On the Vibration of Clamped Rhombic Isotropic Plates I

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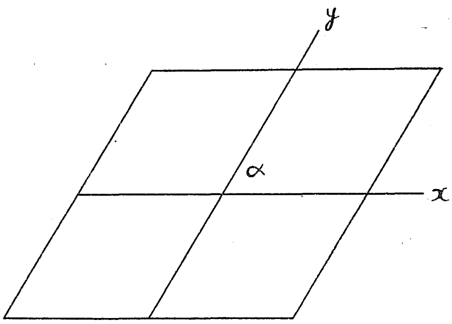
1. Introduction

The vibration of rectangular plates was discussed by W. Ritz[1], S. Iguchi[2], D. Young[3] and M. V. Barton[4]. The vibration of skew cantilever plates was discussed by M. V. Barton[4]. In this paper, we shall calculate the smallest frequency of the vibration of clamped rhombic isotropic plates by the Ritz method.

2. Let the oblique coordinates axes (x,y) be taken in the middle plane of the plate of uniform thin thickness in such a way that the origin coincides with the center of the plate and that the axes are parallel to the sides. Let us denote the length of sides, the thickness, the density, Young's modulus, and Poisson's ratio of the plate by $2a$, h, ρ, E and ν respectively. Let the angle between coordinates axes be denoted by α. If w be the transverse displacement of a point on the middle plane and t the time, the differential equation for w is



$$\frac{D}{\sin^4\alpha}\left[\frac{\partial^4 w}{\partial x^4}-4\cos\alpha\frac{\partial^4 w}{\partial x^3\partial y}+2(1+2\cos^2\alpha)\frac{\partial^4 w}{\partial x^2\partial y}-4\cos\alpha\frac{\partial^4 w}{\partial x\partial y^3}+\frac{\partial^4 w}{\partial y^4}\right]+\rho h\frac{\partial^2 w}{\partial t^2}=0 \quad (1)$$

where $D=\dfrac{Eh^3}{12(1-\nu^2)}$.

The boundary conditions are

$$w=0 \quad \text{and} \quad \frac{\partial w}{\partial n}=0 \qquad \text{at } x=\pm a,\ y=\pm a. \quad (2)$$

where $\dfrac{\partial}{\partial n}$ denotes differentiation along the normal.

As the problem is so complicated that we shall use the Ritz method. According this method, the mode of the vibration is assumed. If we assume the following mode

$$w=W(x,\ y)\cos pt, \quad (3)$$

equation (1) is transformed into

$$\frac{\partial^4 W}{\partial x^4}-4m\frac{\partial^4 W}{\partial x^3\partial y}+2(1+2m^2)\frac{\partial^4 W}{\partial x^2\partial y^2}-4m\frac{\partial^4 W}{\partial x\partial y^3}+\frac{\partial^4 W}{\partial y^4}-kW=0 \quad (4)$$

with the boundary conditions

$$W=0 \text{ and } \frac{\partial W}{\partial n}=0 \quad at\ x=\pm a,\ y=\pm a. \tag{5}$$

in which $m=\cos\alpha$,

and $k=\dfrac{\rho h p^2(1-m^2)^2}{D}$.

In order to apply the Ritz method, we consider the following integral :

$$U=\frac{D}{2\sin^3\alpha}\int_{-a}^{a}\int_{-a}^{a}\left\{(\triangle W)^2-4m\frac{\partial^2 W}{\partial x\partial y}\triangle W+4m^2\left(\frac{\partial^2 W}{\partial x\,\partial y}\right)^2-kW^2\right\}dxdy \tag{7}$$

We assume for the expression of W which satisfies the boundary conditions (5), that

$$W=(a^2-x^2)^2(a^2-y^2)^2(a_{00}+a_{11}xy+a_{20}x^2+a_{02}y^2+a_{31}x^3y+a_{13}xy^3+a_{22}x^2y^2), \tag{8}$$

in which a_{00}, a_{11}, ········ and a_{22} are arbitrary constants.

