REGULARLY TOTALLY ORDERED SEMIGROUPS I

by Miyuki YAMADA

(*Received Nov.* 30, 1956)

山田深雪：正則全順序準群 1

By an ordered semigroup we mean a semigroup S in which a binary relation $\leqq$ is defined as follows ;

(1) $a \leqq a$ for every $a \in S$,

(2) $a \leqq b$, $b \leqq a$ imply $a=b$,

(3) $a \leqq b$, $b \leqq c$ imply $a \leqq c$,

(4) $a \leqq b$ implies both $ac \leqq bc$ and $ca \leqq cb$ for every $c \in S$.

We write usually $a<b$ if $a \leqq b$ but $a \neq b$. Especially, we say S to be a totally ordered semigroup if the binary relation $\leqq$ is a totally ordering. Ordered semigroups have been studied by A. H. Clifford, O. Hörder, F. Klein-Barmen, O. Nakada and many other mathematicians. The present paper takes a small portion of this study in parallel with Clifford [1] and Klein-Barmen [2], [3], [4]. Let S be a commutative semigroup. Then we shall call S to be a *regularly totally ordered semigroup* (r. t. o. semigroup) if S satisfies the following two conditions;

(1) for any different $a, b \in S$, either $aS \subset bS$ or $bS \subset aS$ holds,

(2) if $aS \subset bS$, then there exists a positive integer n such that $a^n \in b^n S$, where the symbol $\subset$ means 'is of a proper subset of'.

In fact, S becomes a totally ordered semigroup if a binary relation $\leqq$ in S is defined as follows; $a \leqq b$ means $aS \supseteq bS$. A naturally totally ordered commutative semigroup (see Clifford [1]) is not necessarily a r. t. o. semigroup, and vice-versa. A linear holoid defined by Klein-Barmen [3] is, however, the same thing as a r. t. o. semigroup with identity element. Moreover, both a dense-in-itself segment in sense of Clifford [1] and an archimedean naturally totally ordered commutative semigroup satisfying the cancellation law are interest examples of r. t. o. semigroups. A r. t. o. semigroup is said to be *locally nilpotent* if S satisfies the following condition; for any element a of S,

$$\bigcap_n a^n S \begin{cases} = \phi & \text{if } S \text{ has no zero element,} \\ = \{o\} & \text{if } S \text{ has zero element } o, \end{cases}$$

where ϕ and $\{o\}$ denote the empty set and the set consisting of only one element zero o

respectively. Moreover S is said to be ***discrete*** or ***non-discrete*** according to whether S contains the least element (i. e. the element e such that $x \geqq e$ for any $x \in S$) or not, respectively.

In §1 we define, for every positive real number α and for every non-negative real number β, a ***closed half line*** $L[\alpha]$ and an ***open half line*** $L(\beta)$, and define their ***indexed subgroups***. In §2 we discuss, in preperation for §3 and §4, on general properties of locally nilpotent r. t. o. semigroups without zero. We devote §3 to show that a discrete, locally nilpotent r. t. o. semigroup without zero is characterized as an indexed subgroup of the closed half line $L[1]$, and §4 to show that a non-discrete, locally nilpotent r. t. o. semigroup without zero is characterized as a β-dense, indexed subgroup of an open half line $L(\beta)$. In the concluding section we show that if a locally nilpotent r. t. o. semigroup S contains zero element o and if S satisfies the cancellation law (i. e. the law; $ax=bx \neq o$ implies $a=b$), then the problem of determing the structure of S is reduced to the problem of determing the structure of either locally nilpotent r. t. o. semigroups without zero or dense-in-itself segments.

Notations. $\{x \mid \cdots\cdots$(proposition about x)$\}$ denotes the set of all elements x such that the proposition about x is correct. If $A \subseteq B$, $B-A$ denotes the complement of A in B. If $\{A_\tau \mid \tau \in I\}$ is a set of classes, $\bullet\sum_{\tau \in I} A_\tau$ denotes their class sum. If A and B are subsets of a semigroup, AB denotes the set $\{xy \mid x \in A, y \in B\}$.

