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Optimality Conditions and Constraint Qualifications for Quasiconvex Programming

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Abstract In mathematical programming, various kinds of optimality conditions have been introduced. In the research of optimality conditions, some types of subdifferentials play an important role. Recently, by using Greenberg-Pierskalla subdifferential and Martínez-Legaz subdifferential, necessary and sufficient optimality conditions for quasiconvex programming have been introduced.

On the other hand, constraint qualifications are essential elements for duality theory in mathematical programming. Over the last decade, necessary and sufficient constraint qualifications for duality theorems have been investigated extensively. Recently, by using the notion of generator, necessary and sufficient constraint qualifications for Lagrange-type duality theorems have been investigated. However, constraint qualifications for optimality conditions in terms of Greenberg-Pierskalla subdifferential and Martínez-Legaz subdifferential have not been investigated yet.

In this paper, we study optimality conditions and constraint qualifications for quasiconvex programming. We introduce necessary and sufficient optimality conditions in terms of Greenberg-Pierskalla subdifferential, Martínez-Legaz subdifferential and generators. We investigate necessary and/or sufficient constraint qualifications for these optimality conditions. Additionally, we show some equivalence relations between duality results for convex and quasiconvex programming.

Keywords quasiconvex programming · optimality condition · constraint qualification · generator of a quasiconvex function

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1 Introduction

In mathematical programming, various kinds of optimality conditions have been introduced; for convex programming [1–6], for generalized convex programming [7–12], for quasiconvex programming [13–22], and so on. In the research of optimality conditions, some types of subdifferentials play an important role. Especially, by the subdifferential in convex analysis, a necessary and sufficient optimality condition for convex programming has been investigated. The optimality condition is an essential tool in convex programming, and has been generalized to various cases. Recently, by using Greenberg-Pierskalla subdifferential and Martínez-Legaz subdifferential, necessary and sufficient optimality conditions for quasiconvex programming have been introduced by Suzuki and Kuroiwa; see [20,21]. Similar results for quasiconvex programming have been investigated; see [14,22].

On the other hand, constraint qualifications are essential elements for duality theory in mathematical programming. Over the last decade, necessary and sufficient constraint qualifications for duality theorems have been investigated extensively; see [1,3,17,19,23–30] and references therein. Especially, in convex programming, necessary and sufficient constraint qualifications for Lagrange duality have been investigated; see [1,3,23–25]. In the research of these constraint qualifications, Fenchel conjugate and the subdifferential play a central role. Recently, a notion, called a generator of a quasiconvex function, was defined by Suzuki and Kuroiwa in [26], which is based on the following property: a quasiconvex function consists of the supremum of quasiaffine functions; in detail, see [31,32]. By using the notion of generator, necessary and sufficient constraint qualifications for Lagrange-type duality theorems have been investigated; see [17,19,26,27,29,30]. However, constraint qualifications for optimality conditions in terms of Greenberg-Pierskalla subdifferential and Martínez-Legaz subdifferential have not been investigated yet.

In this paper, we study optimality conditions and constraint qualifications for quasiconvex programming. We introduce necessary and sufficient optimality conditions in terms of Greenberg-Pierskalla subdifferential, Martínez-Legaz subdifferential and generators. We investigate necessary and/or sufficient constraint qualifications for these optimality conditions. Additionally, we show some equivalence relations between duality results for convex and quasiconvex programming.

The rest of the present paper is organized as follows. In Section 2, we give preliminaries. In Section 3, we study necessary and sufficient optimality conditions and related constraint qualifications. In Section 4, we discuss about our optimality conditions and constraint qualifications. Section 5 is the Conclusions.

2 Preliminaries

Let $\langle v, x\rangle$ denote the inner product of two vectors v and x in the n-dimensional Euclidean space $\mathbb{R}^n$. Given a nonempty set A, we denote the closure, the convex hull, and the conical hull, generated by A, by cl$A$, conv$A$ and cone$A$, respectively. By convention, we define cone $\emptyset = \{0\}$. The normal cone of A at $x \in A$ is denoted by $N_A(x) := \{v \in \mathbb{R}^n : \forall y \in A, \langle v, y - x\rangle \leq 0\}$. The indicator function δ_A of A is defined by

$$\delta_A(x) := \begin{cases} 0, & x \in A, \\ \infty, & otherwise. \end{cases}$$

