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Generators of the Poincaré Group for the Dirac and the Proca Fields in terms of Normal Coordinates

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For the quantized Dirac and Proca fields ten generators of the Poincaré group, energy and momenta, three angular momenta and three boost generators, are brought into diagonal form in the spin and momentum space. The results are quite simple and physically sensible. Some applications are also given.

§1. Introduction and Summary

The principle of invariance under the inhomogeneous Lorentz group (the Poincaré group) plays a fundamental role to describe all of the physical phenomena except for the gravitational one in a cosmological scale. It gives the conservation laws of energy-momentum and angular momentum as well as the law of Lorentz covariance for any isolated system. It is a rather remarkable fact [1, 2] that this principle determines all of the possible types of free particles in the universe with the help of the quantum mechanics. In order to guarantee the Poincaré invariance the existence of the generators of this group with well-known commutation relations is essential. Ten generators of this group have also important physical meanings.

In the relativistic quantum field theory these generators can be written in terms of quantized field operators [3, 4]. To get a particle picture we must decompose these generators into normal mode in terms of occupation number representation in momentum space. *Every* text book [5, 6, 7, 8] on quantum field theory describes four generators, energy and three momenta, in normal mode. But *no* text book except one [2] describes other six generators, three angular momenta and three Lorentz boosts, in normal mode, as far as we know. In his excellent book [2] Ohnuki develops a general theory of the representations of the Poincaré group. He uses the Bargmann Wigner amplitudes and gives ten generators in normal mode for local quantized fields with arbitrary spin.

Owing to their fundamental character it will be worth while to describe a different procedure from that in reference 2 to decompose the six generators into normal mode. In this paper we follow the standard Lagrangian theory of quantized field, and decompose the field into Fourier components, thereby fixing the spin quantization axis arbitrarily. We express the six generators in terms of creation and annihilation operators in momentum space for the Dirac and the Proca fields in §§ 3 and 4, respectively. The calculation is rather tedious, but the results are quite simple and coincide with that in reference 2 for the case of the Dirac field. We use the traditional field theory with the familiar physical meanings, therefore, if we consider the interaction with other fields, we would have more transparent physical picture than that of reference 2. As a simple application of § 3 we examine the transformation properties of the creation and annihilation operators of the spinar particle in § 5.

§ 2. The Generators of the Poincaré Group

In the Lagrangian quantum field theory [3, 4] the ten generators P^{μ} and $M^{\mu\nu}$ can be written*) as

$$P^{\mu}=:\int \theta^{0\mu}d\boldsymbol{x}:, \tag{1}$$

and

$$M^{\mu\nu}=:\int M^{0\mu\nu}d\boldsymbol{x}: \tag{2}$$

with

$$M^{\lambda\mu\nu}=x^{\mu}\,\theta^{\lambda\nu}-x^{\nu}\,\theta^{\lambda\mu}-i\pi^{\lambda}\,\Sigma^{\mu\nu}\,\varphi \tag{3}$$

and

$$\pi^{\lambda}=\frac{\delta\mathscr{L}}{\delta\partial_{\lambda}\varphi}\,.$$

In Eq's (1), (2) and (3) we denote φ (multi components) the quantized field, π^0 its canonical momentum and $\theta^{\mu\nu}$ canonical energy momentum tensor, respectively and : : means Wick's normal product. $\theta^{\mu\nu}$ is defined through Lagrangian $\mathscr{L}$ by

$$\theta^{\mu\nu}=\pi^{\mu}\partial^{\nu}\varphi-g^{\mu\nu}\mathscr{L} \tag{4}$$

and satisfies conservation law

$$\partial_{\mu}\,\theta^{\mu\nu}=0 \tag{5}$$

by virtue of the Euler equation for φ. In Eq. (3) $\Sigma^{\mu\nu}$ is the spin matrix of φ and is defined by the transformation property

$$\varphi(x)\longrightarrow\varphi'(x')=\left(1-\frac{i}{2}\,\varepsilon_{\mu\nu}\,\Sigma^{\mu\nu}\right)\varphi(x)$$

for the infinitesimal Lorentz transformation

$$x^{\mu}\longrightarrow x'^{\mu}=x^{\mu}+\varepsilon^{\mu\nu}x_{\nu}.$$

*) We use the metric (1, −1, −1, −1).

