this theorem is complete. The latter half of the theorem follows from Theorem 1.

A. H. Clifford [1] proved that a band of semigroups of type T is a semilattice of semigroups each of which is a matrix of semigroups of type T.

Using this result and Theorem 2, we obtain

COROLLARY 1. *A periodic semigroup satisfying Condition B is a semilattice of semigroups each of which is a matrix of unipotent homogroups.*

THEOREM 3. *A periodic semigroup is decomposable into a semilattice of unipotent homogroups if and only if it satisfies Condition C.*

Further, in this case such a decomposition is uniquely determined.

Proof. Let S satisfy Condition C. By Lemma 2, S satisfies Condition B. Therefore, by Theorem 2, S is decomposed uniquely into a band J of unipotent homogroups H_α. Let a and b be any elements of S. Then, by Condition C, there exist two integers s and t such that $(ab)^s = (ba)^t$. Hence, there exists H_α which contains both ab and ba. This implies that J is a semilattice.

Conversely, assume that S is decomposed into a semilattice of unipotent homogroups H_α. Let a and b be any elements of S. Let n and m be any integers. Since S satisfies Condition B, there exist integers r_1, r_2, r_3 and r_4 such that $(ab)^{r_1} = (a^n b^m)^{r_2}$ and $(ba)^{r_3} = (b^m a^n)^{r_4}$.

On the other hand, it follows from our assumption on S that ab and ba are contained in the same unipotent homogroup, say H_α, under the decomposition. Hence, there exist two integers t_1 and t_2 such that $(ab)^{t_1} = (ba)^{t_2} = e_\alpha$, where e_α is the idempotent of H_α. Consequently, we have

$$(ab)^{t_1 t_2 r_1 r_3} = (a^n b^m)^{t_1 t_2 r_2 r_3} = (b^m a^n)^{t_1 t_2 r_1 r_4}.$$

If we restrict our consideration to a periodic semigroup in which the set of all idempotents constitutes a band, then Theorem 2 is sharpened as follows:

THEOREM 4. *Let S be a periodic semigroup in which the set I of all idempotents constitutes a band. Then, S is decomposable into a band I of unipotent homogroups if and anly if S satisfies Condition B.*

Finally, we have the following interesting theorem.

THEOREM 5. *A periodic semigroup S is strongly reversible if and only if the set I*

of all idempotents of S constitutes a semilattice and S is decomposable into a semilattice I of unipotent homogroups.

Proof. Assume that S is strongly reversible. By Lemmas 2 and 3, S satisfies Condition B. Let e and f be any elements of I. By strong reversibility, there exist three integers r, s and t such that $(ef)^r=e^sf^t=f^te^s$. Since e and f are idempotents, this implies $ef=fe$. Therefore, I is a semilattice. Consequently, the "only if" part of this theorem follows from Theorem 4.

Conversely, assume that the set I of all idempotents of S constitutes a semilattice, and that S is decomposed into a semilattice I of unipotent homogroups H_α. Let a and b be any elements of S. There exist idempotents e, f and integers s, t such that $a^s=e$ and $b^t=f$. Hence, $a^sb^t=b^ta^s=ef$ follows from commutativity in I.

On the other hand, since ab and ef are contained in the same unipotent homogroup under the decomposition, there exists an integer r such that $(ab)^r=ef$. Thus we have $(ab)^r=a^sb^t=b^ta^s$.

3. *Examples.*

(1) Let S be a periodic semigroup with elements e, f, g, a whose multiplication table is

	e	f	g	a
e	e	f	g	g
f	e	f	g	f
g	e	f	g	f
a	e	f	g	f

S is then the class sum of mutually disjoint unipotent homogroups, but S is not a band of unipotent homogroups.

(2) Let S be a periodic semigroup with elements e, f, g, a whose multiplication table is

	e	f	g	a
e	e	a	g	a
f	g	f	g	g
g	g	g	g	g
a	g	a	g	g

S is then a band of unipotent homogroups, but the set of all idempotents of S does not constitute a band.

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