§1. Closed half lines. Open half lines. Indexed subgroups.

Let $\| \xi \|$ and $[\xi]$ be, for any non-negative real number ξ, the integral part and the decimal part of ξ respectively; i. e., if ξ is expressed in the form $\xi = \alpha_1\alpha_2\cdots\cdots\alpha_n.\beta_1\beta_2\cdots\cdots$ by the decimal system, then $\| \xi \| = \alpha_1\alpha_2\cdots\cdots\alpha_n$ and $[\xi] = o.\beta_1\beta_2\cdots\cdots$. Take up a real number $\alpha > o$ and set $I[\alpha] = \{x \mid \alpha \leqq x < \alpha+1, x$ is a real number$\}$. Then $I[\alpha]$ becomes a group if we define a binary relation o in $I[\alpha]$ as follows; $x o y = [x+y-\alpha]+\alpha$. We shall denote by $G[\alpha]$ the above-mentioned group. Next, we set $P[\alpha] = \{(x,n) \mid x \in G[\alpha]$, n is a non-negative integer$\}$ and define a binary relation $\times$ in $P[\alpha]$ as follows; $(x, n) \times (y, m) = (x o y,\ n+m+\| x+y-\alpha \|)$. $P[\alpha]$ becomes then a semigroup, which we shall call a closed half line $L[\alpha]$. Let $G^*[\alpha]$ be a subgroup of $G[\alpha]$, and set $L^*[\alpha] = \{(x, n) \mid x \in G^*[\alpha]$, n is a non-negatve integer$\}$. Then it is obvious that $L^*[\alpha]$ is a subsemigroup of $L[\alpha]$. We shall call such $L^*[\alpha]$ an indexed subgroup (I-subgroup) of $L[\alpha]$.

Similarly, we define open half lines and their I-subgroups as follows. Take up a real number $\beta \geqq o$, and set $I(\beta) = \{x \mid \beta < x \leqq \beta+1$, x is a real number$\}$. Then $I(\beta)$ becomes a group if we define a binary relation ◎ in $I(\beta)$ as follows;

$$x \circledcirc y = [x+y-\beta] + \beta + \varphi(x+y-\beta),$$

where φ is a real function such that

$$\varphi(z) = o \text{ if } [z] \neq o,$$
$$= 1 \text{ if } [z] = o.$$

We shall denote by $G(\beta)$ the above-mentioned group. Next, we set $P(\beta) = \{(x, n) \mid x \in G(\beta), n$ is a non-negative integer$\}$ and define a binary relation $\divideontimes$ in $P(\beta)$ as follows; $(x,n) \divideontimes (y,m) = (x \circledcirc y,\ n+m+\| x+y-\beta \| - \varphi(x+y-\beta))$.

$P(\beta)$ becomes then a semigroup, which we shall call an open half line $L(\beta)$. Let $G^*(\beta)$ be a subgroup of $G(\beta)$, and set $L^*(\beta) = \{(x,n) \mid x \in G^*(\beta),\ n$ in a non-negative integer$\}$. Then it is easy to see that $L^*(\beta)$ is a subsemigroup of $L(\beta)$. We shall call such $L^*(\beta)$ an indexed subgroup (I-subgroup) of $L(\beta)$. Especially an indexed subgroup $L^*(\beta)$ of $L(\beta)$ is said to be *β-dense* if it satisfies the following condition;

for any $(x,o) \in L(\beta)$, there exists $(y,o) \in L^*(\beta)$ such that $y < x$.

The reason for the term "a closed half line $L[\alpha]$" ["an open half line $L(\beta)$"] is that $L[\alpha]$ [$L(\beta)$] is isomorphic with the additive semigroup consisting of all real numbers $x \geqq \alpha$ [$x > \beta$].

By a half line we shall mean a semigroup which is either a closed half line or an open half line. In conclusion of this section we present the next theorem, omitting its proof.