By using the canonical commutation relations

$$[\pi^0(x^0, \boldsymbol{x}), \varphi(x^0, \boldsymbol{y})]_\pm = -i\delta(\boldsymbol{x}-\boldsymbol{y}), \qquad (6)$$
$$[\pi^0(x^0, \boldsymbol{x}), \pi^0(x^0, \boldsymbol{y})]_\pm = [\varphi(x^0, \boldsymbol{x}), \varphi(x^0, \boldsymbol{y})]_\pm = 0.$$

we can show that the ten generators P^μ and $M^{\mu\nu} = -M^{\nu\mu}$ satisfy the following commutation relations:

$$[P^\mu, P^\nu] = 0, \qquad \text{(i)}$$
$$[P^\lambda, M^{\mu\nu}] = i(g^{\lambda\mu}P^\nu - g^{\lambda\nu}P^\mu), \qquad \text{(ii)} \quad (7)$$
$$[M^{\mu\nu}, M^{\sigma\tau}] = i(g^{\mu\tau}M^{\nu\sigma} + g^{\nu\sigma}M^{\mu\tau} - g^{\mu\sigma}M^{\nu\tau} - g^{\nu\tau}M^{\mu\sigma}). \qquad \text{(iii)}$$

Eq's (7) are the defining equations of the Lie algebra of the Poincaré group.

As is well known, from the conservation laws (5) we can deduce the time independence of P^μ in accordance with (7) (i). From the relativistic invariance of the Lagrangian $\mathscr{L}$, we deduce [4]

$$\pi^\mu \partial^\nu\varphi - \pi^\nu\partial^\mu\varphi = i\partial_\lambda\pi^\lambda \Sigma^{\mu\nu} \varphi + i\pi^\lambda \Sigma^{\mu\nu} \partial_\lambda\varphi. \qquad (8)$$

By using (8) and (5), we obtain

$$\partial_\lambda M^{\lambda\mu\nu} = 0. \qquad (9)$$

Therefore we also deduce the time independence of $M^{\mu\nu}$ defined by (2) in accordance with (7)(ii). It should be noted that owing to the explicit time dependence of M^{0k} we must use the following equation

$$\frac{d}{dx^0} M^{0k} = \frac{\partial M^{0k}}{\partial x^0} + i[P^0, M^{0k}] = P^k + i[P^0, M^{0k}] = 0. \qquad (10)$$

In the remaining of this section we shall discuss the physical meaning of M^{0k}. The symmetrical energy momentum tensor (the Belinfante tensor) $\theta_B^{\mu\nu}$ is given [3, 4] by

$$\theta_B^{\mu\nu} = \theta^{\mu\nu} + \frac{1}{2}\partial_\lambda X^{\lambda\mu\nu} \qquad (11)$$

with

$$X^{\lambda\mu\nu} = -X^{\mu\lambda\nu} = -i(\pi^\lambda \Sigma^{\mu\nu} \varphi - \pi^\mu \Sigma^{\lambda\nu} \varphi - \pi^\nu \Sigma^{\lambda\mu} \varphi). \qquad (12)$$

Since the difference between $\theta_B^{\mu\nu}$ and $\theta^{\mu\nu}$ is total divergence of an anti-symmetric tensor, P^μ can also be rewritten as

$$P^\mu = :\int \theta_B^{0\mu} d\boldsymbol{x}:. \qquad (1')$$

We can also rewrite [3, 4]

$$M^{\mu\nu} = :\int M_B^{0\mu\nu} d\boldsymbol{x}: \tag{2'}$$

where

$$M_B^{\lambda\mu\nu} = x^\mu \theta_B^{\lambda\nu} - x^\nu \theta_B^{\lambda\mu} . \tag{3'}$$

From (2′) and (1′) we have

$$\begin{aligned} M^{0k} &= x^0 : \int \theta_B^{0k} \, d\boldsymbol{x} : - : \int x^k \theta_B^{00} \, d\boldsymbol{x} : \\ &= x^0 P^k - : \int x^k \, \theta_B^{00} \, d\boldsymbol{x} : . \end{aligned} \tag{13}$$

