ON INVERSIBLE SEMIGROUPS

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Let S be a semigroup, and let I be the totality of all idempotents of S. Then S is said to be inversible if S satisfies the following two conditions ; (1) to each $a \in S$ there exists $a^* \in S$ such that $aa^* = a^*a \in I$; (2) I is a subsemigroup of S. For instance, idempotent semigroups (accordingly completely non-commutative semigorups) [3] [4], left (right) regular and right (left) simple semigroups [2] and commutative inverse semigroups [5] are clearly inversible semigroups.

T. Tamura showed that if I is consisting of only one idempotent (he defined such a semigroup to be an 'unipotent semigroup') S has the minimal two sided ideal K (Suschkewitsch kernel [7])which is the same as the maximal subgroup of S. Moreover, under the same restriction he points out that the Rees factor semigroup $Z=S/K$ [6] is a zero-semigroup and that the structure of S is completely determined by K, Z and a ramified homomorphism f of Z into K [8] .

The main purpose of this paper is to show, among other things, that the above-mentioned Tamura's results are extended to an inversible semigroup whose idempotents are primitive.

Throughout the whole paper the operation $\dot{+}$ $(\bullet\Sigma)$ will denote the class sum, i. e. , disjoint sum of sets.

§ 1. i-components and structure of inversivle semigroups.

Let G be a semigroup containing at least one idempotent. For any idempotent e of G, by the ' i-component of G at e' we shall mean the greatest subset $C(e)$ of G such that to each element x of $C(e)$ there exists $x^* \in G$ which satisfies the relation $xx^* = x^*x = e$.

In this paragraph S will always denote an inversible semigroup, and I will denote the totality of all idempotents of S. $N(e)$ will always denote the i-component of S at e.

Lemma 1. 1. *Every $N(e)$ is an inversible subsemigroup of S.*

Proof. Let a, b be any two elements of $N(e)$. Then there exist a^*, $b^* \in S$, such that $aa^* = a^*a = e$ and $bb^* = b^*b = e$ respectively. If we put $c = b^*a^*$, the following relations hold ; $(ab)c = abb^*a^* = aea^* = aa^*aa^* = e$, $c(ab) = b^*a^*ab = b^*eb = b^*bb^*b = e$. Hence $ab \in N(e)$. This implies $N(e)$ to be a subsemigroup of S. Since $N(e)$ is clearly inversible, we obtain this lemma.

Lemma 1. 2. *If e is an idempotent of S, then e commutes with any element of $N(e)$. Accordingly the relation $eN(e) = N(e)e$ holds.*

Proof. Take up any element $x \in N(e)$. There exists $x^* \in S$ such that $xx^* = x^*x = e$. Hence $ex = (xx^*)x = x(x^*x) = xe$.

Lemma 1.3. *If e is an idempotent of S, then $eN(e)$ is the greatest subgroup of $N(e)$ as well as an ideal of $N(e)$. Accordingly $N(e)$ is a subhomogroup* [*] *of S.*

Proof. It is obvious that e is the unit of $eN(e)$. Let x be any element of $N(e)$. Then there exists $x^* \in N(e)$ such that $xx^*=x^*x=e$. ex^* is clearly an element of $eN(e)$. Therefore the element ex^* becomes the inverse element of ex in $eN(e)$, since the relations $exex^*=e$, $ex^*ex=e$ hold. Hence $eN(e)$ is a subgroup of $N(e)$, It is easy to see that $eN(e)$ becomes the greatest subgroup and an ideal of $N(e)$, and so we save the trouble to prove it.

Remark. [*] . A homogroup is a semigroup having an ideal which is also a subgroup. It is obvious that for any homogroup such an ideal is uniquely determined. By a' group ideal' of a homogroup we shall mean an ideal which is also a subgroup. Hence $eN(e)$ is the group ideal of $N(e)$.

Let G be a semigroup containing at least one idempotent, and let I_G be the totality of all idempotents of G. An idempotent e of G is said to be primitive if it satisfies the relation $eI_Ge=\{e\}$ [3] .

Theorem 1.1. *Any homogroup has at most one primitive idempotent. Moreover if a homogroup* H *has a primitive idempotent e, then e is the unit of the group ideal of* H.

Proof. Let K, e be the group ideal of H and the unit of K respectively. If e' is any primitive idempotent of H , then the relation $e'ee'=e'$ follows from the definition of the primitivity. Accordingly $e=e'$ holds, since $e'ee'$ is contained in K and is an idempotent of H.

Lemma 1.4. *For any $N(e)$, e is a primitive idempotent of $N(e)$ itself.*

Proof. Let e' be any idempotent of $N(e)$. Then $ee'=e$, since ee' is an idempotent and is contained in the subgroup eN(e) of N(e). Hence we have $ee'e=e$.

Lemma 1.3 and Lemma 1.4 imply ;

Theorem 1.2. *Every* N (e) *is an inversible subhomogroup, which has one and only one primitive idempotent of* N(e) *itself.*

Lemma 1.5. *The group ideals of* N(e_1), N(e_2) *are mutually disjoint if e_1 and e_2 are different two idempotents of S.*

Proof. If e_1N(e_1) *and* e_2N(e_2) contain x in common, then there exist two elements $x_1^* \in e_1$N$(e_1$ and $x_2^* \in e_2$ N(e_2) such that $xx_1^*=x_1^*x=e_1$ and $xx_2^*=x_2^*x=e_2$ respectively. Hence the following relations hold successively ;

$$e_1 = x_1^*x = x_1^*xe_2 = e_1e_2,$$

$$e_2 = xx_2^* = e_1xx_2^* = e_1e_2,$$

$$e_1 = e_2.$$

This is contrary to our assumption.

