Vibration of Clamped Parallelogrammic Istropic Flat Plates with aspect ratio 1/4

by Misao HASEGAWA

(*Received Nov.* 30, 1956)

長谷川 節 ： 縦横比1/4の周辺固定平行四辺形板の振動

1. Introduction.

Vibration of square and rectangular plates was discussed by W. Ritz[1], S. Tomotika[2], S. Iguchi[3], D. Young[4] and M. V. Barton[5]. Vibrations of clamped parallelogramic flat plates with aspect ratios 1 and 1/2 was discussed by the author[6]. In this paper, we shall calculate the smallest frequency of the vibration of clamped parallelo-grammic isotropic flat plates with aspect ratio 1/4.

2. The basic equations.

Let the oblique coordinates axes (x, y) be taken in the middle plane of the plate of uniform thin thickness in such a way that the origin coincides with the center of the plate and that the axes are parallel to the sides (Fig. 1). Let us denote the lengths of edges, thickness, the density, Young's modulus, and Poisson's ratio of the plate by $2a$, $2b$, h, ρ, E and ν respectively.

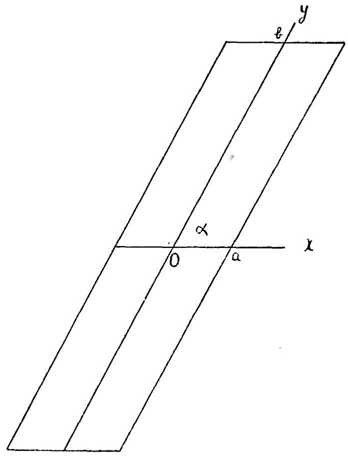

Fig. 1

Let the angle between coordinates axes be denoted by α.

If w be the transverse displacement of a point on the middle plane and t the time, the differential equation for w is

$$\frac{D}{\sin^4}\left[\frac{\partial^4 w}{\partial x^4} - 4\cos\alpha\frac{\partial^4 w}{\partial x^3\partial y} + 2(1+2\cos^2\alpha)\frac{\partial^4 w}{\partial x^2\partial y^2} - 4\cos\alpha\frac{\partial^4 w}{\partial x\partial y^3} + \frac{\partial^4 w}{\partial y^4}\right]$$

$$+\rho h\frac{\partial^2 w}{\partial t^2} = 0 \qquad \cdots\cdots\cdots(1)$$

where $D=\dfrac{Eh^3}{12(1-\nu^2)}$.

The boundary conditions are

$$w = 0 \text{ and } \frac{\partial w}{\partial n} = 0 \qquad \text{at } x = \pm a \qquad \text{and } y = \pm b \ , \qquad \cdots\cdots\cdots(2)$$

where $\dfrac{\partial}{\partial n}$ denotes differentiation along the normal.

As the problem is so complicated that we shall use the Ritz method. According this method, the mode of vibration is assumed. If we assume the following mode

$$w = W(x,y)\cos pt \ . \qquad \cdots\cdots\cdots(3)$$

Eq. (1) is transformed into

$$\frac{\partial^4 W}{\partial x^4} - 4m\frac{\partial^4 W}{\partial x^3\partial y} + 2(1+2m^2)\frac{\partial^4 W}{\partial x^2\partial y^2} - 4m\frac{\partial^4 W}{\partial x\partial y^3} + \frac{\partial^4 W}{\partial y^4} - K' W = 0 \ \cdots\cdots(4)$$

with the boundary conditions

$$W = 0 \text{ and } \frac{\partial W}{\partial n} = 0 \qquad \text{at } x = \pm a \qquad \text{and} \qquad y = \pm b \qquad \cdots\cdots\cdots(5)$$

in which we put $m = \cos\alpha$

and $$K' = \frac{\rho h p^2 \sin^4\alpha}{D} \qquad \cdots\cdots\cdots(6)$$

In order to apply the Ritz method, we consider the following integral :

$$2U = \frac{D}{\sin^3\alpha}\int_{-a}^{a} dx \int_{-b}^{b} \Big[(\triangle W)^2 - 4m\frac{\partial^2 W}{\partial x\partial y}\triangle W + 4m^2\Big(\frac{\partial^2 W}{\partial x\partial y}\Big)^2$$
$$+2(1-\nu)(1-m^2)\Big\{\Big(\frac{\partial^2 W}{\partial x\partial y}\Big)^2 - \frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial y^2}\Big\} - K' W^2\Big]dy \ . \qquad \cdots\cdots\cdots(7)$$

We assume for thr expression of W which satisfies the boundary conditions (5), that

$$W=(a^2-x^2)^2(b^2-y^2)^2(a_{00}+a_{11}xy+a_{20}x^2+a_{02}y^2+a_{31}x^3y+a_{13}xy^3+a_{22}x^2y^2) \qquad \cdots\cdots\cdots(8)$$

in which a_{00}, a_{11}, $\cdots$ and a_{22} are arbitrary constants.