Let f be a function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, where $\overline{\mathbb{R}} := [-\infty, \infty]$. We denote the domain of f by dom$f$, that is, dom$f := \{x \in \mathbb{R}^n : f(x) < \infty\}$. The epigraph of f is defined as epi$f := \{(x, r) \in \mathbb{R}^n \times \mathbb{R} : f(x) \leq r\}$, and f is said to be convex, iff epi$f$ is convex. The Fenchel conjugate of f, $f^* : \mathbb{R}^n \to \overline{\mathbb{R}}$, is defined as $f^*(v) := \sup_{x\in\mathbb{R}^n}\{\langle v, x\rangle - f(x)\}$. Define the level sets of f with respect to a binary relation $\diamond$ on $\overline{\mathbb{R}}$ as

$$\text{lev}(f, \diamond, \beta) := \{x \in \mathbb{R}^n : f(x) \diamond \beta\}$$

for any $\beta \in \mathbb{R}$. A function f is said to be quasiconvex, iff lev$(f, \leq, \beta)$ is a convex set for all $\beta \in \mathbb{R}$. Any convex function is quasiconvex, but the opposite is not true.

A function f is said to be essentially quasiconvex, iff f is quasiconvex and each local minimizer $x \in$ dom$f$ of f in $\mathbb{R}^n$ is a global minimizer of f in $\mathbb{R}^n$. Clearly, all convex functions are essentially quasiconvex. It is known that a pseudoconvex differentiable function is essentially quasiconvex; see [7, 33, 34] for more details. It is shown that a real-valued continuous quasiconvex function is essentially quasiconvex, if and only if it is semistrictly quasiconvex; see Theorem 3.37 in [13]. In [35], the notion of neatly quasiconvex function is introduced. A function f is said to be neatly quasiconvex, iff it is quasiconvex and for every $x \in \mathbb{R}^n$ with $f(x) > \inf_{y\in\mathbb{R}^n} f(y)$, the sets lev$(f, \leq, f(x))$ and lev$(f, <, f(x))$ have the same closure. By Proposition 4.1 in [35], a real-valued quasiconvex f is neatly quasiconvex, if and only if f is essentially quasiconvex.

A function f is said to be quasiaffine, iff it is quasiconvex and quasiconcave. It is known that f is lower semicontinuous (lsc) quasiaffine, iff there exist $k \in Q$ and $w \in \mathbb{R}^n$ such that $f = k \circ w$, where

$$Q := \{h : \mathbb{R} \to \overline{\mathbb{R}} : h \text{ is lsc and non-decreasing}\}.$$

Furthermore, f is lsc quasiconvex, iff there exists $\{(k_j, w_j) : j \in J\} \subset Q \times \mathbb{R}^n$ such that $f = \sup_{j\in J} k_j \circ w_j$; see [31, 32] for more details. This result indicates that a lsc quasiconvex function f consists of a supremum of a family of lsc quasiaffine functions. A set $G = \{(k_j, w_j) : j \in J\} \subset Q \times \mathbb{R}^n$ is said to be a generator of f, iff $f = \sup_{j\in J} k_j \circ w_j$. All lsc quasiconvex functions have at least one generator. In particular, when f is a proper lsc and convex function,

$B_f := \{(k_v, v) : v \in \mathrm{dom} f^*, k_v(t) = t - f^*(v), \forall t \in \mathbb{R}\} \subset Q \times \mathbb{R}^n$ is a generator of f. Actually, for all $x \in \mathbb{R}^n$,

$$f(x) = f^{**}(x) = \sup\{\langle v, x\rangle - f^*(v) : v \in \mathrm{dom} f^*\} = \sup_{v \in \mathrm{dom} f^*} k_v(\langle v, x\rangle).$$

We call the generator B_f "the basic generator" of a convex function f. The concept of the basic generator is very important for the comparison of convex and quasiconvex programming; in detail, see [26,29,31,32].

The following function h^{-1} is said to be the hypo-epi-inverse of a non-decreasing function h:

$$h^{-1}(a) := \inf\{b \in \mathbb{R} : a < h(b)\} = \sup\{b \in \mathbb{R} : h(b) \leq a\}.$$

It is known that, if h has the inverse function, then the inverse and the hypo-epi-inverse of h are the same; see [32]. In the present paper, we denote the hypo-epi-inverse of h by h^{-1}.

In [29], we study the following constraint qualifications. These constraint qualifications are necessary and sufficient condition for Lagrange-type duality theorems. Let $\{g_i : i \in I\}$ be a family of lsc quasiconvex functions from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, $\{(k_{(i,j)}, w_{(i,j)}) : j \in J_i\} \subset Q \times \mathbb{R}^n$ a generator of g_i for each $i \in I$, $T = \{t = (i, j) : i \in I, j \in J_i\}$, $A = \{x \in \mathbb{R}^n : \forall i \in I, g_i(x) \leq 0\}$, and C a closed and convex subset of $\mathbb{R}^n$. Assume that $A \cap C$ is non-empty.

