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The Structure of Quasi-orthodox Semigroups*)

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A regular semigroup S is said to be quasi-orthodox if there exist an inverse semigroup Γ with basic semilattice Λ (that is, Λ is the semilattice of idempotents of Γ) and a surjective homomorphism $f: S \rightarrow \Gamma$ such that λf^{-1} is a completely simple subsemigroup of S for each $\lambda \in \Lambda$. In this paper, the structure of quasi-orthodox semigroups is studied.

§0. Introduction

As generalizations of groups, there are two important classes of regular semigroups. One is the class of inverse semigroups, and the other is the class of completely simple semigroups. The structure of inverse semigroups has been firstly investigated by Vagner [11] and Preston [8], and successively many papers concerning this class have appeared. On the other hand, a structure theorem for completely simple semigroups has been established by Rees [10]. He has shown that every completely simple semigroup can be obtained, up to isomorphism, as a matrix semigroup called a Rees matrix semigroup over a group. These two classes are generalized to the class of orthodox semigroups and the class of completely regular semigroups[1]) respectively, and quite a lot of papers concerning these two classes have appeared during the last two decades (for example, see Hall [3], [4], [5] and the author [13], [14] etc. for orthodox semigroups; and Clifford [1], Petrich [9] and Lallement [7] etc. for completely regular semigroups). As a class containing both the class of orthodox semigroups and the class of completely regular semigroups, we introduce the class of quasi-orthodox semigroups in this paper and discuss the structure of these semigroups. Throughout this paper, we shall use the following notations and terminology: For a completely regular semigroup M, the notation $M \sim \Sigma\{M_\lambda: \lambda \in \Lambda\}$ means that M is a semilattice Λ of completely simple semigroups $\{M_\lambda: \lambda \in \Lambda\}$ (that is, $M \sim \Sigma\{M_\lambda: \lambda \in \Lambda\}$ means *the structure decomposition of* M). Hereafter, the term "a completely regular semigroup $M \sim \Sigma\{M_\lambda: \lambda \in \Lambda\}$" means that M is a completely regular semigroup and has $M \sim \Sigma\{M_\lambda: \lambda \in \Lambda\}$ as the structure decomposition. If an inverse semigroup Γ has Λ

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1) A semigroup S is said to be completely regular if S is a union of groups. In this case, S is uniquely decomposed to a semilattice of completely simple semigroups; and this decomposition is called the structure decomposition.

as the semilattice of idempotents of Γ (that is, the basic semilattice), we shall denote it by $\Gamma(\Lambda)$. For a regular semigroup S, the notation $E(S)$ denotes the set of idempotents of S. In particular, if S is an orthodox semigroup then $E(S)$ denotes the band of idempotents of S.

§1. Basic properties

Let S be a regular semigroup. If there exist an inverse semigroup $\Gamma(\Lambda)$ and a surjective homomorphism $\phi: S\to\Gamma(\Lambda)$ such that $\lambda\phi^{-1}=S_\lambda$ is a completely simple subsemigroup of S for each $\lambda\in\Lambda$, then S is said to be *quasi-orthodox*. In this case, it is obvious that $M=\cup\{S_\lambda: \lambda\in\Lambda\}$ is a completely regular subsemigroup of S, and the structure decomposition of M is $M\sim\Sigma\{S_\lambda: \lambda\in\Lambda\}$. T. E. Hall has shown the following result (see [19]): *A regular semigroup S is quasi-orthodox if and only if the subsemigroup $\langle E(S)\rangle$ of S generated by $E(S)$ is completely regular.*[2)]

Now, we have the following results concerning quasi-orthodox semigroups:

LEMMA 1.1. *Any homomorphic image of a quasi-orthodox semigroup is a quasi-orthodox semigroup.*

PROOF. Let Γ be a homomorphic image of S. Then there exists a surjective homomorphism $\psi: S\to\Gamma$. Let $\bar{a}$ denotes $a\psi$. For any idempotent $\varepsilon\in\Gamma$, there exists $e\in E(S)$ such that $\bar{e}=\varepsilon$. Hence, $E(\Gamma)=\overline{E(S)}=\{\bar{e}: e\in E(S)\}$. Therefore, the subsemigroup $\langle E(\Gamma)\rangle$ of Γ generated by $E(\Gamma)$ is the same to $\overline{\langle E(S)\rangle}=\{\bar{w}: w\in\langle E(S)\rangle\}$. That is, $\langle E(\Gamma)\rangle=\overline{\langle E(S)\rangle}$. Since $\langle E(S)\rangle$ is a union of groups, $\overline{\langle E(S)\rangle}$ is also a union of groups. Hence, Γ is quasi-orthodox.

LEMMA 1.2. *If ρ is a congruence on a quasi-orthodox semigroup, then S/ρ is a quasi-orthodox semigroup.*

PROOF. This is obvious from Lemma 1.1.