Substituting Eq. (8) into U, we have the following equation:

$$2U = \frac{63536D}{\sin^3\alpha}\Big[\Big\{\frac{a^5b^9+a^9b^5}{3150}+\frac{2a^7b^7(1+2m^2)}{11025}\Big\}a_{00}^2 + \frac{2m(a^7b^9+a^9b^7}{11025}a_{00}a_{11}$$
$$+\Big(\frac{a^7b^9}{11025}+\frac{a^{11}b^5}{17325}\Big)a_{00}a_{20} + \Big(\frac{a^5b^{11}}{17325}+\frac{a^9b^7}{11025}\Big)a_{00}a_{02} + 2m\Big(\frac{a^9b^9}{33075}+\frac{a^{11}b^7}{40425}\Big)a_{00}a_{31}$$
$$+2m\Big(\frac{a^7b^{11}}{40425}+\frac{a^9b^9}{33075}\Big)a_{00}a_{13} + \frac{a^7b^{11}+a^{11}b^7}{121275}a_{00}a_{22} + \Big\{\frac{a^7b^{11}+a^{11}b^7}{48510}+\frac{2a^9b^9(1+2m^2)}{99225}\Big\}a_{11}^2$$

$$-2m\left(\frac{a^9b^9}{33075}+\frac{a^{11}b^7}{121275}\right)a_{11}a_{20}-2m\left(\frac{a^7b^{11}}{121275}+\frac{a^9b^9}{33075}\right)a_{11}a_{02}+\Big\{\frac{a^9b^{11}}{72765}+\frac{a^{13}b^7}{105105}$$

$$+\frac{8a^{11}b^9(1+2m^2)}{1091475}\Big\}a_{11}a_{31}+\Big\{\frac{a^7b^{13}}{105105}+\frac{a^{11}b^9}{72765}+\frac{8a^9b^{11}(1+2m^2)}{1091475}\Big\}a_{11}a_{13}$$

$$+2m\left(\frac{a^9b^{11}+a^{11}b^9}{363825}\right)a_{11}a_{22}+\Big\{\frac{a^9b^9}{7350}+\frac{a^{13}b^5}{150150}+\frac{2a^{11}b^7(1+2m^2)}{121275}\Big\}a_{20}^2$$

$$+\frac{a^7b^{11}+a^{11}b^7}{121275}a_{20}a_{02}+2m\left(\frac{a^{11}b^9}{121275}+\frac{a^{13}b^7}{525525}\right)a_{20}a_{31}-2m\left(\frac{a^9b^{11}}{121275}+\frac{a^{11}b^9}{363825}\right)a_{20}a_{13}$$

$$+\left(\frac{a^9b^{11}}{40425}+\frac{a^{13}b^7}{525525}\right)a_{20}a_{22}+\Big\{\frac{a^5b^{13}}{150150}+\frac{a^9b^9}{7350}+\frac{2a^7b^{11}(1+2m^2)}{121275}\Big\}a_{02}^2$$

$$-2m\left(\frac{a^9b^{11}}{363825}+\frac{a^{11}b^9}{121275}\right)a_{02}a_{31}+2m\left(\frac{a^7b^{13}}{525525}+\frac{a^9b^{11}}{121275}\right)a_{02}a_{13}+\Big(\frac{a^7b^{13}}{525525}$$

$$+\frac{a^{11}b^9}{40425}\Big)a_{02}a_{22}+\Big\{\frac{13a^{11}b^{11}}{1600830}+\frac{a^{15}b^7}{630630}+\frac{34a^{13}b^9(1+2m^2)}{14189175}\Big\}a_{31}^2+\Big\{\frac{a^9b^{13}+a^{13}b^9}{315315}$$

$$+\frac{16a^{11}b^{11}(1+2m^2)}{12006225}\Big\}a_{31}a_{13}-2m\left(\frac{a^{11}b^{11}}{1334025}+\frac{a^{13}b^9}{1576575}\right)a_{31}a_{22}+\Big\{\frac{a^7b^{15}}{630630}+\frac{13a^{11}b^{11}}{1600830}$$

$$+\frac{34a^9b^{13}(1+2m^2)}{14189175}\Big\}a_{13}^2-2m\left(\frac{a^9b^{13}}{1576575}+\frac{a^{11}b^{11}}{1334025}\right)a_{13}a_{22}+\Big\{\frac{a^9b^{13}+a^{13}b^9}{350350}$$