Definition 2.1 [29] The inequality system $\{g_i(x) \leq 0 : i \in I\}$ is said to satisfy the closed cone constraint qualification for quasiconvex programming (Q-CCCQ) w.r.t. $G = \{(k_t, w_t) : t \in T\}$ relative to C , iff

$$\text{cone conv} \bigcup_{t \in T} \{(w_t, \delta) \in \mathbb{R}^n \times \mathbb{R} : k_t^{-1}(0) \leq \delta\} + \mathrm{epi}\delta_C^*$$

is closed.

Definition 2.2 [29] The inequality system $\{g_i(x) \leq 0 : i \in I\}$ is said to satisfy the basic constraint qualification for quasiconvex programming (Q-BCQ) with respect to $\{(k_t, w_t) : t \in T\}$ relative to C at $x \in A \cap C$, iff

$$N_{A \cap C}(x) = \text{cone conv} \bigcup_{t \in T(x)} \{w_t\} + N_C(x),$$

where $T(x) = \{t \in T : \langle w_t, x\rangle = k_t^{-1}(0)\}$. $\{g_i(x) \leq 0 : i \in I\}$ is said to satisfy the Q-BCQ w.r.t. $\{(k_t, w_t) : t \in T\}$ relative to C if for all $x \in A \cap C$, $\{g_i(x) \leq 0 : i \in I\}$ satisfies the Q-BCQ w.r.t. $\{(k_t, w_t) : t \in T\}$ at x.

In [29], we show the following two theorems concerned with Q-CCCQ and Q-BCQ.

Theorem 2.1 *The following statements are equivalent:*

(i) $\{g_i(x) \leq 0 : i \in I\}$ *satisfies Q-CCCQ w.r.t.* $\{(k_t, w_t) : t \in T\}$ *relative to* C*,*

(ii) *for each real-valued continuous convex function f on $\mathbb{R}^n$, there exist a finite subset $\bar{T} = \{t_1, \ldots, t_m\} \subset T$ and $\bar{\lambda} \in \mathbb{R}^m_+$ such that $k_{t_j}^{-1}(0) \in \mathbb{R}$ for each $j \in \{1, \ldots, m\}$, and*

$$\inf_{x \in A \cap C} f(x) = \inf_{x \in C} \left\{ f(x) + \sum_{j=1}^{m} \bar{\lambda}_j (w_{t_j}(x) - k_{t_j}^{-1}(0)) \right\}.$$

Theorem 2.2 *Let $x_0 \in A \cap C$. Then, the following conditions are equivalent:*

(i) *$\{g_i(x) \leq 0 : i \in I\}$ satisfies Q-BCQ w.r.t. $\{(k_t, w_t) : t \in T\}$ relative to C at x_0,*
(ii) *for each real-valued continuous convex function f on $\mathbb{R}^n$ that attains its infimum value at x_0, there exist a finite subset $\bar{T} = \{t_1, \ldots, t_m\} \subset T$ and $\bar{\lambda} \in \mathbb{R}^m_+$ such that $k_{t_j}^{-1}(0) \in \mathbb{R}$ for each $j \in \{1, \ldots, m\}$, and*

$$f(x_0) = \min_{x \in A \cap C} f(x) = \inf_{x \in C} \left\{ f(x) + \sum_{j=1}^{m} \bar{\lambda}_j (w_{t_j}(x) - k_{t_j}^{-1}(0)) \right\}.$$

In the research of constraint qualifications for Lagrange strong duality, set containment characterizations are very important. In this paper, we need the following set containment characterization in [29].

Theorem 2.3 *Consider the pair $(v, \alpha) \in \mathbb{R}^n \times \mathbb{R}$. Then, the following statements are equivalent:*

(i) $A \cap C \subset \{x \in \mathbb{R}^n : \langle v, x \rangle \leq \alpha\}$,
(ii) $(v, \alpha) \in \mathrm{cl}\left(\mathrm{cone\ conv} \bigcup_{t \in T} \{(w_t, \delta) \in \mathbb{R}^n \times \mathbb{R} : k_t^{-1}(0) \leq \delta\} + \mathrm{epi}\delta_C^* \right)$.