The physical meaning of Eq. (13) is as follows: we define the center of the mass operator [9] X^k by

$$X^k P^0 = : \int x^k \, \theta_B^{00} \, d\boldsymbol{x} : . \tag{14}$$

By differentiating (14) with x^0, we have

$$\begin{aligned} \dot{X}^k P^0 &= : \int x^k \, \dot{\theta}_B^{00} \, d\boldsymbol{x} : \; = - : \int x^k \, \partial_k \, \theta_B^{k0} \, d\boldsymbol{x} : \\ &= : \int \theta_B^{k0} \, d\boldsymbol{x} : = P^k . \end{aligned} \tag{15}$$

By differentiating (13) with respect to x^0 and using (13), (14) and (15) we again obtain Eq. (10). The explicit x^0-dependence of M^{0k} is canceled by the x^0-dependence of X^k.

By using above discussions we have

$$J^k \equiv M^{ij} = : \int \pi^0 (x^i \partial^j - x^j \partial^i - i \Sigma^{ij}) \varphi \big|_{x^0=0} d\boldsymbol{x} :$$

$$(i, j, k \text{ is a cyclic permutation of } 1, 2, 3.) \tag{16}$$

$$M^{0k} = : \int (-x^k \, \theta^{00} - i\pi^0 \, \Sigma^{0k} \, \varphi)_{x^0=0} \, d\boldsymbol{x} : \tag{17}$$

Our next task is to express (16) and (17) in terms of normal mode. This will be done in the next two sections for the Dirac and the Proca fields.

§ 3. The Dirac Field

For the spin 1/2 field we have the following relations:

$$\begin{aligned} \mathscr{L}(x) &= \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi , \\ \pi_\alpha^0 &= \frac{\delta \mathscr{L}}{\delta \dot{\psi}_\alpha} = i(\bar{\psi}\gamma^0)_\alpha = i\psi_\alpha^\dagger . \end{aligned} \tag{18}$$

In order to satisfy the Dirac equation $\psi(x)$ and $\bar{\psi}(x)$ are expanded into Fourier components:

$$\psi(x)=\sum_r \int \frac{d\boldsymbol{p}}{(2\pi)^3 2p^0}\,(a_r(\boldsymbol{p})u^{(r)}(\boldsymbol{p})\,e^{-ipx}+b_r^\dagger(\boldsymbol{p})v^{(r)}(\boldsymbol{p})\,e^{ipx}),$$

$$(p^0=\sqrt{\boldsymbol{p}^2+m^2} \text{ and } r,\, s=\pm 1/2) \qquad (19)$$

$$\bar{\psi}(x)=\sum_s \int \frac{d\boldsymbol{p}}{(2\pi)^3 2p^0}\,(b_s(\boldsymbol{p})\bar{v}^{(s)}(\boldsymbol{p})\,e^{-ipx}+a_s^\dagger(\boldsymbol{p})\bar{u}^{(s)}(\boldsymbol{p})\,e^{ipx}).$$

The four component Dirac spinors $u^{(r)}(\boldsymbol{p})$ and $v^{(s)}(\boldsymbol{p})$ satisfy

$$(\not{p}-m)u^{(r)}(\boldsymbol{p})=\bar{u}^{(r)}(\boldsymbol{p})(\not{p}-m)=0,$$

$$(\not{p}+m)v^{(r)}(\boldsymbol{p})=\bar{v}^{(r)}(\boldsymbol{p})(\not{p}+m)=0$$

with the ortho-normalization relations

$$\bar{u}^{(r)}(\boldsymbol{p})u^{(s)}(\boldsymbol{p})=-\bar{v}^{(r)}(\boldsymbol{p})v^{(s)}(\boldsymbol{p})=2m\delta_{rs},$$

$$\bar{v}^{(r)}(\boldsymbol{p})u^{(s)}(\boldsymbol{p})=\bar{u}^{(r)}(\boldsymbol{p})v^{(s)}(\boldsymbol{p})=0.$$