$$+\frac{2a^{11}b^{11}(1+2m^2)}{1334025}\Big\}a_{22}^2-K'\Big[\Big\{\frac{a^9b^9}{99225}a_{00}^2+\frac{2(a^{11}b^9+a^9b^{11})}{1091475}a_{00}a_{20}+\frac{2a^{11}b^{11}}{12006225}(a_{00}a_{22}$$

$$+a_{20}a_{02})+\frac{a^{11}b^{11}}{12006225}a_{11}^2+\frac{2}{52026975}\{a^{13}b^{11}a_{11}a_{31}+a^{11}b^{13}a_{11}a_{13}\}$$

$$+\frac{1}{4729725}(a^{13}b^9a_{20}^2+a^9b^{13}a_{02}^2)+\frac{2}{52026975}(a^{13}b^{11}a_{20}+a^{11}b^{13}a_{02})a_{22}$$

$$+\frac{1}{156080925}(a^{15}b^{11}a_{31}^2+a^{11}b^{15}a_{13}^2)+\frac{2}{225450225}a^{13}b^{13}a_{31}a_{13}\Big\}\Big]. \qquad \cdots\cdots\cdots(9)$$

Since the constants are to be determined so as to make U a minimum by the Ritz method, we must have

$$\frac{\partial U}{\partial a_{00}}=0\,;\quad \frac{\partial U}{\partial a_{11}}=0\,;\quad \frac{\partial U}{\partial a_{20}}=0\,;\quad \frac{\partial U}{\partial a_{02}}=0\,;$$

$$\frac{\partial U}{\partial a_{31}}=0\,;\quad \frac{\partial U}{\partial a_{13}}=0\,;\quad \frac{\partial U}{\partial a_{22}}=0\,. \qquad \cdots\cdots\cdots(10)$$

These equations are linear in the A' s and are found to be :

$$
\left.\begin{array}{l}
B_{100}A_{00}+B_{111}A_{11}+B_{120}A_{20}+B_{102}A_{02}+B_{131}A_{31}+B_{113}A_{13}+B_{122}A_{22} = 0 \\
B_{200}A_{00}+B_{211}A_{11}+ \cdots\cdots\cdots\cdots\cdots + B_{222}A_{22} = 0 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
B_{700}A_{00}+B_{711}A_{11}+B_{720}A_{20}+B_{702}A_{02}+B_{731}A_{31}+B_{713}A_{13}+B_{722}A_{22} = 0
\end{array}\right\} \cdots\cdots(11)
$$

where

$$A_{ij}=a^{4+i}\,b^{4+j}\,a_{ij}\ , \qquad k=\frac{b}{a}, \qquad K=K'a^4=\frac{\rho h a^4 p^2 \sin^4\alpha}{D}\ ,$$

$$B_{100}=\frac{1+k^4}{1575}+\frac{4k^2(1+2m^2)}{11025}-\frac{2Kk^4}{99225}\ , \quad B_{111}=B_{200}=\frac{2m(k^3+k)}{11025}\ ,$$

$$B_{120}=B_{300}=\frac{k^4}{11025}+\frac{1}{17325}-\frac{2Kk^4}{1091475}\ , \quad B_{102}=B_{400}=\frac{k^4}{17325}+\frac{1}{11025}-\frac{2Kk^4}{1091475}$$

$$B_{131}=B_{500}=2m\left(\frac{k^3}{33075}+\frac{k}{40425}\right)\ , \quad B_{113}=B_{600}=2m\left(\frac{k^3}{40425}+\frac{k}{33075}\right)$$

$$B_{122}=B_{302}=B_{420}=B_{700}=\frac{k^4+1}{121275}-\frac{2Kk^4}{12006225}\ ,$$

$$B_{211}=\frac{k^4+1}{24255}+\frac{4k^2(1+2m^2)}{99225}-\frac{2Kk^4}{12006225}\ ,$$

$$B_{220}=B_{311}=-2m\left(\frac{k^3}{33075}+\frac{k}{121275}\right)\ , \quad B_{202}=B_{411}=-2m\left(\frac{k^3}{121275}+\frac{k}{33075}\right)\ ,$$

$$B_{231}=B_{511}=\frac{k^4}{72765}+\frac{1}{105105}+\frac{8k^2(1+2m^2)}{1091475}-\frac{2Kk^4}{52026975}\ , \quad B_{222}=B_{711}=\frac{2m(k^3+k)}{363825}\ ,$$

$$B_{213}=B_{611}=\frac{k^4}{105105}+\frac{1}{72765}+\frac{8k^2(1+2m^2)}{1091475}-\frac{2Kk^4}{52026975}\ ,$$