Theorem 1. *Every I-subgroup of any half line is a locally nilpotent r. t. o. semigroup without zero. Especially, every I-subgroup of $L[1]$ is discrete, locally nilpotent r. t. o. semigroup without zero, while every β-dense, I-subgroup of $L(\beta)$ is a non-discrete, locally nilpotent semigroup without zero.*

§ 2. Locally nilpotent r. t. o. semigroups without zero.

Throughout this section S will denote a locally nilpotent r. t. o. semigroup without zero.

Lemma 1. *S is archimedean, i. e. for any $a, b \in S$ there exist positive integers m, n such that $a^n \geqq b$ and $b^m \geqq a$.*

Poof. Take up any two elements a, b from S. We may show that there exist positive integers n, m such that $a^n S \subseteq bS$ and $b^m S \subseteq aS$. Were $b^i S \supset aS$ for every positive integer i, we would have $\bigcap_i b^i S \supseteq aS$, hence $aS = \phi$, contrary to $aS \neq \phi$. Hence, there exists an integer m such that $b^m S \subseteq aS$. Similarly, there exists an integer n such that $a^n S \subseteq bS$.

Lemma 2. *If $a < b$, then there exists an integer n such that $a^{n+1} \leqq b^n$.*

Proof. $a < b$ implies $aS \supset bS$, hence $b^i \in a^i S$ for some integer i.

We have therefore $b^i=a^it$ for some element t of S. On the other hand, there exists an integer k such that $t^k\geqq a$. Hence we have $b^{ik}=a^{ik}t^k\geqq a^{ik+1}$, and hence $b^{ik}\geqq a^{ik+1}$.

Lemma 3. *S contains no idempotents.*

Proof. Obviousness.

Lemma 4. *S is a positively ordered semigroup, i. e. for any $a, b\in S$, $a<ab$ holds.*

Proof. $a\leqq ab$ is obvious by the definition of the ordering $\leqq$. Were $a=ab$, we would have $a=ab^i$ for every positive integer i. From Lemma 1 we obtain $b^j\geqq a$ for some integer j. Consequently, we have $a=ab^j\geqq a^2$, hence $a=a^2$, contrary to Lemma 3.

Lemma 5. *For any $a<b$ and for any $c\in S$ there exist integers n, m such that $a^n<c^m<b^n$.*

Proof. By Lemma 2, $a^{i+1}\leqq b^i$ holds for some integer i. Since $a^k\geqq c^2$ is also satisfied for some integer k, we have $a^{k(i+1)}\leqq b^{ki}$, therefore $a^{ki}c^2\leqq b^{ki}$. From Lemma 1, we are able to show that $c^j\leqq a^{ki}<c^{j+1}$ holds for some integer $j\geqq 2$. Accrdingly we have $a^{ki}<c^{j+1}<c^{j+2}\leqq a^{ki}c^2\leqq b^{ki}$. Putting $n=ki$ and $m=j+1$, we obtain the desirable relation $a^n<c^m<b^n$.

Lemma 6. *S satisfies the cancellation law, i. e. $ax=bx$ implies $a=b$.*

Proof. Assume that $ax=bx$ but $a\neq b$. Since $a<b$ or $b<a$ we may assume $a<b$. By Lemma 2, we have $a^{m+1}\leqq b^m$ for some integer m. Hence $a^{m+1}x^m\leqq b^mx^m=a^mx^m$. Putting $a^mx^m=\xi$, we obtain $a\xi\leqq\xi$, contrary to Lemma 4.