In order to satisfy (6) $a_r(\boldsymbol{p})$, $a_s^\dagger(\boldsymbol{q})$, $b_r^\dagger(\boldsymbol{p})$ and $b_s(\boldsymbol{q})$ must satisfy the following anti commutation relations:

$$[a_r(\boldsymbol{p}),\, a_s^\dagger(\boldsymbol{q})]_+=[b_s(\boldsymbol{p}),\, b_r^\dagger(\boldsymbol{q})]_+=(2\pi)^3 2p^0\delta_{rs}\delta(\boldsymbol{p}-\boldsymbol{q})$$

$$[a_r(\boldsymbol{p}),\, a_s(\boldsymbol{q})]_+=[a_r(\boldsymbol{p}),\, b_s(\boldsymbol{q})]_+=\cdots=0.$$

Substituting (18) and (19) into (16), we obtain

$$\begin{aligned} M^{ij}=&\sum_r \int \frac{d\boldsymbol{p}}{(2\pi)^3 2p^0}\,[a_r^\dagger(\boldsymbol{p})l^{ij}a_r(\boldsymbol{p})+b_r^\dagger(\boldsymbol{p})l^{ij}b_r(\boldsymbol{p})]\\ &+\sum_{r,s}\int \frac{d\boldsymbol{p}}{(2\pi)^3 4p_0^2}:(b_s(-\boldsymbol{p})v^{(s)\dagger}(-\boldsymbol{p})+a_s^\dagger(\boldsymbol{p})u^{(s)\dagger}(\boldsymbol{p}))\times \qquad (20)\\ &\times\left[a_r(\boldsymbol{p})\left(l^{ij}+\frac{\sigma^{ij}}{2}\right)u^{(r)}(\boldsymbol{p})+b_r^\dagger(-\boldsymbol{p})\left(l^{ij}+\frac{\sigma^{ij}}{2}\right)v^{(r)}(-\boldsymbol{p})\right]: \end{aligned}$$

with $l^{ij}=-i\left(p^i\frac{\partial}{\partial p^j}-p^j\frac{\partial}{\partial p_i}\right)$ and $\sigma^{ij}=\frac{i}{2}[\gamma^i,\gamma^j]$. To further simplify Eq. (20), we must use the explicit form of Dirac spinors $u^{(r)}(\boldsymbol{p})$ and $v^{(r)}(-\boldsymbol{p})$:

$$u^{(r)}(\boldsymbol{p})=\begin{bmatrix}\sqrt{p^0+m}\;w^{(r)}\\ \sqrt{p^0-m}\;(\boldsymbol{\sigma n})\;w^{(r)}\end{bmatrix},\quad \boldsymbol{n}\equiv\boldsymbol{p}/|\boldsymbol{p}|,$$

$$v^{(r)}(-\boldsymbol{p})=\begin{bmatrix}-\sqrt{p^0-m}\;(\boldsymbol{\sigma n})\;w'^{(-r)}\\ \sqrt{p^0+m}\;w'^{(-r)}\end{bmatrix}. \qquad (21)$$

In Eq. (21) $w^{(r)}$ is a two component Pauli spinor with arbitrary spin-quantization-axis $\boldsymbol{e}$. Explicitly,

$$w_{\boldsymbol{e}}^{(\pm 1/2)}=e^{-i\frac{\varphi}{2}\sigma_z}e^{-i\frac{\theta}{2}\sigma_y}\times\begin{cases}\begin{pmatrix}1\\0\end{pmatrix},\\ \begin{pmatrix}0\\1\end{pmatrix},\end{cases}$$

with $\boldsymbol{e}$: $(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$. From the consideration for charge conjugation invariance [8] we choose $w'^{(r)}=-\sigma_y w^{(-r)*}$ for the Pauli spinor for anti-particle. Note that $w'^{(-s)\dagger}\boldsymbol{\sigma}w'^{(-r)}=-w^{(r)\dagger}\boldsymbol{\sigma}w^{(s)}$. By using (21) we get