Theorem 2.3 means that

$$\mathrm{epi}\delta_{A \cap C}^* = \mathrm{cl}\left(\mathrm{cone\ conv} \bigcup_{t \in T} \{(w_t, \delta) \in \mathbb{R}^n \times \mathbb{R} : k_t^{-1}(0) \leq \delta\} + \mathrm{epi}\delta_C^* \right)$$

In quasiconvex analysis, various types of subdifferentials have been introduced; Greenberg-Pierskalla subdifferential [20,22,36], Martínez-Legaz subdifferential [21,31], Q-subdifferential with a generator [17–19], Moreau's generalized conjugation [43], and so on; see [15,16,32,37–45]. In [20,21], we study necessary and sufficient optimality conditions for quasiconvex programming in terms of the following subdifferentials.

In [36], Greenberg and Pierskalla introduced the Greenberg-Pierskalla subdifferential of f at $x_0 \in \mathbb{R}^n$ as follows:

$$\partial^{GP} f(x_0) := \{v \in \mathbb{R}^n : \langle v, x \rangle \geq \langle v, x_0 \rangle \text{ implies } f(x) \geq f(x_0)\}.$$

Martínez-Legaz subdifferential of f at $x \in \mathbb{R}^n$ is defined as follows:

$$\partial^M f(x) := \{(v, t) \in \mathbb{R}^{n+1} : \inf\{f(y) : \langle v, y \rangle \geq t\} \geq f(x), \langle v, x \rangle \geq t\}.$$

Martínez-Legaz subdifferential is introduced by Martínez-Legaz in [31] as a special case of c-subdifferential in Moreau's generalized conjugation in [43].

The following theorems are concerned with necessary and sufficient optimality conditions for quasiconvex programming. In [20], we show the following necessary and sufficient optimality condition in terms of Greenberg-Pierskalla subdifferential.

Theorem 2.4 *Let f be a upper semicontinuous (usc) essentially quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, F a convex subset of $\mathbb{R}^n$, and $x \in F$.*

Then, the following statements are equivalent:

(i) $f(x) = \min_{y \in F} f(y)$,

(ii) $0 \in \partial^{GP} f(x) + N_F(x)$.

In [21], we show the following necessary and sufficient optimality condition in terms of Martínez-Legaz subdifferential.

Theorem 2.5 *Let f be an usc quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, F a convex subset of $\mathbb{R}^n$, and $x \in F$.*

Then, the following statements are equivalent:

(i) $f(x) = \min_{y \in F} f(y)$,

(ii) $0 \in \partial^{M} f(x) + \text{epi}\delta_F^*$.

3 Optimality Conditions and Related Constraint Qualifications

Throughout this paper, let I be an index set, $\{g_i : i \in I\}$ a family of lsc quasiconvex functions from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, $\{(k_{(i,j)}, w_{(i,j)}) : j \in J_i\} \subset Q \times \mathbb{R}^n$ a generator of g_i, $T = \{t = (i,j) : i \in I, j \in J_i\}$, $G = \{(k_t, w_t) : t \in T\}$, f an usc quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, and $A = \{x \in \mathbb{R}^n : \forall i \in I, g_i(x) \le 0\}$. Assume that A is non-empty.

In this section, we study the following quasiconvex programming problem:

$$\text{minimize } f(x), \text{ subject to } x \in A.$$

We show two types of necessary and sufficient optimality conditions for the problem and related constraint qualifications.

At first, we assume that f is usc essentially quasiconvex. We introduce the following optimality condition:

$$0 \in \partial^{GP} f(x_0) + \text{cone conv} \bigcup_{t \in T(x_0)} \{w_t\}.$$

In the following theorem, we show a necessary and sufficient optimality condition for essentially quasiconvex programming and related necessary and sufficient constraint qualification, Q-BCQ.

Theorem 3.1 *Let $x_0 \in A$. The following statements are equivalent:*

(i) $\{g_i(x) \le 0 : i \in I\}$ *satisfies Q-BCQ w.r.t.* G *relative to* $\mathbb{R}^n$ *at* x_0*,*

(ii) for each extended real-valued usc essentially quasiconvex function f *on* $\mathbb{R}^n$*,* x_0 *is a global minimizer of* f *in* A*, if and only if*

$$0 \in \partial^{GP} f(x_0) + \text{cone conv} \bigcup_{t \in T(x_0)} \{w_t\}.$$

Proof Assume that $\{g_i(x) \le 0 : i \in I\}$ satisfies Q-BCQ w.r.t. G relative to $\mathbb{R}^n$ at x_0, and let f be an extended real-valued usc essentially quasiconvex function on $\mathbb{R}^n$. By Theorem 2.4, x_0 is a global minimizer of f in A, if and only if

$$0 \in \partial^{GP} f(x_0) + N_A(x_0).$$

By Q-BCQ at x_0,

$$\begin{aligned} &\partial^{GP} f(x_0) + N_A(x_0) \\ &= \partial^{GP} f(x_0) + \text{cone conv} \bigcup_{t \in T(x_0)} \{w_t\} + N_{\mathbb{R}^n}(x_0) \\ &= \partial^{GP} f(x_0) + \text{cone conv} \bigcup_{t \in T(x_0)} \{w_t\}. \end{aligned}$$

This shows that (ii) holds.