Lemma 7. *Let a be any element of S. Then S is formularized as follows; $S=\dot{\sum\limits_{n=0}^{\infty}}a^nS(a)$, where $S(a)$ denotes the set $S-aS$ and $a^0S(a)$ means the set $S(a)$.*

Proof. It is obvious that S is partitioned such as $S=\dot{\sum\limits_{n=0}^{\infty}}(a^nS-a^{n+1}S)$. Accordingly we may show only that $a^nS(a)=a^nS-a^{n+1}S$ holds for each non-negative integer n. Take up any $a^ny\in a^nS(a)$, where y is an element of $S(a)$. Were $a^ny\in a^{n+1}S$, we would have $a^ny=a^{n+1}z$ for some $z\in S$. Hence, by Lemma 6, we obtain $y=az$, which contradicts to our assumption $y\in S(a)$. Accordingly we have $a^ny\notin a^{n+1}S$, which induces the relation $a^nS(a)\subseteq a^nS-a^{n+1}S$. Conversely, let y be any element of $a^nS-a^{n+1}S$. Then $y=a^nt$ for some element $t\in S$. If $t\notin S(a)$ we have $t\in aS$, which implies $at'=t$ for some $t'\in S$. Accordingly we have $y=a^{n+1}t'$, which is contrary to our assumption $y\notin a^{n+1}S$. We obtain therefore $t\in S(a)$, hence $y\in a^nS(a)$, which induces the relation $a^nS-a^{n+1}S\subseteq a^nS(a)$.

Lemma 8. *Let a be any element of S. Then every element y of S is uniquely expressed in the form $y=a^nx$, where n is a non-negative integer, x is an element of S and a^0x means x itself.*

Proof. From Lemma 7, it is easy to see that y is expressed in the form $y=a^nx$, $x\in S(a)$. Therefore we may show only the uniqueness of such a decomposition. Assume that

y is expressed in two ways such that $y=a^n x$ and $y=a^m z$, where $x,z \in S(a)$. Were $n>m$, by Lemma 5 we would have $a^{n-m}x=z$ and $n-m>o$, contrary to our assumption $z \in S(a)$. Hence we have $n \not> m$, and similarly $m \not> n$. Consequently $n=m$ is satisfied. Since S satisfies the cancellation law, we conclude $x=z$ from the relation $a^n x=a^m z$.

Lemma 9. *If b is an element of $S(a)$, then $b<az$ holds for any element z of S.*

Proof. $b \in S-aS$ is obvious by the definition of $S(a)$. Assume that there exists an element z satisfying $az \leq b$. Then $bS \subseteq azS$, hence $bz=azy$ for some $y \in S$. By Lemma 5, we obtain $b=ay$, contrary to $b \notin aS$.

Let e be any element of S. Then, there exists, for every $x \in S$, an integer j satisfying $e \leq x^j$. From the above-mentioned lemmas it is easy to see that an integer $x(n)$ satisfying $e^{x(n)} \leq x^n < e^{x(n)+1}$ is uniquely determined for every integer $n \geq j$. Since we can easily prove the existence of $\lim_{n\to\infty} \frac{x(n)}{n}$, we set $\lim_{n\to\infty} \frac{x(n)}{n}$ as follows; $\lim_{n\to\infty} \frac{x(n)}{n} =[x]$.

We shall call $[x]$ "*the coordinate of x which is induced by the base point e*".

Lemma 10. $[e]=1$.

Proof. Obviousness.

Lemma 11. *$a<b$ implies $[a]<[b]$.*

Proof. By Lemma 4, $a^n < e^j < b^n$ is satisfied for some integers $n,j \geq 2$. Morover, by Lemma 2 $e^{j(i+1)} \leq b^{ni}$ is satisfied for some integer i. Accordingly $a^{ni} \leq e^{ji} < e^{ji+1} < e^{j(i+1)} \leq b^{ni}$. Putting $ni=k$, we have $a^k \leq e^{ji} < e^{j(i+1)} \leq b^k$. Take up two integers $a(k)$ and $b(k)$ such that $e^{a(k)} \leq a^k < e^{a(k)+1}$ and $e^{b(k)} \leq b^k < e^{b(k)+1}$. $a(k)$ and $b(k)$ must then satisfy the relations $a(k) \leq ji$ and $b(k) \geq j(i+1)$.