$$\begin{aligned}
&u^{(s)\dagger}(\boldsymbol{p})\left(\boldsymbol{l}+\frac{\boldsymbol{\sigma}}{2}\right)2u^{(r)}(\boldsymbol{p})=2p^0\left\langle\frac{\boldsymbol{\sigma}}{2}\right\rangle_{sr},\\
&v^{(s)\dagger}(-\boldsymbol{p})\left(\boldsymbol{l}+\frac{\boldsymbol{\sigma}}{2}\right)u^{(r)}(\boldsymbol{p})=0,\\
&v^{(s)\dagger}(\boldsymbol{p})\left(\boldsymbol{l}+\frac{\boldsymbol{\sigma}}{2}\right)v^{(r)}(\boldsymbol{p})=-2p^0\left\langle\frac{\boldsymbol{\sigma}}{2}\right\rangle_{rs}.
\end{aligned}\tag{22}$$

Substituting (22) into (20) we get the result

$$J^k=\sum_{r,s}\int\frac{d\boldsymbol{p}}{(2\pi)^3 2p^0}\left[a_s^\dagger(\boldsymbol{p})\langle J^k\rangle_{sr}a_r(\boldsymbol{p})+b_s^\dagger(\boldsymbol{p})\langle J^k\rangle_{sr}b_r(\boldsymbol{p})\right]\tag{23}$$

where

$$\begin{aligned}
&\langle J^k\rangle_{sr}=\delta_{sr}l^k+\frac{1}{2}\langle\sigma^k\rangle_{sr},\\
&l^k=-i\left(p^i\frac{\partial}{\partial p^j}-p^j\frac{\partial}{\partial p^i}\right)\quad\text{and}\quad\langle\sigma^k\rangle_{sr}=w^{(s)\dagger}\sigma^k w^{(r)}.
\end{aligned}\tag{24}$$

For the helicity state we must choose $\boldsymbol{e}=\boldsymbol{p}/|\boldsymbol{p}|$, but be careful that the differential operation l^{ij} in Eq. (20) does not act on $w^{(r)}$, since the momentum and the spin belong to different freedoms.

For the boost operator M^{0k} the similar calculation as (20) leads to

$$\begin{aligned}
M^{0k}=&\sum_r\int\frac{d\boldsymbol{p}}{(2\pi)^3 2i}\left(a_r^\dagger(\boldsymbol{p})\frac{\partial}{\partial p^k}a_r(\boldsymbol{p})+b_r^\dagger(\boldsymbol{p})\frac{\partial}{\partial p^k}b_r(\boldsymbol{p})\right)\\
&+\sum_{r,s}\int\frac{d\boldsymbol{p}}{(2\pi)^3 4p_0^2}:(b_s(-\boldsymbol{p})v^{(s)\dagger}(-\boldsymbol{p})+a_s^\dagger u^{(s)\dagger}(\boldsymbol{p}))\times\\
&\times\left[a_r(\boldsymbol{p})\left(-ip^0\frac{\partial}{\partial p^k}+\frac{\sigma^{0k}}{2}\right)u^{(r)}(\boldsymbol{p})+b_r^\dagger(-\boldsymbol{p})\left(ip^0\frac{\partial}{\partial p^k}+\frac{\sigma^{0k}}{2}\right)v^{(r)}(-\boldsymbol{p})\right]:.
\end{aligned}\tag{25}$$

By using (21), noting $\sigma^{0k}=i\gamma^0\gamma^k=i\alpha^k$, we have the following relations:

$$u^{(s)\dagger}(\boldsymbol{p})\left(-ip^0\frac{\partial}{\partial \boldsymbol{p}}+i\frac{\boldsymbol{\alpha}}{2}\right)u^{(r)}(\boldsymbol{p})=\frac{2p^0}{p^0+m}\left\langle\frac{\boldsymbol{\sigma}\times\boldsymbol{p}}{2}\right\rangle_{sr},$$

$$v^{(s)\dagger}(-\boldsymbol{p})\left(-ip^0\frac{\partial}{\partial \boldsymbol{p}}+i\frac{\boldsymbol{\alpha}}{2}\right)u^{(r)}(\boldsymbol{p})=0\,, \tag{26}$$

$$v^{(s)\dagger}(\boldsymbol{p})\left(-ip^0\frac{\partial}{\partial \boldsymbol{p}}+i\frac{\boldsymbol{\alpha}}{2}\right)v^{(r)}(\boldsymbol{p})=-\frac{2p^0}{p^0+m}\left\langle\frac{\boldsymbol{\sigma}\times\boldsymbol{p}}{2}\right\rangle_{rs}.$$