Eq. (8) is introduced into U and then we have

$$\left\{\frac{1}{1575}+\frac{2(1+2m^2)}{11025}\right\}a^{14}a_{00}^2+\left(\frac{1}{11025}+\frac{1}{17325}\right)a^{16}(a_{20}+a_{02})a_{00}+\frac{2}{121275}a^{18}a_{00}a_{22}$$

$$+\left\{\frac{1}{7350}+\frac{1}{150150}+\frac{2(1+2m^2)}{121275}\right\}a^{18}(a_{20}^2+a_{02}^2)+\frac{2}{121275}a^{18}a_{20}a_{02}$$

$$+\left(\frac{1}{40425}+\frac{1}{525525}\right)a^{20}(a_{20}+a_{02})a_{22}+\left\{\frac{1}{175175}+\frac{2(1+2m^2)}{1334025}\right\}a^{22}a_{22}^2$$

$$+\left\{\frac{1}{24255}+\frac{2(1+2m^2)}{99225}\right\}a^{18}a_{11}^2+\left\{\frac{1}{72765}+\frac{1}{105105}+\frac{8(1+2m^2)}{1091475}\right\}a^{20}(a_{31}+a_{13})a_{11}$$

$$+\left\{\frac{13}{1600830}+\frac{1}{630630}+\frac{34(1+2m^2)}{14189175}\right\}a^{22}(a_{31}^2+a_{13}^2)+\left\{\frac{2}{315315}+\frac{16(1+2m^2)}{12006225}\right\}a^{22}a_{31}a_{13}$$

$$+\frac{4m}{11025}a^{16}a_{00}a_{11}-2m\left(\frac{1}{33075}+\frac{1}{121275}\right)a^{18}(a_{20}+_{02})a_{11}$$

$$+2m\left(\frac{1}{33075}+\frac{1}{40425}\right)a^{18}(a_{31}+a_{13})a_{00}+\frac{4m}{363825}a^{20}a_{11}a_{22}$$

$$-2m\left(\frac{1}{363825}+\frac{1}{121275}\right)a^{20}(a_{31}a_{02}+a_{13}a_{20})+2m\left(\frac{1}{121275}+\frac{1}{525525}\right)a^{20}(a_{31}a_{20}+a_{13}a_{02})$$

$$-2m\left(\frac{1}{1334025}+\frac{1}{1576575}\right)a^{22}(a_{31}+a_{13})a_{22}$$

$$-K\left\{\frac{1}{99225}a^{18}a_{00}^2+\frac{2}{1091475}a^{20}(a_{20}+a_{02})a_{00}+\frac{2}{12006225}a^{22}(a_{00}a_{22}+a_{20}a_{02})\right.$$

$$+\frac{1}{4729725}a^{22}(a_{20}^2+a_{02}^2)+\frac{2}{52026975}a^{24}(a_{20}+a_{02})a_{22}+\frac{1}{225450225}a^{26}a_{22}^2$$

$$+\frac{1}{12006225}a^{22}a_{11}^2+\frac{2}{52026975}a^{24}(a_{31}+a_{13})a_{11}+\frac{1}{156080925}a^{26}(a_{31}^2+a_{13}^2)$$

$$\left.+\frac{2}{225450225}a^{26}a_{31}a_{13}\right\}=0 \tag{9}$$

where

$$K = ka^4 = \frac{\rho h a^4 p^2 \sin^4 \alpha}{D} .$$

Since the constants are to be determined so as to make U a minimun by the Ritz method, we must have

$$\left.\begin{aligned} &\frac{\partial U}{\partial a_{00}} = 0 \ ; \quad \frac{\partial U}{\partial a_{11}} = 0 \ ; \quad \frac{\partial U}{\partial a_{20}} = 0 \quad \frac{\partial U}{\partial a_{02}} = 0 \\ &\frac{\partial U}{\partial a_{31}} = 0 \ ; \quad \frac{\partial U}{\partial a_{13}} = 0 \ ; \quad \frac{\partial U}{\partial a_{22}} = 0 . \end{aligned}\right\} \quad (10)$$

These equations are linear in the A' s and are found to be :

$$\left.\begin{aligned} &B_{100} A_{00} + B_{111} A_{11} + B_{120} A_{20} + B_{102} A_{02} + B_{131} A_{31} + B_{113} A_{13} + B_{122} A_{22} = 0 \\ &B_{200} A_{00} + B_{211} A_{11} + B_{220} A_{20} + B_{202} A_{02} + B_{231} A_{31} + B_{213} A_{13} + B_{222} A_{22} = 0 \\ &B_{300} A_{00} + \cdots\cdots\cdots + B_{322} A_{22} = 0 \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &B_{700} A_{00} + B_{711} A_{11} + B_{720} A_{20} + B_{702} A_{02} + B_{731} A_{31} + B_{713} A_{13} + B_{722} A_{22} = 0 \end{aligned}\right\} \quad (11)$$