By Lemma 1.5 and Theorem 1.2, we get the following

Theorem 1.3. *S is represented as a sum of inversible subhomogroups* $\{S_\alpha\}_\alpha$ *such that*

(1) *every* S_α *has one and only one primitive idempotent* ,

(2) *the group ideals of* $\{S_\alpha\}_\alpha$ *are mutually disjoint.*

Moreover we can prove the next interesting theorem.

Theorem 1.4. *S is uniquely decomposed into the class sum of mutually disjoint subgroups if and only if the following relation is solvable for any given a of S.*

$$(R)\quad axa = a \text{ and } ax = xa \in I.$$

Proof. To prove the first half of this theorem, we assume that S has a solution of (R) for any given a of S. We shall show, first of all, that any element of S is contained in the group ideal of some i-component of S. Take up any element $a \in S$. By the assumption, there exists x such that $axa = a$ and $ax = xa \in I$. If we put $e = ax = xa$, then a and x are elements of $N(e)$. Hence $ea \in N(e)$, and hence $a \in N(e)$, since $ea = a$ holds. This shows a to be an element of the group ideal of $N(e)$. Accordingly it follows from Lemma 1.5 that S is decomposed into the class sum of mutually disjoint subgroups; $S = \circ\sum_{e\in I} eN(e)$. We prove next the uniqueness of such decompositions. Assume that there exists a decomposition φ , $S = \circ\sum_{\alpha} \mathcal{G}_\alpha$ of S into the class sum of mutually disjoint subgroups $\mathcal{G}_\alpha$. Then each $\mathcal{G}_\alpha$ contains clearly one and only one idempotent of S. Let $\mathcal{G}(e) = \mathcal{G}_\alpha$ if $\mathcal{G}_\alpha$ contains an idempotent e. Then S is represented as follows ; $S = \circ\sum_{e\in I} \mathcal{G}(e)$. If x is an element of $\mathcal{G}(e)$, there exists $x^* \in \mathcal{G}(e)$ such that $xx^* = x^*x = e$. Hence $x \in eN(e)$, and therefore $\mathcal{G}(e) \subset eN(e)$. This implies $\circ\sum_{e\in I} \mathcal{G}(e) = \circ\sum_{e\in I} eN(e)$. The latter half of this theorem is obvious by the properties of g oups.

Corollary 1.1 *If the relation* $ax = xa \in I$ *has a unique solution for any given element* a *of* S, *then S is uniquely decomposed into the class sum of mutually disjoint subgroups, and it is the decomposition into the class sum of all i-components of S.*

Proof. Take up any element a of S. Then by the assumption, there exists one and only one $x \in S$ such that $ax = xa \in I$. Let $e = ax = xa$. Since $eax = xea = e$, the relation $ea = a$ follows from the uniqueness of an element y such that $yx = xy \in I$.

Hence $N(e) = eN(e)$ and $axa = a$. Therefore by Lemma 1.5 and Theorem 1.4, we have our corollary.

Corollary 1.2. *If S is left (right) cancellable, then S is uniquely decomposed into the class sum of mutually disjoint subgroups, and it is the decomposition into the class sum of* *all i-components of S.*

Proof. We may prove this corollary only when S is left cancellable, because in the other case we can prove it by the same process. Let S be left cancellable. Take up any element $a\in S$, and assume that there exist two elements x_1 and x_2 which satisfy the relations $ax_1=x_1a\in I$ and $ax_2=x_2a\in I$ respectively. We set $e_1=ax_1$ and $e_2=ax_2$. Then the following relations follow successively ; $e_1x_1a=e_1$, $x_1e_1=e_1x_1$, $x_1e_1a=e_1=x_1a$ and $e_1a=a$. Since $ax_1=x_1a_1=e_1$ implies $e_1a=ae_1$, the relation $ae_1=e_1a=a$ is concluded. Similarly we have $ae_2=e_2a=a$ by the same procedure. Thus $ae_1=ae_2$, and consequently $e_1=e_2$, $x_1=x_2$ by the left cancellability of S. This means that the proof of our corollary is reduced to one of corollary 1. 1.

Remark. If S satisfies the condition of Corollary 1. 1 or Corollary 1. 2, then S is, in effect, isomorphic to the direct product [11] $\mathcal{G}\times L\times R$ of a group $\mathcal{G}$, a left singular semigroup L [2] and a right singular semigroup R [2] (see the paragraph 3) . Hereafter by a ' quasi-$\mathfrak{Gr}$-group ' we shall mean a semigroup which is isomorphic to the direct product $\mathcal{G}\times L\times$R of a group $\mathcal{G}$, a left singular semigroup L and a right singular semigroup R [9] .

§ 2. Special middle unitary inversible semigroups.

If an element a of a semigroup G satisfies the relation $xay=xy$ for any elements x, $y\in G$, then a is said to be a middle unit of G. By a ' middle unitary semigroup' we shall mean a semigroup having at least one middle unit, and especially a ' special middle unitary semigroup ' will mean a middle unitary semigroup whose idempotents are middle units [10] . Moreover by a ' special middle unitary inversible semigroup ' we shall mean a special midde unitary semigroup which is inversible. Of course, middle units are not necessarily idempotents even in a special middle unitary inversible semigroup. In this paragraph we shall determine the structure of special middle unitary inversible semigroups. To save repetition, we shall adhere throughout this paragraph to the following notations. V will denote a special middle unitary inversible semigroup. I_V will denote the totality of all idempotents of V. M will denote the totality of all middle units of V. I_V and M are obviously subsemigroups of V, and I_V is contained in M by the above definitions.