From Eq's (25) and (26) we get the result

$$M^{0k}=-\sum_{r,s}\int\frac{d\boldsymbol{p}}{(2\pi)^3 2p^0}\,[a_s^\dagger(\boldsymbol{p})\langle K^k\rangle_{sr}a_r(\boldsymbol{p})+b_s^\dagger(\boldsymbol{p})\langle K^k\rangle_{sr}b_r(\boldsymbol{p})]\,, \tag{27}$$

with

$$\langle K^k\rangle_{sr}=\delta_{sr}ip^0\frac{\partial}{\partial p^k}-\frac{1}{p^0+m}\langle\boldsymbol{\sigma}\times\boldsymbol{p}\rangle_{sr}/2$$

and

$$\langle\boldsymbol{\sigma}\times\boldsymbol{p}\rangle_{sr}=w^{(s)\dagger}(\boldsymbol{\sigma}\times\boldsymbol{p})w^{(r)}. \tag{28}$$

Eq's (25) and (27) coincide with that given in reference 2. Finally, for the sake of completeness we write P^μ in our notations:

$$P^\mu=\sum_r\int\frac{d\boldsymbol{p}}{(2\pi)^3 2p^0}\,p^\mu(a_r^\dagger(\boldsymbol{p})a_r(\boldsymbol{p})+b_r^\dagger(\boldsymbol{p})b_r(\boldsymbol{p})) \tag{29}$$

with p^μ: $(p^0=\sqrt{\boldsymbol{p}^2+m^2},\boldsymbol{p})$.

§ 4. The Proca Field

For the neutral vector field φ^μ, we have the following relations:

$$\mathscr{L}=-\frac{1}{4}(\partial_\mu\varphi_\nu-\partial_\nu\varphi_\mu)(\partial^\mu\varphi^\nu-\partial^\nu\varphi^\mu)+\frac{m^2}{2}\varphi_\nu\varphi^\nu\,,$$

$$\pi_k=\frac{\delta\mathscr{L}}{\delta\dot\varphi^k}=\dot\varphi^k+\nabla^k\varphi^0\equiv(\boldsymbol{\Pi})^k\,,$$

$$\varphi^0=\frac{1}{m^2}\nabla\boldsymbol{\Pi}\,, \tag{30}$$

$$\theta^{00}=\frac{1}{2}(\boldsymbol{\Pi}^2+m^2\boldsymbol{\varphi}^2+(\nabla\times\boldsymbol{\varphi})^2+(\nabla\boldsymbol{\Pi})^2/m^2)\,,$$

$$(\Sigma^{\mu\nu})_{\alpha\beta}=i(g^\mu{}_\alpha g^\nu{}_\beta-g^\nu{}_\alpha g^\mu{}_\beta)\,.$$

The Fourier expansion of $\varphi^\mu(x)$ is written

$$\varphi^{\mu}(x)=\int\frac{d\boldsymbol{k}}{(2\pi)^3 2k^0}\left(a^{\mu}(\boldsymbol{k})\,e^{-ikx}+a^{\mu\dagger}(\boldsymbol{k})\,e^{ikx}\right),$$

with $$a^0(\boldsymbol{k})=\boldsymbol{k}\boldsymbol{a}(\boldsymbol{k})/k^0, \tag{31}$$

$$\boldsymbol{\Pi}(x)=\int\frac{d\boldsymbol{k}}{(2\pi)^3 2i}\left[\left(\boldsymbol{a}(\boldsymbol{k})-\frac{\boldsymbol{k}a^0(k)}{k^0}\right)e^{-ikx}-\left(\boldsymbol{a}^{\dagger}(\boldsymbol{k})-\frac{\boldsymbol{k}a^{0\dagger}(\boldsymbol{k})}{k^0}\right)e^{ikx}\right]. \tag{32}$$

The commutation relations between $a^{\lambda}(\boldsymbol{k})$ and $a^{\mu\dagger}(\boldsymbol{k}')$ is read from (6), (31) and (32) as

$$[a^{\mu}(\boldsymbol{k}),\ a^{\nu\dagger}(\boldsymbol{k}')]=-(2\pi)^3 2k^0\delta(\boldsymbol{k}-\boldsymbol{k}')\left(g^{\mu\nu}-\frac{k^{\mu}k^{\nu}}{m^2}\right),\quad (k^0=\sqrt{\boldsymbol{k}^2+m^2}) \tag{33}$$

others are zero.