Next, we show that (ii) implies (i). Assume that for each extended real-valued usc essentially quasiconvex function f on $\mathbb{R}^n$, x_0 is a global minimizer of f in A, if and only if

$$0 \in \partial^{GP} f(x_0) + \text{cone conv} \bigcup_{t \in T(x_0)} \{w_t\}.$$

We need to show that

$$N_A(x_0) = \text{cone conv} \bigcup_{t \in T(x_0)} \{w_t\} + N_{\mathbb{R}^n}(x_0).$$

We can check easily that one inclusion always holds. Let $\bar{t} = (\bar{i}, \bar{j}) \in T(x_0)$ and $y \in A$,

$$k_{\bar{t}}(\langle w_{\bar{t}}, y\rangle) \le \sup_{j \in J_{\bar{i}}} k_{(\bar{i},j)} \circ w_{(\bar{i},j)}(y) = g_{\bar{i}}(y) \le 0.$$

Hence, $\langle w_{\bar{t}}, y\rangle \le \langle w_{\bar{t}}, x_0\rangle$ because $\langle w_{\bar{t}}, x_0\rangle = k_{\bar{t}}^{-1}(0) = \sup\{b \in \mathbb{R} : k_{\bar{t}}(b) \le 0\}$. This shows that $w_{\bar{t}} \in N_A(x_0)$. Since $N_{\mathbb{R}^n}(x_0) = \{0\}$ and $N_A(x_0)$ is a convex cone,

$$N_A(x_0) \supset \text{cone conv} \bigcup_{t \in T(x_0)} \{w_t\} + N_{\mathbb{R}^n}(x_0).$$

Let $v \in N_A(x_0)$. If $v = 0$, then it is clear that

$$v \in \text{cone conv} \bigcup_{t \in T(x_0)} \{w_t\} + N_{\mathbb{R}^n}(x_0).$$

Hence, we assume that $v \neq 0$. Since $\langle v, y - x_0 \rangle \leq 0$ for each $y \in A$, x_0 is a global minimizer of $-v$ in A. Since $-v$ is a continuous essentially quasiconvex function on $\mathbb{R}^n$, by the statement (ii),

$$0 \in \partial^{GP}(-v)(x_0) + \text{cone conv} \bigcup_{t \in T(x_0)} \{w_t\}.$$

By the definition of Greenberg-Pierskalla subdifferential,

$$\begin{aligned}\partial^{GP}(-v)(x_0) &= \{w \in \mathbb{R}^n : \langle w, x \rangle \geq \langle w, x_0 \rangle \text{ implies } \langle -v, x \rangle \geq \langle -v, x_0 \rangle\} \\ &= \{-\lambda v : \lambda > 0\}.\end{aligned}$$

Hence there exists, $\lambda_0 > 0$ such that

$$0 \in -\lambda_0 v + \text{cone conv} \bigcup_{t \in T(x_0)} \{w_t\}.$$

Since cone conv $\bigcup_{t \in T(x_0)} \{w_t\}$ is a cone,

$$\begin{aligned}v &= \frac{1}{\lambda_0} \lambda_0 v \\ &\in \text{cone conv} \bigcup_{t \in T(x_0)} \{w_t\} \\ &= \text{cone conv} \bigcup_{t \in T(x_0)} \{w_t\} + N_{\mathbb{R}^n}(x_0).\end{aligned}$$

This shows that (i) holds, and completes the proof. □

Next, we assume that f is usc quasiconvex, not necessary essentially quasiconvex. We introduce the following optimality condition:

$$0 \in \partial^M f(x_0) + \text{cone conv} \bigcup_{t \in T} \{(w_t, \delta) \in \mathbb{R}^n \times \mathbb{R} : k_t^{-1}(0) \leq \delta\} + \{0\} \times [0, \infty[.$$

In the following theorem, we show a necessary and sufficient optimality condition for quasiconvex programming and related constraint qualification, Q-CCCQ.