LEMMA 1.3. *A regular subsemigroup of a quasi-orthodox semigroup S is quasi-orthodox.*

PROOF. Let A be a regular subsemigroup of a quasi-orthodox semigroup S. Since $\langle E(S)\rangle$ is a union of groups, $\langle E(S)\rangle$ is a semilattice Λ of completely simple semigroups $\{M_\lambda: \lambda\in\Lambda\}$. Now, $\langle E(A)\rangle\subset\langle E(S)\rangle$. Let $a\in M_\lambda\cap\langle E(A)\rangle$. Since $\langle E(A)\rangle$ is regular, there exists an inverse a^* of a in $\langle E(A)\rangle$. Since $\langle E(S)\rangle\supset\langle E(A)\rangle$, it follows that $a^*\in\langle E(S)\rangle$. Suppose that $a^*\in M_\mu$. Then, $a=aa^*a\in M_\lambda M_\mu M_\lambda\subset M_{\lambda\mu}$, while $a^*=a^*aa^*\in M_\mu M_\lambda M_\mu\subset M_{\lambda\mu}$. Hence, $\lambda=\lambda\mu=\mu$. Therefore, $a^*\in M_\lambda\cap\langle E(A)\rangle$. This implies that $M_\lambda\cap\langle E(A)\rangle$ is regular. Since any idempotent of M_λ is primitive in M_λ, every idempotent of $M_\lambda\cap\langle E(A)\rangle$ is of course primitive in the regular semigroup

2) It follows from [2] that $\langle E(S)\rangle$ is regular.

$M_\lambda \cap \langle E(A)\rangle$. Hence, $M_\lambda \cap \langle E(A)\rangle$ is completely simple. Since $\langle E(A)\rangle = \cup \{M_\lambda \cap \langle E(A)\rangle : \lambda \in \Lambda\}$, $\langle E(A)\rangle$ is a union of groups. Thus, A is quasi-orthodox.

Hereafter, for any element a of a regular semigroup S, $V(a)$ denotes the set of inverses of a.

LEMMA 1.4. *Let A and B be regular subsemigroups of a quasi-orthodox semigroup S. If*

(1.1) *$A \cap B \neq \square$ and $A \cap B \ni a$ implies $a^* \in A \cap B$ for some $a^* \in V(a)$,*

then $A \cap B$ is a quasi-orthodox subsemigroup of S.

PROOF. This is obvious from Lemma 1.3.

Let R and T be regular semigroups, and $f: R \to T$ a homomorphism. If the condition

(1.2) for any $a \in R$ and for any $(af)^* \in V(af)$, there exists $a^* \in V(a)$ such that $a^*f = (af)^*$

is satisfied, then f is called *$*$-homomorphism.*

LEMMA 1.5. *Let S and T be quasi-orthodox semigroups, and $f: S \to T$ a surjective $*$-homomorphism. For any regular subsemigroup K of T, Kf^{-1} is a regular subsemigroup of S. Hence, Kf^{-1} is a quasi-orthodox subsemigroup of S.*

PROOF. Let $a \in Kf^{-1}$. Then, $af \in K$. There exists an inverse $(af)^*$ of af in K. Hence, there exists an inverse a^* of a in S such that $a^*f = (af)^* \in K$. Hence, $a^* \in Kf^{-1}$. It is clear that Kf^{-1} is a subsemigroup of S. Therefore, Kf^{-1} is a regular subsemigroup of S, and it follows from Lemma 1.3 that Kf^{-1} is quasi-orthodox.

If every H-class of a semigroup S consists of a single element, then S is said to be *H-degenerated.*

THEOREM 1.6. *An H-degenerated quasi-orthodox semigroup is an orthodox semigroup.*

PROOF. Let S be an H-degenerated quasi-orthodox semigroup. Then, $\langle E(S)\rangle$ is a union of groups $\{G_\lambda : \lambda \in \Lambda\}$. However, each G_λ is contained in an H-class of S, and accordingly G_λ must be a single element. That is, $\langle E(S)\rangle$ is a band. This implies that $E(S) = \langle E(S)\rangle$. That is, S is orthodox.

Let S be a regular semigroup. A completely regular subsemigroup $G \sim \Sigma\{S_\lambda : \lambda \in \Lambda\}$ is called *a kernel normal system of S* if $G \supset E(S)$ and if there exists a congruence σ on S such that each S_λ $(\lambda \in \Lambda)$ is a complete σ-class. In this case, such a congruence σ is unique and is called *the congruence determined by $G \sim \Sigma\{S_\lambda : \lambda \in \Lambda\}$.* Of course, σ is an inverse semigroup congruence on S. If ρ is an inverse semigroup congruence on a quasi-orthodox semigroup such that $e\rho$ is a completely simple subsemigroup of S

for each $e\in E(S)$, then $G=\cup\{e\rho: e\in E(S)\}$ is a kernel normal system of S and has $G\sim\Sigma\{S_\lambda: \lambda\in\Lambda\}$ as its structure decomposition (where each S_λ is a ρ-class). In this case, the kernel normal system $G\sim\Sigma\{S_\lambda: \lambda\in\Lambda\}$ is called *the kernel normal system determined by* ρ, and conversely ρ is called *an inverse semigroup congruence with kernel normal system (abbrev., k.n.s.).* The completely regular subsemigroup $M\sim\Sigma\{S_\lambda: \lambda\in\Lambda\}$ in Page 2, Line 10 is a kernel normal system, which is called *the kernel normal system determined by* ϕ.