$$B_{320}=\frac{k^4}{3675}+\frac{1}{75075}+\frac{4k^2(1+2m^2)}{121275}-\frac{2Kk^4}{4729725}\ , \quad B_{302}=B_{420}=\frac{k^4+1}{121275}-\frac{2Kk^4}{12006225}\ ,$$

$$B_{331}=B_{520}=2m\left(\frac{k^3}{121275}+\frac{k}{525525}\right)\ , \quad B_{313}=B_{620}=-2m\left(\frac{k^3}{121275}+\frac{k}{363825}\right)\ ,$$

$$B_{322}=B_{720}=\frac{k^4}{40425}+\frac{1}{525525}-\frac{2Kk^4}{52026975}\ ,$$

$$B_{402}=\frac{k^4}{75075}+\frac{1}{3675}+\frac{4k^2\,(1+2m^2)}{121275}-\frac{2Kk^4}{4729725}\ ,$$

$$B_{431}=B_{502}=-2m\left(\frac{k^3}{363825}+\frac{k}{121275}\right), \quad B_{413}=B_{602}=2m\left(\frac{k^3}{525525}+\frac{k}{121275}\right),$$

$$B_{422}=B_{702}=\frac{k^4}{525525}+\frac{1}{40425}-\frac{2Kk^4}{52026975},$$

$$B_{531}=\frac{13k^4}{800415}+\frac{1}{315315}+\frac{68k^2(1+2m^2)}{14189175}-\frac{2Kk^4}{156080925},$$

$$B_{513}=B_{631}=\frac{k^4+1}{315315}+\frac{16k^2(1+2m^2)}{12006225}-\frac{2Kk^4}{225450225}, \quad B_{522}=B_{731}=-2m\left(\frac{k^3}{1334025}+\frac{k}{1576575}\right),$$

$$B_{613}=\frac{k^4}{315315}+\frac{13}{800415}+\frac{68k^2(1+2m^2)}{14189175}-\frac{2Kk^4}{156080925},$$

$$B_{622}=B_{713}=-2m\left(\frac{k^3}{1576575}+\frac{k}{1334025}\right),$$

$$B_{722}=\frac{k^4+1}{175175}+\frac{4k^2(1+2m^2)}{1334025}-\frac{2Kk^4}{225450225}.$$

If we eliminate A_{00}, $A_{11}\cdots\cdots$, A_{22} from Eqs. (11), we have the following determinantal equation

$$D=\begin{vmatrix} B_{100} & B_{111} & B_{120} & B_{102} & B_{131} & B_{113} & B_{122} \\ B_{200} & B_{211} & \cdots & \cdots & \cdots & \cdots & \cdots \\ B_{300} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & & & & & & \\ B_{700} & B_{711} & B_{720} & B_{702} & B_{731} & B_{713} & B_{722} \end{vmatrix}=0. \qquad \cdots\cdots\cdots\cdots\cdots\cdots(12)$$

3. Numerical calculations and conclusions.

Eq. (12) determines the values of K i. e., the frequencies of vibration. The smallest roots of K are calculated for several values of α for $k=4$ and are shown in Table 1.

Table 1

α	55°	60°	70°	75°	90°
K	32. 2015	32. 2590	32. 3804	32. 4304	32. 4972

The true values of K are smaller than those given in the above Table and these values give each upper limit. We have thus determined the approximate values of K of the plate with aspect ratio 1/4 for α = 55°, 60°, 70°, 75° and 90°.

The author would like to thank Miss Y. Nagasawa for her aid with the computational aspects of the problem.

References

1. Ritz, W., Theorie der Transversalschwingungen einer quadratischen Platten mit freien Randern, Ann der Physik, Vol. 28, p. 737, 1909

2. Tomotika, S., On the Transverse Vibration of a Square Plate with Four Clamped Edges, Report of the Aeronautical Research Institute, Tokyo Imp. Univ. Vol. X, p. 301, 1935

3. Iguchi, S., Die Eigenwertsprobleme für die elastische rechteckige Platte, Memoirs of the Faculty of Engineering Hokkaido Imperial University, Vol. 4, p. 305, 1938

4. Young, D., Vibration of Rectangular Plates by the Ritz Method, Jour. App. Mech. Trans. ASME. Vol. 72, p. 448, 1950

5. Barton, M. V., Vibration of Rectangular and Skew Cantilever Plates, Jour. App. Mech. Trans. ASME. Vol. 73, p. 129, 1951

6. Hasegawa. M., On the Vibration of Clamped Clamped Rhombic Isotropic Plates I., Bulletin of the Shimane University, Vol. 6, p. 21, 1956

Vibration of Clamped Parallelogrammic Isotropic Isotropic Flates, Jour. Aero. Sci. Vol. 24, p. 145, 1957