Theorem 3.2 *The following statement (i) implies the statement (ii):*

(i) $\{g_i(x) \leq 0 : i \in I\}$ *satisfies Q-CCCQ w.r.t.* G *relative to* $\mathbb{R}^n$*,*
(ii) for each extended real-valued usc quasiconvex function f *on* $\mathbb{R}^n$ *and* $x_0 \in A$*,* x_0 *is a global minimizer of* f *in* A*, if and only if*

$$0 \in \partial^M f(x_0) + \text{cone conv} \bigcup_{t \in T} \{(w_t, \delta) \in \mathbb{R}^n \times \mathbb{R} : k_t^{-1}(0) \leq \delta\} + \{0\} \times [0, \infty[.$$

Furthermore, if A *is compact, then (i) and (ii) are equivalent.*

Proof Assume that $\{g_i(x) \le 0 : i \in I\}$ satisfies Q-CCCQ w.r.t. G relative to $\mathbb{R}^n$. Let f be a real-valued usc quasiconvex function on $\mathbb{R}^n$ and $x_0 \in A$. By Theorem 2.5, x_0 is a global minimizer of f in A, if and only if

$$0 \in \partial^M f(x_0) + \text{epi}\delta_A^*.$$

By Q-CCCQ, Theorem 2.3, and $\text{epi}\delta_{\mathbb{R}^n}^* = \{0\} \times [0,\infty[$, the above condition is equivalent to

$$0 \in \partial^M f(x_0) + \text{cone conv} \bigcup_{t \in T} \{(w_t,\delta) \in \mathbb{R}^n \times \mathbb{R} : k_t^{-1}(0) \le \delta\} + \{0\} \times [0,\infty[.$$

This shows that (ii) holds.

Next, we show that (ii) implies (i) assuming that A is compact. We need to show that

$$\text{cone conv} \bigcup_{t \in T} \{(w_t,\delta) \in \mathbb{R}^n \times \mathbb{R} : k_t^{-1}(0) \le \delta\} + \text{epi}\delta_{\mathbb{R}^n}^*$$

is w^*-closed. By the set containment characterization in Theorem 2.3, we can see that

$$\text{epi}\delta_A^* = \text{cl}\left(\text{cone conv} \bigcup_{t \in T} \{(w_t,\delta) \in \mathbb{R}^n \times \mathbb{R} : k_t^{-1}(0) \le \delta\} + \text{epi}\delta_{\mathbb{R}^n}^*\right).$$

This shows that Q-CCCQ is satisfied, if and only if

$$\text{epi}\delta_A^* = \text{cone conv} \bigcup_{t \in T} \{(w_t,\delta) \in \mathbb{R}^n \times \mathbb{R} : k_t^{-1}(0) \le \delta\} + \text{epi}\delta_{\mathbb{R}^n}^*.$$

Hence, we show the above equality. We can check that one inclusion always holds. Indeed, let $t \in T$ and $\delta \ge k_t^{-1}(0)$, then $k_t \circ w_t(x) \le 0$ for each $x \in A$. Hence,

$$\delta_A^*(w_t) = \sup_{x \in A} \langle w_t, x\rangle \le k_t^{-1}(0) \le \delta.$$

This shows that $\{(w_t,\delta) \in \mathbb{R}^n \times \mathbb{R} : k_t^{-1}(0) \le \delta\} \subset \text{epi}\delta_A^*$. Since $\text{epi}\delta_A^*$ is a convex cone,

$$\text{epi}\delta_A^* \supset \text{cone conv} \bigcup_{t \in T} \{(w_t,\delta) \in \mathbb{R}^n \times \mathbb{R} : k_t^{-1}(0) \le \delta\} + \text{epi}\delta_{\mathbb{R}^n}^*.$$

Consider the pair $(v,\alpha) \in \text{epi}\delta_A^*$. If $v = 0$, then, clearly, $\alpha \ge 0$ and

$$(v,\alpha) \in \text{cone conv} \bigcup_{t \in T} \{(w_t,\delta) \in \mathbb{R}^n \times \mathbb{R} : k_t^{-1}(0) \le \delta\} + \text{epi}\delta_{\mathbb{R}^n}^*.$$

Hence, we assume that $v \ne 0$. Since A is compact and $-v$ is continuous, there exists $x_0 \in A$ such that x_0 is a global minimizer of $-v$ in A. By the statement (ii),

$$0 \in \partial^M(-v)(x_0) + \text{cone conv} \bigcup_{t \in T} \{(w_t,\delta) \in \mathbb{R}^n \times \mathbb{R} : k_t^{-1}(0) \le \delta\} + \{0\} \times [0,\infty[.$$