Lemma 2. 1. *If the relation $xa=ax\in M$ has two solutions x_1, x_2 in V for given element a of V, then $x_1a=x_2a$ holds.*

Proof. If we set $e_1=x_1a=ax_1$ and $e_2=a_2x_1=x_2a$, then $ex_2=x_2ax_1=x_2e_1$ and successively $ax_1=ae_2x_1=ax_2e_1=x_2ae_1$, $ax_1=x_2a\,(x_1a)=x_2\,(ax_1)\,a=x_2a=ax_2$.

We define an equivalence relation between elements a and b of V as follows ;

$a\sim b$ if and only if there exists an element $x\in V$ such that $ax\in M$ and $bx\in M$.

Then the following relations hold ;

(1) $a\sim a$ for every $a\in V$.

(2) $a \sim b$ implies $b \sim a$.

(3) $a \sim b$, $b \sim c$ imply $a \sim c$.

(4) $a \sim b$, $c \sim d$ imply $ac \sim bd$.

Let Ω be the factor semigroup of V mod $(\sim)$ and let $\overline{\overline{a}}$ be the residue class of V which contains the element a. Then we have

Lemma 2. 2. *Ω is a group, and its unit class concurres with M.*

Proof. It is obvious that the relation $\overline{\overline{e}} = M$ holds for any element e of M. Let e be any element of M and let $\overline{\overline{x}}$ be any element of Ω. Then $\overline{\overline{e}} \circ \overline{\overline{x}} = \overline{\overline{ex}}$ and $\overline{\overline{x}} \circ \overline{\overline{e}} = \overline{\overline{xe}}$ hold. On the other hand, there exists x^* such that $xx^* = x^*x \in I_\Gamma \subset M$. Hence, it follows that $xx^* \in M$. $ex \cdot x^* \in M$ and $xe \cdot x^* \in M$. Consequently $\overline{\overline{e}} \circ \overline{\overline{x}} = \overline{\overline{ex}} = \overline{\overline{x}}$ and $\overline{\overline{x}} \circ \overline{\overline{e}} = \overline{\overline{xe}} = \overline{\overline{x}}$, which imply $\overline{\overline{e}}$ to be the unit element of Ω. Moreover we have $\overline{\overline{x}} \circ \overline{\overline{x^*}} = \overline{\overline{xx^*}} = \overline{\overline{e_1}} = \overline{\overline{e}} = \overline{\overline{e_1}} = \overline{\overline{x^*}} \circ \overline{\overline{x}}$, where e_1 denotes the element xx^*. Therefore there exists an inverse element for any element of Ω. This completes the proof of our lemma.

Since I_Γ is the totality of all idempotent middle units of V, it is isomorphic to the direct-product $L \times R$ of a left singular semigroup L and a right singular semigroup R [9]. Accordingly there exists an isomorphism ξ of I_Γ onto $L \times R$;

$$I_\Gamma \underset{\overline{\overline{}}}{\overset{\xi}{}} L \times R \qquad (A).$$

On the other hand the mapping φ, which is the correspondence $a \longrightarrow (\overline{\overline{a}}, aa^*)$, is a homomorphism of V onto $\Omega \times I_\Gamma$ (where a^* is an element of V such that $aa^* = a^*a \in M$. Such an element aa^* is uniquely determined by Lemma 2. 1);

$$V \underline{\quad \varphi \quad} \Omega \times I_\Gamma \qquad (B).$$

From (A) and (B), we have $V \underline{\quad \xi\varphi \quad} \Omega \times L \times R$.

If we denote by $V(g, l, r)$ the inverse image of $(g, l, r) \in \Omega \times L \times R$ by $\xi\varphi$, $V(g_1, l_1, r_1)$ $V(g_2, l_2, r_2)$ is a set consisting of only one element for any two elements (g_1, l_1, r_1), (g_2, l_2, r_2) of $\Omega \times L \times R$.

Summarizing the above mentioned results, we obtain;

Lemma 2. 3. *There exists a quasi-$\mathfrak{Gr}$-group Γ and a collection $\{V_\alpha \mid \alpha \in \Gamma\}$ of subsets of V which satisfy the following conditions;*

(C. 1) $V = \sum_{\alpha \in \Gamma} V_\alpha$,

(C. 2) *for any β, $\gamma \in \Gamma$, $V_\beta V_\gamma$ is a set consisting of only one element of $V_{\beta\gamma}$.*

Proof. Let $\Gamma = \Omega \times L \times R$ and let V_α be the inverse image of $\alpha \in \Gamma$ by $\xi\varphi$. Then this lemma follows from the above observations.

Now we prove the most important assertion;

Theorem 2.1. *There exist a quasi -$\mathfrak{G}^{t}$- group Γ, a collection $\{V_\alpha \mid \alpha\in\Gamma\}$ of subsets of V and a subset $\{p_\alpha \mid \alpha\in\Gamma\}$ of V, such that*

(C. 1) $V=\sum_{\alpha\in\Gamma} V\alpha$,

(C. 2) $p_\alpha\in V_\alpha$ *for any* $\alpha\in\Gamma$,

(C. 3) $V_\beta V_\gamma = p_{\beta\gamma}$ *for any* $\beta,\ \gamma\in\Gamma$.