Hence $[a] \leq \frac{a(k)+1}{k} \leq \frac{ji+1}{k}$ and $[b] \geq \frac{b(k)}{k} \geq \frac{ji+j}{k}$.

These imply the desired relation $[a]<[b]$.

Lemma 12. $[ab]=[a]+[b]$, *for any elements a,b of S.*

Proof. By Lemma 1, $e \leq a^k$ and $e \leq b^j$ hold for some integers k,j.
Let $a(n)$ and $b(n)$ be, for every integer $n>max(k,j)$, two integers such that $e^{a(n)} \leq a^n < e^{a(n)+1}$ and $e^{b(n)} \leq b^n < e^{b(n)+1}$. Then $e^{a(n)+b(n)} \leq (ab)^n < e^{a(n)+b(n)+2}$. Accordingly

$\lim_{n\to\infty} \frac{a(n)+b(n)}{n} \leq [ab] \leq \lim_{n\to\infty} \frac{a(n)+b(n)+2}{n}$. Hence we have $[a]+[b]=[ab]$.

Let $R[\alpha]$, $R(\alpha)$ be two additive semigroups consisting of all real numbers x such that $x \geq \alpha$ and $x>\alpha$ respectively. If we set $\alpha = \inf_{S \ni x} [x]$, then it is obvious by Lemmas 11, 12 that S is embedded in $R[\alpha]$ or $R(\alpha)$ according to whether S is discrete or not. In §3 and §4, we shall yet discuss on the structure of S more precisely.

§ 3. Discrete, locally nilpotent r. t. o. semigroups without zero.

Throughout this section S will denote a discrete, locally nilpotent r. t. o. semigroup without zero, and e the least element of S. $S(e)$ will denote the set $S-eS$. As was seen in § 2, every element x of S has a coordinate $[x]$ which is induced by the base point e. Set $G^*=\{[x] \mid x\in S(e)\}$. Then we have the following

Lemma 13. *G^* is a subgroup of $G[1]$.*

Proof. $[e]=1$ is obviously by Lemma 10. Take up any element $[x]\in G^*$. $x<e^2$ is then satisfied by Lemma 9. Hence $[x]<2$, which implies $[x]\in G[1]$. This implies the relation $G^*\subseteq G[1]$. Let $[x],[y]$ be any elements of G^*. Since $e^2\leqq xy$, $xy\notin S(e)$ holds. Accordingly xy is expressed as follows; $xy=e^iz$, where $i\geqq 1$ and $z\in S(e)$. $[z]$ is clearly contained in G^*. On the other hand, we have $[x]\circ[y]=([x]+[y]-1)+1=(i+[z]-1)$

$$+1=([z]-1)+1=[z].$$

Consequently $[x]\circ[y]\in G^*$, which implies G^* to be closed under the binary relation o. It is easy to see that $[e]$ is an identity element in G^*. Finally we prove, for each element $[x]$ of G^*, the existence of an inverse element of $[x]$. In case $x=e$ the existence of an inverse element of $[x]$ is trivial. We may, therefore, consider it in case $x\neq e$. Let $x\neq e$. Since $xS\supset e^2S$ is satisfied by Lemma 9, there exists an element y of S such that $xy=e^3$. $xy=e^3$ implies $[x]+[y]=3$, hence $[y]<2$, and hence $y\in S(e)$. Hence $[y]\in G^*$. On the other hand, we have $[x]\circ[y]=([x]+[y]-1)+1=1=[e]$.

Set $L^*[1]=\{(x, n) \mid x\in G^*$, n is a non-negative integer$\}$. Since G^* is a subgroup of $G[1]$, the set $L^*[1]$ is clearly an I-subgroup of $L[1]$. Let x be any element of S. Then x is uniquely expressed in the form $x=e^ny$, where n is an integer and y is an element of $S(e)$. We define a mapping ψ of S into $L^*[1]$ as follows;

$$\psi\ ;\ x\longrightarrow([y],n),\ \text{if}\ x=e^ny,\ y\in S(e).$$

Then it is easy to see that ψ is an isomorphism of S onto $L^*[1]$.