Let S be a quasi-orthodox semigroup. Let G_e be a subgroup (of S) containing e for each $e\in E(S)$ such that $G=\cup\{G_e: e\in E(S)\}$ is a subsemigroup of S. In this case, $G\sim\Sigma\{S_\lambda: \lambda\in\Lambda\}$ is not necessarily a kernel normal system of S. However, at least we have the following:

LEMMA 1.7. *Let S be a regular semigroup, and G_e a subgroup (of S) containing e for each $e\in E(S)$. If $G=\cup\{G_e: e\in E(S)\}$ is a subsemigroup of S, then S is quasi-orthodox.*

PROOF. Suppose that $G=\cup\{G_e: e\in E(S)\}$ is a subsemigroup of S. Of course $G\supset E(S)$, and hence $\langle E(S)\rangle\subset G$. On the other hand, $\langle E(S)\rangle$ is a regular subsemigroup of the completely regular semigroup G. Hence, $\langle E(S)\rangle$ is completely regular (it is well known that any regular subsemigroup of a completely regular semigroup is completely regular). Hence, S is quasi-orthodox.

LEMMA 1.8. *Let S be a quasi-orthodox semigroup. Let σ be an inverse semigroup congruence on S with k.n.s.*

(1) *If $\rho\supset\sigma$ is an inverse semigroup congruence on S with k.n.s., then the relation $\bar\rho$ on S/σ defined by*

$$x\sigma\,\bar\rho\,y\sigma \quad \textit{if and only if} \quad x\,\rho\,y \tag{1.3}$$

is an idempotent separating congruence on S/σ.

(2) *Conversely, if $\bar\rho$ is an idempotent separating congruence on S/σ, then the relation ρ on S defined by*

$$x\,\rho\,y \quad \textit{if and only if} \quad x\sigma\,\bar\rho\,y\sigma \tag{1.4}$$

is an inverse semigroup congruence on S with k.n.s. and satisfies $\sigma\subset\rho$.

PROOF. (1) Assume that $e\sigma\bar\rho f\sigma$ for $e, f\in E(S)$. Since $e\rho f$ and the ρ-class $e\rho$ containing e is a completely simple subsemigroup, $efe\in e\rho$ and there exists the group inverse $(efe)^{-1}$ of efe in a maximal subgroup A of $e\rho$; that is, $efe(efe)^{-1}=(efe)^{-1}efe=g$ and $efe(efe)^{-1}efe=efe$, where g is the identity of the maximal subgroup A containing efe. Since $ge=eg=g$, $g\in e\rho$ and $e\rho$ is completely simple, it follows that $g=e$. Hence, the maximal subgroup M of S/σ containing $e\sigma$ contains $(efe)\sigma$. Similarly, the maximal subgroup N of S/σ containing $f\sigma$ contains $(fef)\sigma$. Since $(efe)\sigma=(fef)\sigma$, it follows that

$N=M$, and hence $e\sigma=f\sigma$.

(2) For $e\in E(S)$, put $e\sigma=\alpha$. Then, $e\rho=\{x\in S\colon e\sigma\bar{\rho}\ni x\sigma\}=\{x\in S\colon \alpha\bar{\rho}\ni x\sigma\}$. Since $\bar{\rho}$ is an idempotent separating congruence, $\alpha\bar{\rho}$ is a subgroup of the H-class (this is a group) of S/σ containing the idempotent α. Hence, for any $x\sigma\in\alpha\bar{\rho}$ there exists $y\sigma\in\alpha\bar{\rho}$ such that $y\sigma x\sigma=x\sigma y\sigma=\alpha$. Since σ is an inverse semigroup congruence, $x^*\sigma=y\sigma$ for any inverse x^* of x. Thus $x^*\in e\rho$. This implies that $e\rho$ is a regular subsemigroup of S. Next, assume that $e\sigma\neq f\sigma$ for $e, f\in E(S)$. Since $\bar{\rho}$ is an idempotent separating congruence on S/σ, $e\rho\neq f\rho$. Hence, $e\rho\cap E(S)=e\sigma\cap E(S)$. Since every idempotent of S contained in $e\sigma$ is primitive, $e\rho$ is a regular semigroup in which every idempotent is primitive. Hence, $e\rho$ is a completely simple subsemigroup of S. It is obvious that S/ρ is an inverse semigroup.

THEOREM 1.9. *If S is a quasi-orthodox semigroup, then there exist a fundamental inverse semigroup $\Gamma(\Lambda)$ and a surjective homomorphism $\psi\colon S\to\Gamma(\Lambda)$ such that $\lambda\psi^{-1}$ is a completely simple subsemigroup of S for each $\lambda\in\Lambda$.*

PROOF. There exist an inverse semigroup $\Gamma(\Lambda)$ and a surjective homomorphism $\phi\colon S\to\Gamma(\Lambda)$ such that $\lambda\phi^{-1}$ is a completely simple subsemigroup of S for each $\lambda\in\Lambda$. Let $\bar{\rho}$ be the maximum idempotent separating congruence on $\Gamma(\Lambda)$, and define ρ on S by