In order to get the normal mode we must expand $a^{\mu}(\boldsymbol{k})$ in terms of the four dimensional polarization vectors [7] $\varepsilon^{(r)\mu}$;

$$\varepsilon^{(r)\mu}=\left(\frac{\boldsymbol{k}\boldsymbol{e}^{(r)}}{m},\ \boldsymbol{e}^{(r)}+\frac{\boldsymbol{k}(\boldsymbol{k}\boldsymbol{e}^{(r)})}{m(k^0+m)}\right),\qquad (r=1,2,3) \tag{34}$$

where $\boldsymbol{e}^{(r)}$ are three orthogonal unit space vectors in the rest system of the time like four vector k^{μ},

$$a^{\mu}(\boldsymbol{k})=\sum_r \varepsilon^{(r)\mu}a^{(r)}(\boldsymbol{k}),\ a^{\mu\dagger}(\boldsymbol{k})=\sum_r \varepsilon^{(r)\mu}a^{(r)\dagger}(\boldsymbol{k}). \tag{35}$$

Denoting

$$\varepsilon^{(0)\mu}\equiv k^{\mu}/m, \tag{36}$$

we have the following relations:

$$\begin{aligned}&\varepsilon^{(\rho)\mu}\varepsilon^{(\sigma)}_{\mu}=g^{\rho\sigma},\\ &\varepsilon^{(0)\mu}\varepsilon^{(0)\nu}-\sum_r \varepsilon^{(r)\mu}\varepsilon^{(r)\nu}=g^{\mu\nu}.\end{aligned} \tag{37}$$

Using (37) we can invert (35) as

$$a^{(r)}(\boldsymbol{k})=-a^{\mu}(\boldsymbol{k})\varepsilon^{(r)}_{\mu}. \tag{38}$$

From (33), (37) and (38) we get

$$[a^{(r)}(\boldsymbol{k}),\ a^{(s)\dagger}(\boldsymbol{k}')]=\delta_{rs}(2\pi)^3 2k^0\delta(\boldsymbol{k}-\boldsymbol{k}'). \tag{39}$$

$a^{(r)}(\boldsymbol{k})$ and $a^{(r)\dagger}(\boldsymbol{k})$ are the usual annihilation and creation operators of a vector meson with momentum $\boldsymbol{k}$ and linear polarization $\boldsymbol{e}^{(r)}$. Substituting (31) and (32) into (16) we have

$$J=\int\frac{dk}{(2\pi)^3 2ik^0}:\Big[\Big(a^\dagger(k)-a(-k)-\frac{k}{k^0}(a^{0\dagger}(k)+a^0(-k))\Big)$$

$$\cdot(k\times\nabla_k)\,(a(k)+a^\dagger(-k))+\Big(a^\dagger(k)-a(-k)$$

$$-\frac{k}{k^0}(a^{0\dagger}(k)+a^0(-k))\Big)\times(a(k)+a^\dagger(-k))\Big]:. \tag{40}$$

By examining the symmetry operation $k\to -k$, we can drop the terms quadratic in a ($a^\dagger$) in (40). Substituting (35) into (40) and making use of the following relations:

$$\sum_l\Big(\varepsilon^{(r)l}-\frac{k^l}{k^0}\varepsilon^{(r)0}\Big)\varepsilon^{(s)l}=\delta_{rs},$$

$$\sum_l\Big(\varepsilon^{(r)l}-\frac{k^l}{k^0}\varepsilon^{(r)0}\Big)(k\times\nabla_k)\varepsilon^{(s)l}+\Big(\vec{\varepsilon}^{(r)}-\frac{\vec{k}}{k^0}\varepsilon^{(r)0}\Big)\times\vec{\varepsilon}^{(s)}=e^{(r)}\times e^{(s)}, \tag{41}$$