Thus we have

Theorem 2. *Let S be a discrete, locally nilpotent r. t. o. semigroup without zero. Then S is isomorphic with an I-subgroup of the closed half line $L[1]$.*

From Theorem 1 and Theorem 2, we conclude that a discrete, locally nilpotent r. t. o. semigroup without zero is essentially the same thing as an indexed subgroup of $L[1]$.

§ 4. Non-discrete, locally nilpotent r. t. o. semigroups without zero.

Throughout this section S will denote a non-discrete, locally nilpotent r. t. o. semigroup without zero. Take up an element e of S. $S(e)$ will denote the set $S-eS$. As was seen in § 2, every element x has a coordinate $[x]$ which is induced by the base point e. Set $G^*=\{[x] \mid x\in S(e)\}$. Since S does not contain the least element, there exists no element z such that $[z]=\inf_{S\ni x}[x]$. Set $\beta=\inf_{S\ni x}[x]$. Then $1>\beta\geqq o$ is obvious.

Lemma 14. *G^* is a subgroup of $G(\beta)$.*

Psoof. Let ξ be an element of S. Then $z<e\xi$ for every element $z\in S(e)$. We obtain therefore $[z]<1+[\xi]$ for every element $z\in S(e)$ and for every element $\xi\in S$. Hence wa have $[z]\leqq \inf_{\xi\in S}(1+[\xi])=1+\beta$.

Consequently we have $G^*\subseteq G(\beta)$. $[e]=1$ is obviously by Lemma 10. Let $[x], [y]$ be any elements of G^*. xy is then uniquely expressed in the form $xy=e^iz$, where $i\geqq o$ and $z\in S(e)$. $[z]$ is crearly contained in G^*.

$$[x]\circledcirc[y]=([x]+[y]-\beta)+\beta+\varphi([x]+[y]-\beta)=(i+[z]-\beta)+\beta+\varphi(i+[z]-\beta)$$
$$=([z]-\beta)+\beta+\varphi([z]-\beta)=[z]$$

Consequently $[x]\circledcirc[y]\in G^*$, which implies G^* to be closed under the binary relation $\circledcirc$. It is easy to see that $[e]$ is an identity element in G^*. Finally we prove, for each element $[x]$ of G^*, the existence of an inverse element of $[x]$. In case $x=e$ the existence of an inverse elenent of $[x]$ is trivial. We may, therefore, consider it in case $x\neq e$. Let $x\neq e$. Since $xS\supset e^2S$ is satisfied by Lemma 9, there exists an element y of S such that $xy=e^3$. On the other hand, y is expressed as follows; $y=e^iz$, where $i\geqq o$ and $z\in S(e)$. Hence $e^3=e^ixz$. Since $i<3$ is obviously, we have $e^{3-i}=xz$ by Lemma 6. Accordingly

$$[x]\circledcirc[z]=([x]+[z]-\beta)+\beta+\varphi([x]+[z]-\beta)=(3-i-\beta)+\beta+\varphi(3-i-\beta)=1=[e].$$

That is, $[z]$ is an inverse element of $[x]$.

Set $L^*(\beta)=\{(x, n) \mid x\in G^*,\ n \text{ is a non-negative integer}\}$. Since G^* is a subgroup of $G(\beta)$, the set $L^*(\beta)$ is clearly an I-subgroup of $L(\beta)$.