Proof. Let $\Gamma=\Omega\times L\times R$, let V_α be the inverse image of $\alpha\in\Gamma$ by $\xi\varphi$, and let $p(g,l,r) = V(g,l,r)\ V(g^{-1},l,r)\ V(g,l,r)$ to each element $(g,l,r)\in\Gamma$. Then the following relation holds for any two elements $(g,l,r,)$, $(g',l',r')\in\Gamma$;

$$
\begin{aligned}
p(g,l,r)\,p(g',l',r') &= \{V(g,l,r)\ V(g^{-1},l,r)\ V(g,l,r)\}\ \{V(g',l',r')\ V(g'^{-1},l',r')\ V(g',l',r')\}\\
&= V(g,l,r)\ \{V(g^{-1},l,r)\ V(g,l,r)\}\ \{V(g',l',r')\ V(g'^{-1},l',r')\}\ V(g',l',r')\ [a]\\
&= V(g,l,r,)\ V(g',l',r')\\
&= V(g,l,r)\ \{V(g',l,r')\ V(g'^{-1},l,r)\}\ \{V(g^{-1},l'r')\ V(g,l,r)\}\ V(g'l'r')\ [a]\\
&= V(gg',l,r')\ V(g'^{-1}g^{-1},l,r')\ V(gg',l,r')\\
&= p(gg',l,r')
\end{aligned}
$$

Hence, by Lemma 2.3 this completes the proof of our theorem.

Remarks. $[a]$. $V(g^{-1},l,r)\ V(g,l,r)$, $V(g',l',r')\ V(g'^{-1},l',r')$, $V(g',l,r')\ V(g'^{-1},l,r)$ and $V(g^{-1},l',r')\ V(g,l,r')$ are consisting of only one middle unit, respectively (see the mapping $\xi\varphi$).

$[b]$. Conversely, it is easy to see that any semigroup satisfying the conditions (C. 1), (C. 2), (C. 3) of Theorem 2.1 becmes a special middle unitary inversible semigroup. Since the set $\{p_\alpha \mid \alpha\in\Gamma\}$ of Theorem 2.1 is clearly a quasi-$\mathfrak{G}^{t}$-group, V is nothing but a quasi-$\mathfrak{G}^{t}$-group in its essence.

Moreover, we have

Theorem 2.2. *If V is simple* [12], *then V is a quasi-$\mathfrak{G}^{t}$-group.*

Proof. Let P be the set $\{p_\alpha \mid \alpha\in\Gamma\}$ of Theorem 2.1. Then P is clearly the minimal ideal of V (Suschkewitsch kernel), Since V is simple, we have the relation $P=V$. This means V to be a quasi-$\mathfrak{G}^{t}$-group.

§ 3. Kernels of inversible semigroups with primitive idempotents.

If an inversible semigroup has at least one primitive idempotent, it has also a minimal ideal (Suschkewitsch kernel). In paragraphs 3, 4, we shall investigate the stucture of inversible semigroups having at least one primitive idempotent. Hereafter $\mathfrak{S}$ will always denote an inversible semigroup with primitive idempotents and $\mathfrak{K}$ will denote the kernel of $\mathfrak{S}$. Moreover $\mathfrak{J}$ and $\mathfrak{P}$ will always denote the totality of all idempotents of $\mathfrak{S}$ and the totality of all primitive idempotents of $\mathfrak{S}$ respectively.

By the definitons of the inversibility and the primitivity it is easy to see that $\mathfrak{J}$ and $\mathfrak{P}$ are subsemigroups of $\mathfrak{S}$, and that $\mathfrak{P}$ becomes an ideal of $\mathfrak{J}$ (see [3]).

Theorem 3. 1. $\mathfrak{S}$ *has a kernel, which contains all primitive idempotents of* $\mathfrak{S}$.

Proof. Let E be any ideal of $\mathfrak{S}$. We shall show first that E contains the set $\mathfrak{P}$. Take up any $x \in E$. Then there exists $x^* \in \mathfrak{S}$ such that $xx^* = x^*x \in \mathfrak{J}$. If we put $e = xx^*$, e is an idempotent contained in E. Hence any primitive idempotent p of $\mathfrak{S}$ is contained in E, since E is an ideal of $\mathfrak{S}$ and since the relation $pep = p$ follows from the definition of the primitivity. This implies $\mathfrak{P} \subset E$. Let $\bigcap_\alpha E_\alpha$ be the intersection of all ideals E_α of $\mathfrak{S}$. Since the relation $\bigcap_\alpha E_\alpha \supset \mathfrak{P}$ follows from the above relation, $\bigcap_\alpha E_\alpha$ is not empty, and hence $\bigcap_\alpha E_\alpha$ becomes the kernel of $\mathfrak{S}$.