$$\sum_l\varepsilon^{(r)l}(k\times\nabla_k)\Big(\varepsilon^{(s)l}-\frac{k^l}{k^0}\varepsilon^{(s)0}\Big)+\vec{\varepsilon}^{(r)}\times\Big(\vec{\varepsilon}^{(s)}-\frac{\vec{k}}{k^0}\varepsilon^{(s)0}\Big)=e^{(r)}\times e^{(s)}.$$

we finally get the following result

$$J=\sum_{r,s}\int\frac{dk}{(2\pi)^3 2k^0}a^{(s)\dagger}(k)\langle J\rangle_{sr}a^{(r)}(k) \tag{42}$$

with

$$\langle J\rangle_{sr}=-i\delta_{sr}(k\times\nabla_k)-i(e^{(s)}\times e^{(r)}).$$

To get the angular momentum representation we must make use of the circular polarization;

$$a^{(\pm)}=\mp(a^{(1)}\mp ia^{(2)})/\sqrt{2},$$

$$a^{(\pm)\dagger}=\mp(a^{(1)\dagger}\pm ia^{(2)\dagger})/\sqrt{2}, \tag{43}$$

$$a^{(0)}=a^{(3)},\quad a^{(0)\dagger}=a^{(3)\dagger}.$$

As for the boost generators M^{0k} a similar calculation as that of J^k yields the following answer

$$M^{0k}=-\sum_{r,s}\int\frac{dk}{(2\pi)^3 2k^0}a^{(s)\dagger}(k)\langle K\rangle_{sr}a^{(r)}(k) \tag{44}$$

with

$$\langle K\rangle_{sr}=\delta_{sr}ik^0\nabla_k+\frac{i}{k^0+m}((ke^{(s)})e^{(r)}-(ke^{(r)})e^{(s)}).$$

§ 5. Application

As a simple application of the result in § 3 we shall examine the transformation property of the annihilation operator $a_r(\boldsymbol{p})$ of the Dirac particle. In this section we denote it $a_{re}(\boldsymbol{p})$ to specify the spin quantization axis explicitly.

We consider a three dimensional rotation with rotation parameters $\boldsymbol{\alpha}=\boldsymbol{u}\alpha$. The unit vector $\boldsymbol{u}$ represents the axis of the rotation and α its magnitude; $-\pi<\alpha\leq\pi$. By this rotation the momentum $\boldsymbol{p}$ is transformed to $\boldsymbol{p}'$ and $\boldsymbol{e}$ to $\boldsymbol{e}'$. The unitary transformation corresponding to this rotation is given by $e^{-iJ\alpha}$ with J^k given by (23). By a straightforward calculation we can prove that

$$e^{-iJ\alpha}\,a_{re}(\boldsymbol{p})\,e^{iJ\alpha}=a_{re'}(\boldsymbol{p}'). \tag{45}$$

Eq. (45) shows that $a_{re}(\boldsymbol{p})$ behaves as a scalar operator in accordance with our intuition.

The proof of (45) goes as follows. Calculating multiple commutators $[J\boldsymbol{\alpha}[\cdots[J\boldsymbol{\alpha},\, a_{re}(\boldsymbol{p})]\cdots]$, we write the l.h.s. of (45) as

$$(e^{i\langle J\alpha\rangle}\,a_e(\boldsymbol{p}))_r$$

where $\langle J^k\rangle$ is the two by two matrix with the matrix element given by (24) and $a_e(\boldsymbol{p})$ is two by one matrix. By using (24) we have the followings;

$$\begin{aligned}\text{l.h.s. of (45)} &= \sum_s (w_e^{(r)\dagger}\, e^{\frac{i\sigma\alpha}{2}}\, w_e^{(s)})\, e^{il\alpha}\, a_{se}(\boldsymbol{p}) \\ &= \sum_s (w_{e'}^{(r)\dagger}\, w_e^{(s)}) a_{se}(\boldsymbol{p}') = a_{re'}(\boldsymbol{p}').\end{aligned}$$

q.e.d.

For the boost operators (27) a similar relation as (45) can be proved.

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