Moreover we can prove the β-desity of $L^*(\beta)$ as follows. Since $1>\beta=\inf_{S\ni x}[x]$ there exists, for any element $(\xi,o)\in L(\beta)$, an element z of S such that $\beta<[z]<\xi$ and $[z]<1$. $[z]<1$ implies $z\notin eS$, hence $[z]\in G^*$, and hence $([z],o)\in L^*(\beta)$. Thus the β-density of $L^*(\beta)$ is proved. Let x be any element of S. Then x is uniquely expressed in the form $x=e^ny$, where n is an integer and y is an element of $S(e)$. We define a mapping ψ of S into $L^*(\beta)$ as follows;

$$\psi\ ; x\longrightarrow([y], n)\ , \text{if } x=e^ny,\ y\in S(e).$$

Then it is easy to see that ψ is an isomorphism of S onto $L^*(\beta)$. Thus we have

Theorem 3. *Let S be a non-discrete, locally nilpotent r. t. o. semigroup without zero. Then S is isomorphic with a β-dense, I-subgroup of an open half line $L(\beta)$, where $\beta<1$.*

From Theorem 1 and Theorem 3, we conclude that a non-discrete, locally nilpotent r. t. o. semigroup without zero is essentially the same thing as a β-dense, indexed subgroup of an open half line $L(\beta)$, where $\beta<1$.

§ 5. Locally nilpotent r. t. o. semigroups with zero.

Let S be a locally nilpotent r. t. o. semigroup with zero o. By a *zero divisor* we shall mean a non-zero element x such that $xy=o$ for some non-zero element y. Moreover, by a *nil element* we shall mean an element z satisfying $z^n=o$ for some integer n.

Theorem 4. *If S has no zero divisor and if the set $S^*=S-\{o\}$ is not the empty set then S^* is a locally nilpotent r. t. o. subsemigroup of S. That is, S^* becomes a subsemigroup of S which is also a locally nilpotent r. t. o. semigroup without zero.*

Proof. It is obvious that S^* is a subsemigroup of S, and that S^* has no zero element in S^* itself. Therefore, we shall next prove the remaining part of this theorem.

(1) For any two different elements $a,b\in S^*$, either $aS\subset bS$ or $bS\subset aS$ holds.

In case $aS\subset bS$ we have $aS^*\subset bS^*$, while in case $bS\subset aS$ we have $bS^*\subset aS^*$.

(2) Let a,b be elements of S^*. $aS^*\subset bS^*$ implies $aS^*+\{o\}\subset bS^*+\{o\}$, hence $aS\subset bS$. Hence $a^n\in b^nS$ for some integer n. Since $a^n\neq o$, we have $a^n\in b^nS^*$.

(3) Let a be an element of S^*. Then $\bigcap_n a^nS=\{o\}$. Since $\bigcap_n a^nS^*\subset \bigcap_n a^nS=\{o\}$, $\bigcap_n a^nS^*$ must be the empty set.

From (1)~(3), we obtain this theorem.

Lemma 15. *Every element of S is a nil-element if S has at least one zero divisor.*

Proof. Let a be a zero divisor of S. Then there exists an element b such that $ab=o$ and $b\neq o$. Take up any element x of S. Were $x^n\leqq b$ for every positive integer n, we would have $\bigcap_n x^nS\supseteq bS$, hence $bS=\{o\}$, contrary to our assumption $b\neq o$. Thus there exists an integer i satisfying $x^i>b$. Similarly, there exists an integer j satisfying $x^j>a$. Hence we have $o=ab\leqq x^{i+j}$, and hence $x^{i+j}=o$. (It is obvious that zero element o is the greatest element of S)

Lemma 16. $xt=x$ *implies* $x=o$.

Proof. Since $t^nx=x$ is satisfied for any positive integer n, we obtain $\{o\}=\bigcap_n t^nS\ni x$, and hence $x=o$.

Lemma 17. $xt=y$, $yt'=x$ *imply* $x=y=o$.

Proof. $xt=y$, $yt'=x$ imply $x(tt')=x$. We obtain therefore $x=y=o$ from Lemma 16.

The author is not able to know whether every locally nilpotent r. t. o. semigroup having zero elment o always satisfies the cancellation law or not, but at least he is able to present the following statement.

Theorem 5. *If S satisfies the cancellation law and if S has at least one zero divisor, then S is a dense-in-itself segment (in sense of Clifford [1])*.*

Proof. We shall prove this theorem in three steps.