Lemma 3. 1. $\mathfrak{K}$ *is formularized as follows* ;

$$\mathfrak{K} = \sum_{e_i, e_j \in \mathfrak{P}} e_i \mathfrak{S} e_j$$

Proof. $\sum_{e_i, e_j \in \mathfrak{P}} e_i \mathfrak{S} e_j \subset \mathfrak{K}$ is obvious by the relation $\mathfrak{P} \subset \mathfrak{K}$. Accordingly we may show the converse of this relation. Take an element e_k of $\mathfrak{P}$. Then we get the relation $\mathfrak{S} e_k \mathfrak{S} = \mathfrak{K}$, since $\mathfrak{S} e_k \mathfrak{S}$ is an ideal of $\mathfrak{S}$ such that $\mathfrak{S} e_k \mathfrak{S} \subset \mathfrak{K}$. Let $xe_k y$ be any element of $\mathfrak{S} e_k \mathfrak{S}$. Then there exist two elements x^*, y^* of $\mathfrak{S}$ such that

$$(xe_k)x^* = x^*(xe_k) \in \mathfrak{J} .$$

$$(e_k y)y^* = y^*(e_k y) \in \mathfrak{J} .$$

Let $(xe_k)x^* = e$ and let $(e_k y)y^* = e'$. Since $xe_k \in C(e)$ and $e_k y \in C(e')$, where $C(e)$ and $C(e')$ denote the i-components of $\mathfrak{S}$ at e and e' respectively, the relations $(xe_k)e = e(xe_k)$ and $(e_k y)e' = e'(e_k y)$ hold. On the other hand, $x^*(xe_k)e_k = ee_k$. Hence $x^*(xe_k) = ee^k$, which implies $e = ee_k \in \mathfrak{P}$. Similarly we have the relation $e' \in \mathfrak{P}$.

Therefore

$$xe_k y = xe_k e_k y = (xe_k)(ee')(e_k y) = (xe_k)ee'(e_k y) = e(xe_k)(e_k y)e'$$

$$= e(xe_k y)e' \in \sum_{e_i, e_j \in \mathfrak{P}} e_i \mathfrak{S} e_j ,$$

and this implies $\mathfrak{K} = \mathfrak{S} e_k \mathfrak{S} \subset \sum_{e_i, e_j \in \mathfrak{P}} e^i \mathfrak{S} e_j$.

Lemma 3. 2. $\mathfrak{P}$ *is formularized as follows* ;

$$\mathfrak{P} = \mathfrak{J} \cap \mathfrak{K} .$$

Proof. $\mathfrak{P} \subset \mathfrak{J} \cap \mathfrak{K}$ is obvious by the relation $\mathfrak{P} \subset \mathfrak{K}$. To show the converse of this relation, take up any element e of $\mathfrak{J} \cap \mathfrak{K}$. Since $e \in \sum_{e_i, e_j \in \mathfrak{P}} e_i \mathfrak{S} e_j$, there exist three elements e_i, x, e_j such that $e_i \in \mathfrak{P}$, $e_j \in \mathfrak{P}$, $x \in \mathfrak{S}$ and $e_i x e_j = e$. If e' is an element of $\mathfrak{J}$, then $e_j e' e_i = e_j e_i$ holds. For the following relations hold successively ;

$$e_j = e_j(e'e_i)e_j = e_j e_i e_j ,$$

$$e_j e' (e_i e_j e_i) = (e_j e_i)(e_j e_i),$$

$$e_j e' e_i = e_j e_i.$$

Since $(e_i x e_j) e' (e_i x e_j) = (e_i x e_j)^2$, we have the relation $e e' e = e$. This implies $e \in \mathfrak{P}$, and hence $\mathfrak{J} \cup \mathfrak{K} \subset \mathfrak{P}$.

Lemma 3. 3. *$\mathfrak{K}$ is an inversible subsemigroup of $\mathfrak{S}$.*

Proof. Let x be any elemnt of $\mathfrak{K}$. Then there exists $x^* \in \mathfrak{S}$ such that $xx^* = x^*x \in \mathfrak{J}$. We put $e = xx^*$. Since e is an idempotent of $\mathfrak{K}$, $e \in \mathfrak{P}$ follows from Lemma 3. 2. Let $x^{**} = ex^*e$. Then x^{**} is an element of $\mathfrak{K}$ such that $xx^{**} = x^{**}x \in \mathfrak{P}$. Thus $\mathfrak{K}$ becomes an invresible subsemigroup of $\mathfrak{S}$.

Lemma 3. 4. *Every primitive idempotent is a middle unit of $\mathfrak{K}$.*

Proof. $e_i e_j e_k = e_i e_k$ holds for any elements e_i, e_j, e_k of $\mathfrak{P}$ as we see in the proof of Lemma 3. 2. Hence this lemma is obvious by Lemma 3. 1.

Since the kernel of a semigroup is simple, above three Lemmas 3. 2, 3. 3 and 3. 4 can be summed up as the following

Theorem 3. 2. *$\mathfrak{K}$ is a simple snd special middle unitary inversible semigroup. Accordingly $\mathfrak{K}$ is a quasi-$\mathfrak{G}^r$-group.*

Corollary 3. 1. *If $\mathfrak{J}$ is a commutative semigroup, then $\mathfrak{K}$ is a subgroup of $\mathfrak{S}$. Accordingly $\mathfrak{S}$ is a homogroup.*

Proof. According to Theorem 2. 2, $\mathfrak{K}$ is a quasi-$\mathfrak{G}^r$-group. On the other hand, $\mathfrak{P}$ is a commtative subsemigroup of $\mathfrak{J}$ such that $\mathfrak{P} \subset \mathfrak{K}$. Hence it is easy to see that $\mathfrak{P}$ is consisting of only one element. This means $\mathfrak{K}$ to be a group, since $\mathfrak{P}$ is the totality of all idempotents of $\mathfrak{K}$.