By the definition of Martínez-Legaz subdifferential,

$$\begin{aligned}\partial^M(-v)(x_0) &= \{(w,t) \in \mathbb{R}^{n+1} : \inf\{\langle -v, y\rangle : \langle w, y\rangle \geq t\} \geq \langle -v, x_0\rangle, \langle w, x_0\rangle \geq t\} \\ &= \{(-\lambda v, -\lambda \langle v, x_0\rangle) \in \mathbb{R}^{n+1} : \lambda > 0\}.\end{aligned}$$

Hence, there exists $\lambda_0 > 0$ such that

$$0 \in -(\lambda_0 v, \langle \lambda_0 v, x_0\rangle) + \text{cone conv} \bigcup_{t\in T} \{(w_t, \delta) \in \mathbb{R}^n \times \mathbb{R} : k_t^{-1}(0) \leq \delta\} + \{0\} \times [0, \infty[.$$

Since cone conv $\bigcup_{t\in T}\{(w_t, \delta) \in \mathbb{R}^n \times \mathbb{R} : k_t^{-1}(0) \leq \delta\} + \{0\} \times [0, \infty[$ is a cone,

$$\begin{aligned}(v, \alpha) &= (v, \langle v, x_0\rangle) + (0, \alpha - \langle v, x_0\rangle) \\ &= \frac{1}{\lambda_0}(\lambda_0 v, \langle \lambda v, x_0\rangle) + (0, \alpha - \langle v, x_0\rangle) \\ &\in \text{cone conv} \bigcup_{t\in T} \{(w_t, \delta) \in \mathbb{R}^n \times \mathbb{R} : k_t^{-1}(0) \leq \delta\} + \{0\} \times [0, \infty[\\ &= \text{cone conv} \bigcup_{t\in T} \{(w_t, \delta) \in \mathbb{R}^n \times \mathbb{R} : k_t^{-1}(0) \leq \delta\} + \text{epi}\delta^*_{\mathbb{R}^n}.\end{aligned}$$

This shows that (i) holds, and completes the proof. □

4 Discussion

In this section, we discuss about our optimality conditions and constraint qualifications. Especially, we show some equivalence relations between duality results via convex and quasiconvex programming.

In the second half of Theorem 3.2, we assume that A is compact. We need the assumption to guarantee that a minimizer of v in A exists. Hence, we can show the theorem under different assumptions; for example, A is an intersection of finitely many closed halfspaces.

In Theorem 2.1 and Theorem 2.2, we show that Q-CCCQ and Q-BCQ are necessary and sufficient constraint qualifications for Lagrange-type duality theorems. By Theorem 3.1 and Theorem 3.2, we show the following corollaries for equivalence relation between Lagrange-type duality and optimality conditions.

Corollary 4.1 *Let $x_0 \in A$. The following statements are equivalent:*

(i) $\{g_i(x) \leq 0 : i \in I\}$ satisfies Q-BCQ w.r.t. G relative to $\mathbb{R}^n$ at x_0,
(ii) for each extended real-valued usc essentially quasiconvex function f on $\mathbb{R}^n$, x_0 is a global minimizer of f in A, if and only if

$$0 \in \partial^{GP} f(x_0) + \text{cone conv} \bigcup_{t\in T(x_0)} \{w_t\},$$

(iii) *for each real-valued continuous convex function f that attains its infimum value at x_0, there exist a finite subset $\bar{T} = \{t_1, \ldots, t_m\} \subset T$ and $\bar{\lambda} \in \mathbb{R}^m_+$ such that $k_{t_j}^{-1}(0) \in \mathbb{R}$ for each $j \in \{1, \ldots, m\}$, and*

$$f(x_0) = \min_{x \in A} f(x) = \inf_{x \in \mathbb{R}^n} \left\{ f(x) + \sum_{j=1}^{m} \bar{\lambda}_j (w_{t_j}(x) - k_{t_j}^{-1}(0)) \right\}.$$

Proof By Theorem 2.2 and 3.1, we can prove the corollary. □

Corollary 4.2 *Assume that A is compact. Then, the following statements are equivalent:*

(i) *$\{g_i(x) \leq 0 : i \in I\}$ satisfies Q-CCCQ w.r.t. G relative to $\mathbb{R}^n$,*
(ii) *for each extended real-valued usc quasiconvex function f on $\mathbb{R}^n$ and $x_0 \in A$, x_0 is a global minimizer of f in A, if and only if*

$$0 \in \partial^M f(x_0) + \text{cone conv} \bigcup_{t \in T} \{(w_t, \delta) \in \mathbb{R}^n \times \mathbb{R} : k_t^{-1}(0) \leq \delta\} + \{0\} \times [0, \infty[,$$