$$x\,\rho\,y \quad \text{if and only if} \quad x\phi\,\bar{\rho}\,y\phi.$$

Then, it follows from Lemma 1.8 that ρ is an inverse semigroup congruence with kernel normal system, and $S/\rho\overset{\Psi}{\cong}\Gamma(\Lambda)/\bar{\rho}$ and $\Gamma(\Lambda)/\bar{\rho}$ is a fundamental inverse semigroup (see [7]), where $\Psi\colon S/\rho\to\Gamma(\Lambda)/\bar{\rho}$ is given by $(x\rho)\Psi=(x\phi)\bar{\rho}$. Let $\upsilon\colon\Gamma(\Lambda)\to\Gamma(\Lambda)/\bar{\rho}$ be the natural homomorphism. For an idempotent $\lambda\bar{\rho}$ of $\Gamma(\Lambda)/\bar{\rho}$, where $\lambda\in\Lambda$, there exists an idempotent y of S such that $\lambda=y\phi$. Now, $S_\lambda=(\lambda\bar{\rho})\upsilon^{-1}\phi^{-1}\ni x\Leftrightarrow x\phi\in(\lambda\bar{\rho})\upsilon^{-1}\Leftrightarrow (x\phi)\upsilon=\lambda\bar{\rho}\Leftrightarrow(x\phi)\bar{\rho}=\lambda\bar{\rho}\Leftrightarrow x\phi\,\bar{\rho}\,y\phi\Leftrightarrow x\,\rho\,y\Leftrightarrow x\in y\rho$. Hence $(\lambda\bar{\rho})\upsilon^{-1}\phi^{-1}=y\rho$. Since y is an idempotent, $y\rho$ is a completely simple semigroup.

THEOREM 1.10. *Let S be a regular semigroup, and G_e a subgroup (of S) containing e for each $e\in E(S)$. If $G=\cup\{G_e\colon e\in E(S)\}$ is a subsemigroup of S, then there exists a kernel normal system $M\sim\Sigma\{S_\lambda\colon\lambda\in\Lambda\}$ of S such that $M\supset G$. Accordingly, S is a quasi-orthodox semigroup.*

PROOF. Suppose that $G=\cup\{G_e\colon e\in E(S)\}$ is a subsemigroup of S. By Lemma 1.7, S is quasi-orthodox. Hence, there exists a surjective homomorphism ψ of S onto a fundamental inverse semigroup $\Gamma(\Lambda)$ such that $\lambda\psi^{-1}=M_\lambda$ is a completely simple subsemigroup for each $\lambda\in\Lambda$. Then, $M=\cup\{M_\lambda\colon\lambda\in\Lambda\}$ is a subsemigroup, and $M\sim\Sigma\{M_\lambda\colon\lambda\in\Lambda\}$ is a kernel normal system of S. Suppose that $G\not\subset\cup\{M_\lambda\colon\lambda\in\Lambda\}=M$. Since $G\psi$ is a subsemigroup of the inverse semigroup $\Gamma(\Lambda)$ and is the union of groups $\{G_e\colon e\in E(S)\}$, $G\psi$ is a semilattice Λ of groups $\{T_\alpha\colon\alpha\in\Lambda\}$. Now, $G\psi\not\subset\Lambda$ since

$G \not\subset \cup \{M_\lambda : \lambda \in \Lambda\}$. Hence, there exists T_α such that T_α is not a single element. Let $\gamma, \delta \in T_\alpha$ such that $\gamma \neq \delta$. Consider the relation $\mu = \{(\xi, \eta) \in \Gamma(\Lambda) \times \Gamma(\Lambda): \xi^{-1}\varepsilon\xi = \eta^{-1}\varepsilon\eta$ for all $\varepsilon \in \Lambda\}$. This μ is the greatest idempotent separating congruence on $\Gamma(\Lambda)$ (see Howie [6]). For γ and δ, $\gamma^{-1}\varepsilon\gamma = \delta^{-1}\varepsilon\delta$ for all $\varepsilon \in \Lambda$ (since both $\gamma^{-1}\varepsilon\gamma$ and $\delta^{-1}\varepsilon\delta$ are idempotents and contained in the same $T_{\alpha\beta}$, if $\varepsilon \in T_\beta$). Therefore, $(\gamma, \delta) \in \mu$. Since $\Gamma(\Lambda)$ is a fundamental inverse semigroup, μ must be trivial. That is, $\gamma = \delta$. This contradicts to $\gamma \neq \delta$. Hence $G \subset \cup \{M_\lambda : \lambda \in \Lambda\} = M$.

Let S be a regular semigroup, and M_e a maximal subgroup (of S) containing e for each $e \in E(S)$. If $M = \cup \{M_e : e \in E(S)\}$ is a subsemigroup of S, then S is said to be *natural regular*. By the result above, in this case $M \sim \Sigma\{S_\lambda : \lambda \in \Lambda\}$ is a kernel normal system of S. Of course, a natural regular semigroup is quasi-orthodox.

For the kernel normal systems of regular subsemigroups of a quasi-orthodox semigroup, we have the following result:

LEMMA 1.11. *Let S be a quasi-orthodox semigroup, and $N \sim \Sigma\{S_\lambda : \lambda \in \Lambda\}$ a kernel normal system of S. Let A be a regular subsemigroup of S, and put $\Lambda' = \{\lambda \in \Lambda : S_\lambda \cap A \neq \square\}$. Then, $A \cap N \sim \Sigma\{A \cap S_{\lambda'} : \lambda' \in \Lambda'\}$ is a kernel normal system of A.*