Corollary 3. 2. *Let S be an inversible semigroup having at most finite idempotents. Then* S *has at least one primitive idempotent, and therefore* S *has a kernel which is a simple and special middle unitary inversible subsemigrop.*

Proof. Let I be the totality of all idempotents of S. By the assumption I is a finite subsemigroup of S. We show first that $pIp = qIq$ implies $p = q$ for any two elements p, q of I. Assume that $pIp = qIq$. Then the following relations hold successively ; $p = pqp$, $q = pqp$, $pq = qpq$, $pq = pqp$ and $p = qpq = pq = pqp = q$. Thus $pIp = qIq$ implies $p = q$. Now since the collecton $\{pIp \mid p \in I\}$ of subsets pIp of I is finite, there exists at least one minimal set $pIp \in \{pIp \mid p \in I\}$. If pIp is not consisting of only one element p, then there exists $q \in pIp$ such that $p \neq q$. Clearly $pIp \supsetneqq qIq$. This is incmpatible, for pIp is a minimal set of $\{pIp \mid p \in I\}$. Thus we have the relation $pIp = \{p\}$, i. e. , p is a primitive idempotent of S.

Remark. N. Kimura showed that a finite semigroup, whose elements are idempotents, has at least one primitive idempotent. The proof of Corollary 3. 2 is due to N. Kimura [3].

Corollary 3. 3. *Any simple and inversible semigroup having at most finite idempotents is a quasi-$\mathfrak{G}$-group.*

Lemma 3. 5. *$\mathfrak{S}$ is simple if and only if it satisfies the foowing conditions ;*

(*C.* 1) *every idempotent is primitive,*

(*C.* 2) *$axa=a$ is solvable for any given $a\in\mathfrak{S}$.*

Proof. If $\mathfrak{S}$ is simple, then according to the definition of the simplicity the relation $\mathfrak{K}=\mathfrak{S}$ holds. Hence the necessity of our conditions is obvious by the definition of a quasi-$\mathfrak{G}$-group. To prove the sufficiency we assume that $\mathfrak{S}$ satisfies the conditions (C. 1) and (C. 2). Let a be any element of $\mathfrak{S}$. According to the assumption, there exists $x\in\mathfrak{S}$ such that $axa=a$. Clearly $ax\in\mathfrak{K}$. Consequently $axa=a\in\mathfrak{K}$. Hence $\mathfrak{S}\subset\mathfrak{K}$, and hence $\mathfrak{S}=\mathfrak{K}$, which completes our proof of this lemma.

Theorem 3. 3. *$\mathfrak{S}$ is a quasi-$\mathfrak{G}$-group if and only if it satisfies the following conditions ;*

(C. 1) *every idempotent is primitive,*

(C. 2) *$axa=a$ is solvable for any given $a\in\mathfrak{S}$.*

Proof. Obvious by Lemma 3. 5 and Theorem 2. 2.

Corollary 3. 4. *Let* S *be an inversible semigroup, and let I be the totality of all idempotents of* S. *If the relation $ax=xa\in I$ has a unique solution for any given element a of* S, *then* S *is a quasi-$\mathfrak{G}$-group.*

Proof. We show first that every idempotent of S is primitive.

Let e and e' be any two idempotents of S. Then the following relations hold successively ;

$$e\cdot ee'e=ee'e\in I,\quad ee'e\cdot e=ee'e\in I,$$

$$e\cdot ee'e=ee'e\cdot e\in I \text{ and } e\cdot e\in I.$$

By the assumption, we have $ee'e=e$. Therefore any idempoent of S is primitive. Using Theorem1. 4, Corollary 1. 1and Theorem 3. 3, we get this corollary.

Corollary 3. 5. *Let* S *be an inversible semigroup. If* S *is left (right) cancellable, then* S *is a quasi-$\mathfrak{G}$-group.*

Proof. This corollary is obvious by Corollary 3. 4, since the relation $ax=xa\in I$ has a unique solution for any given element a of S if S is left (right) cancellable (see the proof of Corollary 1. 2).

4. Construction of inversible semigroups with primitive idempotents.

Let G be any semigroup, and let E be any ideal of G. Rees bifines the factor semigroup G/E essentially that obtined by collapsing E into a single zero element 0, while the remaining elements of G retain their identity. Thus the G/E-product of two nonzero elements is defined to be 0 if their G-product lies in E, and otherwise to be the same as defined in G [6]. Now the Rees factor semigroup $\mathfrak{S}/\mathfrak{R}$ becomes clearly an inversible semigroup with zero.

Conversely we consider the problem of constructing, for given quasi-$\mathfrak{g}_t$-group Q and given inversible semigroup D with zero, every possible inversible semigroup $\mathfrak{S}^*$ with primitive idempotents which satisfies the following conditions ;

(P. 1) $\mathfrak{S}^* = Q \overset{\bullet}{+} D^*$,

(P. 2) Q is an ideal of $\mathfrak{S}^*$ (hence Q is the kernel of $\mathfrak{S}^*$, since it is a quasi-$\mathfrak{g}_t$-group),

$$(P. 3)\quad A \circ B \begin{cases} \in Q \text{ if } A,\ B \in D^* \text{ and if } AB = 0 \\ = AB \text{ if } A,\ B \in D^* \text{ and if } AB \neq 0, \end{cases}$$

where $\circ$ denotes the $\mathfrak{S}^*$-product and D^* denotes the set of nonzero elements of D. We shall call such a $\mathfrak{S}^*$ an ' i. p-extension ' of Q by D.

A. H. Clifford considered the problem of constructing, for given semigroup G and given semigroup T with 0, every possible semigroup $\sum$ containing G as an ideal, such that $\sum/G$ is isomorphic with T. He calls such a $\sum$ an ' extension ' of G by T. He showed the following result [1].