(1) *S is a naturally totally ordered commutative semigroup.*

We first prove the relation $S=S^2$. Assume the contrary, and take up any element $t \in S-S^2$. Then $txyS \subseteq t^2S$ holds for any elements x, y of S. Thus we have $tS^3 \subseteq t^2S$, hence $tS^{2n-1} \subseteq t^nS$ for every integer $n \geqq 2$. Since t is a nil-element, there exists an integer $i \geqq 2$ satisfying $t^i=o$. Hence we have $tS^{2i-1}=\{o\}$. Let j be an integer such that $tS^{j-1} \neq \{o\}$ and $tS^j=\{o\}$. Then there exist elements $x_1, x_2, \cdots\cdots, x_{j-1}$ such that $tx_1x_2\cdots\cdots x_{j-1} \neq o$. On the other hand, we have $tx_1x_2\cdots\cdots x_{j-1}S \subseteq tS^j=\{o\}$, hence $tx_1x_2\cdots\cdots x_{j-1}=o$. Consequently $S=S^2$ holds.

Now, it is sufficient to prove that for any different elements x, y of S at least one of relations $xt=y$ and $yt=x$ holds for some element t of S, since from Lemma 17 it is impossible that both $xt=y$ and $yt=x$ happen at the same time. Assume $x<y$. In case $y=o$ the above assertion is trivial. We assume therefore $y \neq o$. Then there exists an element t satisfying $yt \neq o$. Moreover, $xy' \leqq y$ is satisfied by some element t' of S. In fact, this is proved as follows. Since $xz>y$ means $xzS \subset yS$ for any element z of S, if $xz>y$ holds for every element z of S we obtain $xS^2 \subseteq yS$, hence $xS \subseteq yS$. This is impossible since $x<y$ means $xS \supset yS$. There exists therefore an element t' satisfying $xt' \leqq y$. Set $min(t, t')=t''$. Then $ytS \subseteq yt''S$, and hence $yt'' \neq o$. Since $xt'' \leqq y$ we have $xt''S \supseteq yS$. There exists therefore an element s such that $xt''s=yt'' \neq o$. Thus we obtain $xs=y$ by using the cancellation law.

Hereafter $S(\leqq)$ will denote S in which the naturally ordering $\leqq$ is defined.

(2) *$S(\leqq)$ is ordinally irreducible (see Clifford [1]).*

Obviousness.

(3) *Every element of $S(\leqq)$ has a finite order.*

Obviously by Lemma 15.

(4) *$S(\leqq)$ is dense-in-itself. That is, for given $x \lessdot y$ there exists an element z satisfying $x \lessdot z \lessdot y$.*

Assume the contrary. $x \lessdot y$ implies $x \neq o$ and $x<y$. In case $y=o$, we have $xt=o$ for any element $t \in S$, hence $xS=\{o\}$, and hence $x=o$. This is contrary to $x \neq o$. In case $y \neq o$, $xt'=xt \neq o$ is satisfied if we take up two elements t, t' such that $xt=y$ and $t'<t$. (The existence of these elements is obvious). By the cancellation law we have therefore $t=t'$, contrary to $t'<t$.

From (1) $\sim$ (4), we conclude that S is a dense-in-itself segment.

REFERENCES.

[1] A. H. Clifford, "Naturally totally ordered commutative semigroups", American Joural of Mathematics, Vol. 76, No. 3, 1954, pp. 631——646.

[2] F. Klein-Barmen, "Über gewisse Halbverbande und kommutative Semigruppen 1", Mathematische Zeitschrift, Vol. 48, 1942-3, pp. 275-288.

[3] 〃 , "Über gewisse Halbverbande und kommutative Semigruppen 2", Mathematische Zeitschrift, Vol. 48, 1942-3, pp. 715-734.

[4] 〃 , "Ein Beitrag zur Theorie der linearen Holoide", Mathematische Zeitschrift, Vol. 51, 1947-9, pp. 355-366.

[*] A naturally totally ordered commutative semigroup S is said to be a segment if S is ordinally irreducible and if each element of S has a finite order.