PROOF. Put $S_\lambda \cap A = A_\lambda$ for each $\lambda \in \Lambda$ such that $S_\lambda \cap A \neq \square$. There exists an inverse semigroup $\Gamma(\Lambda)$ and a surjective homomorphism $\psi: S \to \Gamma(\Lambda)$ such that $\lambda\psi^{-1} = S_\lambda$ for $\lambda \in \Lambda$. If $a \in A_\lambda$, then $a\psi = \bar{a}$ is an idempotent. There exists an inverse a^* of a in A. Since $\bar{a}^* = a^*\psi$ is an inverse of $\bar{a}$, it follows that $\bar{a} = \bar{a}^*$. Hence, $a^* \in A_\lambda$. Since $a^* \in S_\lambda \cap A = A_\lambda$, A_λ is a regular subsemigroup of A. Hence, $N \cap A$ is a regular subsemigroup of the completely regular semigroup N, and accordingly $N \cap A$ is completely regular. Of course, $N \cap A \sim \Sigma\{S_{\lambda'} \cap A : \lambda' \in \Lambda'\}$, where $\Lambda' = \{\lambda \in \Lambda : S_\lambda \cap A \neq \square\}$, is the structure decomposition of $N \cap A$. Now, consider $\bar{\psi}: A \to \Gamma(\Lambda)$ defined by $a\bar{\psi} = a\psi$ for $a \in A$. Then, $\bar{\psi}$ is a surjective homomorphism of A onto $\Gamma'(\Lambda')$, where $\Gamma'(\Lambda') = A\bar{\psi}$. For any $\lambda \in \Lambda'$, $\lambda\bar{\psi}^{-1} = \lambda\psi^{-1} \cap A = S_\lambda \cap A = A_\lambda$. Therefore, $N \cap A \sim \Sigma\{S_{\lambda'} \cap A : \lambda' \in \Lambda'\}$ is a kernel normal system of A.

Let S be a regular semigroup. Let $M \sim \Sigma\{M_\lambda : \lambda \in \Lambda\}$ be a completely regular semigroup, and $\Gamma(\Lambda)$ an inverse semigroup. If

(1) M is a subsemigroup of S, and
(2) there exists a surjective homomorphism $f: S \to \Gamma(\Lambda)$ such that $\lambda f^{-1} = M_\lambda$ for each $\lambda \in \Lambda$,

then S is called *a regular extension of $M \sim \Sigma\{M_\lambda : \lambda \in \Lambda\}$ by $\Gamma(\Lambda)$*.

Next, we show some characterizations of a quasi-orthodox semigroup:

THEOREM 1.12. *For a regular semigroup S, the following five conditions are equivalent:*

(1) *S is a quasi-orthodox semigroup.*
(2) *The subsemigroup $\langle E(S) \rangle$ is completely regular.*

(3) *There exists a completely regular subsemigroup C of S such that $S \supset C \supset E(S)$.*

(4) *S is a regular extension of a completely regular semigroup $M \sim \Sigma\{S_\lambda: \lambda \in \Lambda\}$ by an (fundamental) inverse semigroup $\Gamma(\Lambda)$.*

(5) *S has a kernel normal system; and accordingly, there exists an inverse semigroup congruence (on S) with kernel normal system.*

PROOF. The part "(1)⇒(2)⇒(3)" is obvious. The part "(3)⇒(1)" follows from Lemma 1.7, while the part "(1)⇒(4)⇒(5)⇒(1)" is also obvious.

Now, we can infer from the results in this section that an analogue to the process used in [13], [3], [14] and [5] for the study of orthodox semigroups will be applicable for the theory of quasi-orthodox semigroups. In the following sections, we shall discuss the structure of quasi-orthodox semigroups under this direction.

§2. Inverse semigroup congruences with k. n. s.

If S is a quasi-orthodox semigroup, then it has been seen from Theorem 1.9 that there exists an inverse semigroup congruence ρ_S with kernel normal system such that S/ρ_S is a fundamental inverse semigroup.

THEOREM 2.1. *ρ_S is the maximum inverse semigroup congruence (on S) with kernel normal system.*

PROOF. Let ρ be an inverse semigroup congruence (on S) with kernel normal system such that $\rho \supset \rho_S$. Then, $\bar{\rho}$ defined by "$x \rho y \Leftrightarrow x\rho_S \bar{\rho}\, y\rho_S$ $(x, y \in S)$" is an idempotent separating congruence on S/ρ_S. Since S/ρ_S is a fundamental inverse semigroup, $\bar{\rho}$ must be the identity congruence. Hence, $\rho = \rho_S$.

Let N_S be the set of all inverse semigroup congruences (on S) with kernel normal system: $N_S = \{\rho_\omega: \omega \in \Omega\}$. It is easily verified that $\eta_S = \cap\, \{\rho_\omega: \omega \in \Omega\}$ is an inverse semigroup congruence on S.

Moreover,

LEMMA 2.2. *η_S is an inverse semigroup congruence with kernel normal system. Accordingly, η_S is the least inverse semigroup congruence (on S) with kernel normal system.*

PROOF. As was stated above, η_S is an inverse semigroup congruence on S. Hence, it is needed only to show that $e\eta_S$ is a completely simple subsemigroup of S for any $e \in E(S)$. Let $x \in e\eta_S$. Then, $x\eta_S = e\eta_S$. Let x^* be an inverse of x. Since $e\eta_S$ is an idempotent of the inverse semigroup S/η_S, $x^*\eta_S = e\eta_S$. Hence, $x^* \in e\eta_S$. This implies that $e\eta_S$ is a regular subsemigroup of S. If $f \in E(S) \cap e\eta_S$, then $f \in e\rho_\omega$ for all $\omega \in \Omega$. Since $e\rho_\omega$ is a completely simple subsemigroup of S, f is a primitive idempotent of $e\rho_\omega$. Since $e\rho_\omega \supset e\eta_S$, f is also a primitive idempotent of $e\eta_S$. Hence, $e\eta_S$ is a regular semigroup in which every idempotent is primitive. Therefore, $e\eta_S$ is completely simple.