Theorem. *Let G satisfy Cond. A, and let T^* be the set of non-zero elements of T. Then every extension of G by T is found as follows. Let $A \longrightarrow \lambda_A$ and $A \longrightarrow \rho_A$ be mappings of T^* into the semigroups $\mathfrak{J}_L$ and $\mathfrak{J}_R$ of left and right translations of G respectively, and let $[A, B]$ be a ramification set of T^* in G, such that Cond.* (C. 1~3) *are satisfied. Then the class sum $\sum = G \overset{\bullet}{+} T^*$ of G and T^* becomes an extension of G by T if product $\circ$ therein is defined by the equations* (N. 1~4) ;

Cond. A. *If $as = bs$ and $sa = sb$ for all s, then $a = b$.*

$$(C. 1)\quad \lambda_A \lambda_B = \begin{cases} \lambda_{AB} \text{ if } AB \neq 0, \\ \lambda_{[A,B]} \text{ if } AB = 0\ [\divideontimes]. \end{cases}$$

$$(C. 2)\quad \rho_A \rho_B = \begin{cases} \rho_{AB} \quad \text{if } AB \neq 0, \\ \rho_{[A,B]} \text{ if } AB = 0\ [\divideontimes]. \end{cases}$$

(C. 3) $s(\lambda_A t) = (s\rho_A)t$ *if* $s,\ t \in G$, *that is*, λ_A *and* ρ_A *are linked.*

$$(N. 1)\quad A \circ B = \begin{cases} AB \quad \text{if } A \in T^*,\ B \in T^* \text{ and if } AB \neq 0, \\ [A,B] \text{ if } A \in T^*,\ B \in T^* \text{ and if } AB = 0. \end{cases}$$

(N. 2) $A \circ s = \lambda_A s$ *if* $s \in G,\ A \in T^*$.

(N. 3) $s \circ A = s\rho_A$ *if* $s \in G,\ A \in T^*$.

(N. 4) $s \circ t = st$ *if* $s,\ t \in G$.

Remark. [$\divideontimes$]. $\lambda_{[A,B]}$ and $\rho_{[A,B]}$ denote the special left and right translations respectively, which are induced by $[A, B]$ [1].

A semigroup is called a 'zero-semigroup' if it contains a zero element but no other idempotent. The Rees factor semigroup $\mathfrak{S}/\mathfrak{R}$ becomes a zero-semigroup if all idempotents of $\mathfrak{S}$ are primitive. In this paragraph, we shall show a result which is closely related to the above theorem.

To save repetition, we shall adhere throughout this paragraph to the following notations.

Q will denote any quasi-$\mathfrak{G}$-group. D will denote any inversible semigroup with zero o, having no elements in common with Q. Especially Z will denote any zero-semigroup, having no elements in common with Q. D^* and Z^* will denote the sets of nonzero elements of D and Z respectively. The small letters a, b, c, d, e, s, t, u, v, will always denote elements of Q. Except in Corollary 4.1, the capitals A, B, C, will denote elements of D^*. These will denote, in Corollary 4.1, elements of Z^*. I_Q will denote the totality of idempotents of Q and I_D will denote the totality of nonzero idempotents of D. $\mathfrak{J}_L$ and $\mathfrak{J}_R$ will denote the semigroups of left and right translations of Q respectively. For any $a \in Q$, λ_a will denote the special left translation induced by a and ρ_a will denote the special right translation induced by a [1].

Theorem 4.1. *Every i. p-extension of Q by D is found as follows.*
Let φ ; $A \longrightarrow \lambda_A$ and ψ ; $A \longrightarrow \rho_A$ be mappings of D^* into $\mathfrak{J}_L$ and $\mathfrak{J}_R$ respectively and let f ; $[A, B]$ be a ramification set of D^* in Q, such that Cond. (C. 1'$\sim$6') are satisfied. Then the class sum $\mathfrak{S}^* = Q \dot{+} D^*$ of Q and D^* becomes an i. p-extension of Q by D if product $\circ$ therein is defined by the equations (N. 1' $\sim$ 4') ;

(C. 1') $\lambda_A \lambda_B = \begin{cases} \lambda_{AB} & \text{if } AB \neq 0, \\ \lambda_{[A,B]} & \text{if } AB = 0. \end{cases}$

(C. 2') $\rho_A \rho_B = \begin{cases} \rho_{AB} & \text{if } AB \neq 0, \\ \rho_{[A,B]} & \text{if } AB = 0. \end{cases}$

(C. 3') $s(\lambda_A t) = (s\rho_A)t$, *that is, λ_A and ρ_A are linked.*

(C. 4') If there exist no elements B such that $AB = BA \in I_D$, then there exists s such that $\lambda_A s = s\rho_A \in I_Q$.

(C. 5') $[E, E'] \in I_Q$ *if* E, $E' \in I_D$ *and* $EE' = 0$.

(C. 6') $\lambda_E e \in I_Q$, $e\rho_E \in I_Q$ *and* $e(\lambda_E e) = (e\rho_E)e = e$ *if* $E \in I_D$, $e \in I_Q$.

(N. 1') $A \circ B = \begin{cases} AB & \text{if } AB \neq 0, \\ [A.B] & \text{if } AB = 0. \end{cases}$

(N. 2') $A \circ s = \lambda_A s$.

(N. 3') $s \circ A = s\rho_A$.

(N. 4') $s \circ t = st$.