(iii) *for each real-valued continuous convex function f on $\mathbb{R}^n$, there exist a finite subset $\bar{T} = \{t_1, \ldots, t_m\} \subset T$ and $\bar{\lambda} \in \mathbb{R}^m_+$ such that $k_{t_j}^{-1}(0) \in \mathbb{R}$ for each $j \in \{1, \ldots, m\}$, and*

$$\inf_{x \in A} f(x) = \inf_{x \in \mathbb{R}^n} \left\{ f(x) + \sum_{j=1}^{m} \bar{\lambda}_j (w_{t_j}(x) - k_{t_j}^{-1}(0)) \right\}.$$

Proof By Theorem 2.1 and 3.2, we can prove the corollary. □

In convex programming, the following necessary and sufficient constraint qualifications for Lagrange duality have been investigated; see [1,3,23–25].

Let I be an index set, g_i proper lsc and convex functions from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, C a closed and convex subset of $\mathbb{R}^n$, $A = \{x \in \mathbb{R}^n : \forall i \in I, g_i(x) \leq 0\}$ and assume that $A \cap C$ is nonempty.

(i) $\{g_i(x) \leq 0 : i \in I, x \in C\}$ is said to satisfy Farkas-Minkowski (FM), iff

$$\text{epi}\delta^*_{A \cap C} = \text{cone conv} \bigcup_{i \in I} \text{epi} g_i^* + \text{epi}\delta^*_C,$$

(ii) $\{g_i(x) \leq 0 : i \in I\}$ is said to satisfy the basic constraint qualification (BCQ) relative to C at $x \in A$, iff

$$N_{A \cap C}(x) = N_C(x) + \text{cone conv} \bigcup_{i \in I(x)} \partial g_i(x).$$

Clearly, FM (BCQ relative to C) is equivalent to Q-CCCQ (Q-BCQ, respectively) w.r.t. basic generator relative to C.

By Theorem 3.1 and Theorem 3.2, we show the following corollaries for optimality conditions and constraint qualifications via quasiconvex minimization problem with convex inequality constraints. These results show that optimality conditions for quasiconvex programming are equivalent to Lagrange duality theorems for convex programming.

Corollary 4.3 *Let $x_0 \in A$, and assume that g_i are real-valued convex. Then, the following statements are equivalent:*

(i) $\{g_i(x) \leq 0 : i \in I\}$ *satisfies BCQ relative to* $\mathbb{R}^n$ *at* x_0,
(ii) *for each extended real-valued usc essentially quasiconvex function* f *on* $\mathbb{R}^n$, x_0 *is a global minimizer of* f *in* A, *if and only if*

$$0 \in \partial^{GP} f(x_0) + \text{cone conv} \bigcup_{i \in I(x_0)} \partial g_i(x_0).$$

Proof Since BCQ relative to $\mathbb{R}^n$ is equivalent to Q-BCQ w.r.t. the basic generator relative to $\mathbb{R}^n$, we can prove the corollary by Theorem 3.1. □

Corollary 4.4 *Assume that A is compact and g_i are real-valued convex. Then, the following statements are equivalent:*

(i) $\{g_i(x) \leq 0 : i \in I, x \in C\}$ *satisfies FM,*
(ii) *for each extended real-valued usc quasiconvex function* f *on* $\mathbb{R}^n$ *and* $x_0 \in A$, x_0 *is a global minimizer of* f *in* A, *if and only if*

$$0 \in \partial^M f(x_0) + \text{cone conv} \bigcup_{i \in I} \text{epi} g_i^* + \{0\} \times [0, \infty[.$$

Proof Since FM is equivalent to Q-CCCQ w.r.t. the basic generator, we can prove the corollary by Theorem 3.2. □

5 Conclusions

In this paper, we study optimality conditions and constraint qualifications for quasiconvex programming. In Theorem 3.1, we show a necessary and sufficient optimality condition for essentially quasiconvex programming and related necessary and sufficient constraint qualification, Q-BCQ. In Theorem 3.2, we show a necessary and sufficient optimality condition for quasiconvex programming and related constraint qualification, Q-CCCQ. Additionally, we discuss about our optimality conditions and constraint qualifications. Especially, we show some equivalence relations between duality results for convex and quasiconvex programming in Corollary 4.3 and Corollary 4.4.

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