REMARK. It is also seen from the proof of Lemma 2.2 that the intersection of any collection of inverse semigroup congruences (on S) with kernel normal system is also an inverse semigroup congruence (on S) with kernel normal system.

Let S be a quasi-orthodox semigroup. It is easily seen that the intersection τ_S of all inverse semigroup congruences on S is the least inverse semigroup congruence on S.

Now,

THEOREM 2.3. $\eta_S = \tau_S$.

PROOF. It is obvious that $\tau_S \subset \eta_S$. It is also easily verified that $e\tau_S$ is a regular subsemigroup of S for each $e \in E(S)$. Since $e\tau_S \subset e\eta_S$ and each idempotent of $e\eta_S$ is primitive, $e\tau_S$ is a regular semigroup in which every idempotent is primitive. Hence, $e\tau_S$ is completely simple. This implies that τ_S is an inverse semigroup congruence (on S) with kernel normal system. Hence, $\eta_S = \tau_S$.

Let σ be an inverse semigroup congruence on S with k.n.s., and put $C_\sigma(S) = \{\rho$: ρ is an inverse semigroup congruence on S with k.n.s. such that $\rho \supset \sigma\}$. On the other hand, let $I_\sigma(S)$ be the set of all idempotent separating congruences on S/σ. Then,

THEOREM 2.4.

(1) *For any* $\rho \in C_\sigma(S)$, *the congruence* $\bar{\rho}$ *defined by*

$$(2.1) \qquad x\sigma\, \bar{\rho}\, y\sigma \quad \textit{if and only if} \quad x\,\rho\, y$$

is an element of $I_\sigma(S)$.

(2) *For any* $\bar{\tau} \in I_\sigma(S)$, *the congruence* τ *defined by*

$$(2.2) \qquad x\,\tau\, y \quad \textit{if and only if} \quad x\sigma\, \bar{\tau}\, y\sigma$$

is an element of $C_\sigma(S)$.

(3) *The mapping* ψ: $C_\sigma(S) \to I_\sigma(S)$ *defined by* $\rho\psi = \bar{\rho}$ *is an order-preserving bijection (where ordering in each of* $C_\sigma(S)$ *and* $I_\sigma(S)$ *is given by the set-inclusion).*

PROOF. This is obvious from Lemma 1.8.

In particular, consider the case where $\sigma = \eta_S$. Then, ψ in Theorem 2.4 is an order-preserving bijection of the set of all inverse semigroup congruences on S with k.n.s. onto the set of all idempotent separating congruences on S/η_S. Hence, if $\bar{\zeta}_S$ is the greatest idempotent separating congruence on S/η_S then ζ_S given by (2.2) is the maximum inverse semigroup congruence on S with k.n.s. Hence, $\zeta_S = \rho_S$.

Next for $\Delta \subset C_{\eta_S}(S)$, put $\rho_\Delta = \cap \{\rho : \rho \in \Delta\}$. Then $\rho_\Delta \in C_{\eta_S}(S)$. Accordingly, if $\sigma \cup \rho$ is defined by $\sigma \cup \rho = \cap \{\tau \in C_{\eta_S}(S) : \tau \supset \sigma, \rho\}$ for $\sigma, \rho \in C_{\eta_S}(S)$ then $C_{\eta_S}(S)$ is a complete lattice with respect to $\cap$, $\cup$. Similarly, $I_{\eta_S}(S)$ is a complete lattice with respect to the ordering defined by set-inclusion.

COROLLARY 2.5. *The complete lattices $C_{\eta_S}(S)$ and $I_{\eta_S}(S)$ are lattice isomorphic.*

PROOF. Obvious.

THEOREM 2.6. *Let a quasi-orthodox semigroup S be a regular extension of a completely regular semigroup $G \sim \Sigma\{S_\lambda : \lambda \in \Lambda\}$ by an inverse semigroup $\Gamma(\Lambda)$. If $\Gamma(\Lambda)$ is H-degenerated, then G is the union of all maximal subgroups of S. Conversely, if G is the union of all maximal subgroups of S then $\Gamma(\Lambda)$ is H-degenerated.*

PROOF. There exists a surjective homomorphism $\phi: S \to \Gamma(\Lambda)$ such that $\lambda\phi^{-1} = S_\lambda$ for every $\lambda \in \Lambda$. Firstly assume that $\Gamma(\Lambda)$ is H-degenerated. For a subgroup M of S, $M\phi$ is a subgroup of $\Gamma(\Lambda)$ since $M\phi$ is a homomorphic image of a group. Since $\Gamma(\Lambda)$ is H-degenerated, $M\phi$ must be a single element. Hence $M \subset \Lambda\phi^{-1} = G$. This implies that G is the union of all maximal subgroups of S. Conversely, assume that G is the union of all maximal subgroups of S. Suppose that there exists an H-class H_λ of $\Gamma(\Lambda)$ which contains $\lambda \in \Lambda$ and at least two elements. Of course, in this case H_λ is a group having λ as its identity. If $H_\lambda\phi^{-1} \ni x$, then there exists $y\phi \in H_\lambda$ such that $(y\phi)(x\phi) = (x\phi)(y\phi) = \lambda$ since H_λ is a group. On the other hand, if x^* is an inverse of x then $x^*\phi$ is an inverse of $x\phi$. Since $\Gamma(\Lambda)$ is an inverse semigroup, $x^*\phi = y\phi$. That is, $x^* \in H_\lambda\phi^{-1}$. Let $H_\lambda\phi^{-1} = G_\lambda$. Then, $xx^*, x^*x \in S_\lambda \subset G_\lambda$ for $x \in G_\lambda$. Let B be an ideal of G_λ. Then, $B \cap S_\lambda \neq \square$, and hence $B \cap S_\lambda$ is an ideal of S_λ. Since S_λ is simple, $S_\lambda \subset B$. For any $x \in G_\lambda$, $xx^* \in S_\lambda \subset B$. Therefore, $x = xx^*x \in B$. Thus, $G_\lambda = B$. This implies that G_λ is a simple subsemigroup of S. Since it is easily seen that $E(G_\lambda) = E(S_\lambda)$, it follows that G_λ is a simple subsemigroup in which every idempotent is primitive. Hence, G_λ is a completely simple semigroup. Therefore, $G_\lambda \subset G$, and hence $G_\lambda\phi = H_\lambda \subset \Lambda$. This contradicts the assumption that H_λ consists of at least two elements. Hence, $\Gamma(\Lambda)$ is H-degenerated.