where

$$\left.\begin{aligned} &A_{00} = a^8 a_{00} , \quad A_{11} = a^{10} a_{11} , \quad A_{20} = a^{10} a_{20} , \quad A_{02} = a^{10} a_{02} , \\ &A_{31} = a^{12} a_{31} , \quad A_{13} = a^{12} a_{13} , \quad A_{22} = a^{12} a_{22} , \\ &B_{100} = \frac{2}{1575} + \frac{4(1+2m^2)}{11025} - \frac{2K}{99225} , \quad B_{111} = B_{200} = \frac{m}{11025} , \\ &B_{120} = B_{102} = B_{300} = B_{400} = \frac{1}{11025} + \frac{1}{17325} - \frac{2}{1091475} K , \\ &B_{131} = B_{113} = B_{500} = B_{600} = 2m \left(\frac{1}{33075} + \frac{1}{40425} \right) , \\ &B_{122} = B_{302} = B_{420} = B_{700} = \frac{2}{121275} - \frac{2}{12006225} K , \\ &B_{211} = \frac{2}{24255} + \frac{4(1+2m^2)}{99225} - \frac{2}{12006225} K , \\ &B_{220} = B_{202} = B_{311} = B_{411} = -2m \left(\frac{1}{33075} + \frac{1}{121275} \right) , \\ &B_{222} = B_{711} = \frac{4m}{363825} \\ &B_{231} = B_{213} = B_{511} = B_{611} = \frac{1}{72765} + \frac{1}{105105} + \frac{8(1+2m^2)}{91475} - \frac{2}{52026975} K , \\ &B_{320} = B_{402} = \frac{1}{3675} + \frac{1}{75075} + \frac{4(1+2m^2)}{121275} - \frac{2}{4729725} K , \\ &B_{331} = B_{413} = B_{520} = B_{602} = 2m \left(\frac{1}{121275} + \frac{1}{525525} \right) , \\ &B_{313} = B_{431} = B_{502} = B_{620} = -2m \left(\frac{1}{121275} + \frac{1}{363825} \right) , \end{aligned}\right\} \quad (12)$$

$$\left.\begin{aligned}
B_{322} &= B_{422} = B_{720} = B_{702} = \frac{1}{40425} + \frac{1}{525525} - \frac{2}{52026975}K, \\
B_{531} &= \frac{13}{800415} + \frac{1}{315315} + \frac{68\,(1+2m^2)}{14189175} - \frac{2}{156080925}K, \\
B_{513} &= B_{631} = \frac{1}{315315} + \frac{16\,(1+2m^2)}{12006225} - \frac{2}{225450225}K, \\
B_{522} &= B_{622} = B_{731} = B_{713} = -2m\left(\frac{1}{1334025} + \frac{1}{1576575}\right), \\
B_{613} &= \frac{2}{315315} + \frac{13}{800415} + \frac{68\,(1+2m^2)}{14189175} - \frac{2}{156080925}K,
\end{aligned}\right.$$

and

$$B_{722} = \frac{2}{175175} + \frac{4\,(1+2m^2)}{1334025} - \frac{2}{225450225}K.$$

If we eliminate A_{00}, A_{11}, $\cdot\cdot\cdot\cdot$ from eq. (1), we have the following determinantal equation :

$$\triangle = \begin{vmatrix}
B_{100} & B_{111} & B_{120} & B_{102} & B_{131} & B_{113} & B_{122} \\
B_{200} & B_{211} & B_{220} & B_{202} & B_{231} & B_{213} & B_{222} \\
B_{300} & \cdots & \cdots & \cdots & \cdots & \cdots & B_{322} \\
B_{400} & \cdots & \cdots & \cdots & \cdots & \cdots & B_{422} \\
B_{500} & \cdots & \cdots & \cdots & \cdots & \cdots & B_{522} \\
B_{600} & \cdots & \cdots & \cdots & \cdots & \cdots & B_{622} \\
B_{700} & B_{711} & B_{720} & B_{702} & B_{731} & B_{713} & B_{722}
\end{vmatrix} = 0. \qquad (13)$$

This equation determines the values of K i. e. , the frequencies of the vibration.

3. The smallest values of K satisfying eq. (13) are calculated in the case $\alpha = 55°$, $60°$ and $90°$ and are shown in Table I.

Table I

α	55°	60°	90°
K	87.255	85.034	80.952

From above Table, we see that the value of K for $\alpha = 90°$ is 80.952. This value of K is in good agreement with $K = 80.9348$ obtained by Prof. S. Tomotika[5]. As we have obtained the approximate values of K, we shall calculate the accurate values of K in the following paper.

References

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