Proof. Let $\mathfrak{S}^*$ be the class sum $Q\dot{+}D^*$ of Q and D^*. Assume that $\mathfrak{S}^*$-product $\circ$ is defined by φ, ψ, f satisfying Cond. (C. 1′~6′) and by the equations (N. 1′~4′). According to the Clifford Theorem, $\mathfrak{S}^*$ becomes an extension of Q by D since Q satisfies Cond. A. To prove $\mathfrak{S}^*$ to be inversible, we show first that the class sum $I=I_Q\dot{+}I_D$ of I_Q and I_D becomes a subsemigroup of $\mathfrak{S}^*$. Let α and β be two elements of I. If both α and β are contained in I_Q or in I_D, then $\alpha\circ\beta$ is contained in I_Q or in I_D according to Cond. (C. 5′). We assume that $\alpha\in I_D$ and $\beta\in I_Q$. Since $\alpha\circ\beta=\lambda_\alpha\ \beta$ by (N. 2′), $\alpha\circ\beta\in I_Q$ follows from Cond. (C. 6′). Similarly we can prove the relation $\alpha\circ\beta\in I_Q$, if $\alpha\in I_Q$ and $\beta\in I_D$. Hence I becomes a subsemigioup of $\mathfrak{S}^*$. Moreover it is obvious, by Cond. (C. 4′), that to each element $\alpha\in\mathfrak{S}^*$ there exists $\beta\in\mathfrak{S}^*$ such that $\alpha\circ\beta=\beta\circ\alpha\in I$. Accordingly $\mathfrak{S}^*$ becomes an i. p-extension of Q by D. Conversely, let $\mathfrak{S}^0$ be an i. p-extension of Q by D. If we take a ramification set f ; $[A, B]$ of D^* in Q and mappings φ, ψ of D^* into $\mathfrak{J}_L$ and $\mathfrak{J}_R$ as follows ;

φ ; $A\longrightarrow\lambda_A$, where λ_A is the left translation of Q such that $\lambda_A t=A\bullet t$ ($\bullet$ *denotes the* $\mathfrak{S}^*$*-product*) for all $t\in Q$,

ψ ; $A\longrightarrow\rho_A$, where ρ_A is the right translation of Q such that $t\rho_A=t\bullet A$ for all $t\in Q$,

f ; $[A, B]=A\bullet B$ if $AB=0$,

then it is easy to see that φ, ψ and f satisfy Cond. (C, 1′~6′) and that $\mathfrak{S}^*$-product $\bullet$ is the same as the product $\circ$ defined by (N. 1′~4′). Thus our theorem is completely proved.

Moreover we obtain the following corollary as a special case of Theorem 4. 1.

Corollary 4. 1. *Every i. p-extension of Q by Z is found as follows.*
Let φ ; $A\longrightarrow\lambda_A$ and ψ ; A——ρ_A be mappings of Z^* into $\mathfrak{J}_L$ and $\mathfrak{J}_R$ respectively and let f ; $[A, B]$ be a ramification set of Z^* in Q, such that Cond. (C. 1′~4′) are satisfied. Then the class sum $\mathfrak{S}^*=Q\dot{+}Z^*$ of Q and Z^* becomes an i. p -extension of Q by Z if product $\circ$ therein is defined by the equations (N. 1′~4′).

Corollary 4. 2. *Let e be any idempotent of Q. Then the class sum $\mathfrak{S}^*=Q\dot{+}D^*$ of Q and D^* becomes an i. p-extension of Q by D if product $\circ$ therein is defined as follows ;*

(1) $A\circ B=\begin{cases} AB \text{ if } AB\neq 0, \\ e \text{ if } AB=0. \end{cases}$ (2) $A\circ s=es$.

(3) $s\circ A=se$. (4) $s\circ t=st$

Proof. Let φ be the mapping ; $A\longrightarrow\lambda_e$ (special left translation), let ψ be the mapping ; $A\longrightarrow\rho_e$ (special right translation) and let f be the ramification set ; $[A, B]=e$. Then this corollary follows from Theorem 4. 1, since these mappings φ, ψ and the ramification set f satisfy Cond. (C. 1′~6′).

Summarizing the results of paragarphs 3, 4, we obtain the following conclusion.

Let G be any inversible semigroup with primitive idempotents. Let Q be the kernel of G, and let D be the Rees factor semigroup G/Q. Then D is an inversible semigroup with zero, and the structure of G is competely determined by Q, D and a pair (φ, ψ, f) of mappings φ, ψ and $f=[A. B]$ saltisfying Cond. (C. $1'\sim6'$) ;

$$G=\{Q, D, (\varphi, \psi, f)\}$$

Especially if we consider G only for inversible semigroups whose all idempotents are primitive, then G/Q is always a zero-semigroup Z and hence we have

$$G=\{Q, Z, (\varphi, \psi, f)\}.$$

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A semigroup G is said to be right (left) singular if $ab=b$ ($ab=a$) for any a, $b\in G$.

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An 'idempotent semigroup' is a semigroup whose all elements are idempotents.

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A semigroup G is said to be completely non-commutative if $xy\neq yx$ for any different two elements x, y of G.

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[11] Let G_1, G_2 be any two semigroups, and let G be the set $\{(x_1, x_2) \mid x_1\in G_1, x_2\in G_2\}$. Then G becomes a semigroup if we define G-product $\circ$ as follows ;

$$(x_1, x_2) \circ (y_1, y_2) = (x_1y_1, x_2y_2).$$

This semigroup G is called the direct product of G_1 and G_2.

[12] A semigroup G is called simple if it contains no other ideals than G itself. This definition of simplicity is equal to the definition of Rees [6] if G has not a zeroelement.

Supplement to Corollary 3. 5. Accordingly S is a right (left) group [2].