If Green's H-relation is a congruence on a semigroup T, then T is said to be *H-compatible.*

THEOREM 2.7. *Let S be an H-compatible regular semigroup. For each $a \in S$, let H_a be the H-class of S containing a. If for any $e, f \in E(S)$ there exists $u \in E(S)$ such that*

$$H_e H_f \subset H_u,$$

then S is a quasi-orthodox semigroup in which the union of maximal subgroups is a kernel normal system.

PROOF. This is obvious from Theorem 1.12 and the fact that $G = \cup \{H_e : e \in E(S)\}$ is a subsemigroup of S.

§ 3. Construction

In this section, we shall consider the construction of quasi-orthodox semigroups. A construction theorem for general quasi-orthodox semigroups has been given by the author [16], but it is given in a somewhat complicated form and we omit to show it in this paper. We shall only consider the construction of some special quasi-orthodox semigroups called

I. an upwards [downwards] directed quasi-orthodox semigroup, and

II. a split quasi-orthodox semigroup.

First, we introduce the concept of a partial chain as follows:

Let Λ be a semilattice, and T_λ a semigroup for each $\lambda \in \Lambda$. If a partial binary operation $\circ$ is defined in $T=\Sigma\{T_\lambda: \lambda \in \Lambda\}$ (disjoint sum)[3] such that

(1) $\lambda \geq \tau$, $a \in T_\lambda$ and $b \in T_\tau$ imply that $a \circ b$ $[b \circ a]$ is defined and $a \circ b$ $[b \circ a] \in T_\tau$,

(2) $a, b \in T_\lambda$ implies $a \circ b = ab$ (the product of a, b in T_λ), and

(3) $\lambda \geq \tau \geq \delta$, $a \in T_\lambda$, $b \in T_\tau$ and $c \in T_\delta$ imply $a \circ (b \circ c) = (a \circ b) \circ c$ $[(c \circ b) \circ a = c \circ (b \circ a)]$,

then the resulting system $T(\circ)$ is called *a lower* [*upper*] *partial chain* Λ *of* $\{T_\lambda: \lambda \in \Lambda\}$. We denote it by $T=LP\{T_\lambda: \lambda \in \Lambda; \circ\}$ $[T=UP\{T_\lambda: \lambda \in \Lambda; \circ\}]$.

First, we consider the construction of all $LP\{T_\lambda: \lambda \in \Lambda; \circ\}$ for a given semilattice Λ and for given right reductive semigroups $\{T_\lambda: \lambda \in \Lambda\}$. This has been given by the author [20] as follows: If G is a right reductive semigroup, then the inner left translation semigroup $\Lambda_0(G)$ of G is a left ideal of the left translation semigroup $\Lambda(G)$ of G and the mapping $\phi: G \to \Lambda(G)$ defined by $a\phi = \lambda_a$ (where λ_a is the inner left translation of G induced by a) is an injective homomorphism. Hereafter, $D(G)$ denotes an isomorphic copy of $\Lambda(G)$ such that $D(G)$ contains G as its left ideal and there exists an isomorphism $\phi_G: D(G) \to \Lambda(G)$ satisfying $a\phi_G = \lambda_a$ for $a \in G$.

Then,

THEOREM. *Let Λ be a semilattice, and S_λ a right reductive semigroup for each $\lambda \in \Lambda$. For every pair (α, β) of $\alpha, \beta \in \Lambda$ with $\alpha \geq \beta$, let $\phi_{\alpha,\beta}: S_\alpha \to D(S_\beta)$ be a homomorphism such that the family $\{\phi_{\alpha,\beta}: \alpha \geq \beta, \alpha, \beta \in \Lambda\}$ satisfies the following* (3.1) *and* (3.2):

(3.1) *$\phi_{\lambda,\lambda}$ is the identity mapping on S_λ for each $\lambda \in \Lambda$,*

(3.2) *$(a\phi_{\alpha,\beta} * b)\phi_{\beta,\gamma} = (a\phi_{\alpha,\gamma}) * (b\phi_{\beta,\gamma})$ for $\alpha \geq \beta \geq \gamma$, $a \in S_\alpha$ and $b \in S_\beta$, where $*$ denotes the multiplication in $D(S_\tau)$ $(\tau \in \Lambda)$,*

then $S=\Sigma\{S_\lambda: \lambda \in \Lambda\}$ becomes a lower partial chain Λ of $\{S_\lambda: \lambda \in \Lambda\}$ under the partial binary operation $\circ$ defined by

3) Hereafter Σ means "disjoint sum".

(3.3) $$a\circ b=(a\phi_{\alpha,\beta})*b \quad \text{for} \quad \alpha\geq\beta,\ a\in S_\alpha,\ b\in S_\beta.$$

Further, every $LP\{S_\lambda: \lambda\in\Lambda; \circ\}$ can be obtained in this way.

Dually, we can construct every $UP\{S_\lambda: \lambda\in\Lambda; \circ\}$ for a given semilattice Λ and left reductive semigroups $\{S_\lambda: \lambda\in\Lambda\}$.

Let S be a quasi-orthodox semigroup, and $G\sim\Sigma\{G_\lambda: \lambda\in\Lambda\}$ a kernel normal system of S. Then, there exist an inverse semigroup $\Gamma(\Lambda)$ and a surjective homomorphism ϕ: $S\rightarrow\Gamma(\Lambda)$ such that $\lambda\phi^{-1}=G_\lambda$ for all $\lambda\in\Lambda$. Put $\gamma\phi^{-1}=S_\gamma$ for $\gamma\in\Gamma(\Lambda)$. Let L_λ and R_λ^* be an L-class and an R-class of G_λ respectively, and put $E(R_\lambda^*)=R_\lambda$. Then, by [16], $L[\Lambda]=\Sigma\{L_\lambda: \lambda\in\Lambda\}$ $[R^*[\Lambda]=\Sigma\{R_\lambda^*: \lambda\in\Lambda\}]$ is a lower upper [partial] chain Λ of $\{L_\lambda: \lambda\in\Lambda\}$ $[\{R_\lambda^*: \lambda\in\Lambda\}]$ with respect to the multiplication in S.

In [16], the following has been proved:

THEOREM. *Let u_γ be a representative of S_γ for $\gamma\in\Gamma(\Lambda)$. Then, for any $a\in S_\gamma$ there exist a unique x and a unique y such that $x\in L_{\gamma\gamma^{-1}}$, $y\in R_{\gamma^{-1}\gamma}$ and $a=xu_\gamma y$.*

I. The construction of upwards [downwards] directed quasi-orthodox semigroups

Let S be a quasi-orthodox semigroup, and $G_1\sim\Sigma\{G_{1,\lambda}: \lambda\in\Lambda\}$, $G_2\sim\Sigma\{G_{2,\delta}: \delta\in\Delta\}$ kernel normal systems of S. Let ρ_1, ρ_2 be the inverse semigroup congruences determined by these kernel normal systems $G_1\sim\Sigma\{G_{1,\lambda}: \lambda\in\Lambda\}$ and $G_2\sim\Sigma\{G_{2,\delta}: \delta\in\Delta\}$ respectively. Let $\rho_1\cap\rho_2=\rho_3$. Then, as was shown above this ρ_3 is also an inverse semigroup congruence with kernel normal system. Let $G_3\sim\Sigma\{G_{3,\omega}: \omega\in\Omega\}$ be the kernel normal system determined by ρ_3. Define a congruence $\bar\rho_i$ $(i=1, 2)$ on S/ρ_3 as follows:

$$a\rho_3\,\bar\rho_i\,b\rho_3 \quad \text{if and only if} \quad a\,\rho_i\,b.$$

As was already shown above, $\bar\rho_i$ is an idempotent separating congruence. Now, if $G_{3,\eta}$, $G_{3,\mu}$ $(\eta\neq\mu)\subset G_{1,\lambda}$ then $e\rho_3\neq f\rho_3$ and $e\rho_3\,\bar\rho_1 f\rho_3$ for $e\in E(G_{3,\eta})$ and $f\in E(G_{3,\mu})$. This contradicts to the fact that $\bar\rho_1$ is an idempotent separating congruence. Hence, for any $\lambda\in\Lambda$ there exists a unique $\lambda'\in\Omega$ such that $G_{3,\lambda'}\subset G_{1,\lambda}$. Therefore, the mapping ψ: $\Lambda\rightarrow\Omega$ defined by $\lambda\psi=\lambda'$ gives an isomorphism, and accordingly we can assume $\Lambda=\Omega$ if λ' is identified with λ. Further, in this case $E(G_{1,\omega})=E(G_{3,\omega})$ holds for all $\omega\in\Omega$. Similarly, we can assume that $\Delta=\Omega$ and $E(G_{2,\omega})=E(G_{3,\omega})$ for all $\omega\in\Omega$. Hence, we have $\Delta=\Omega=\Lambda$ and $E(G_{1,\omega})=E(G_{2,\omega})$ for all $\omega\in\Omega$.

Thus, we have the following result:

LEMMA 3.1. *Let S be a quasi-orthodox semigroup, and $G\sim\Sigma\{G_{1,\lambda}: \lambda\in\Lambda\}$, $G_2\sim\Sigma\{G_{2,\delta}: \delta\in\Delta\}$ kernel normal systems of S. Then there exists an isomorphism ψ: $\Lambda\rightarrow\Delta$ such that $E(G_{1,\lambda})=E(G_{2,\lambda\psi})$ for all $\lambda\in\Lambda$.*

Let S be a quasi-orthodox semigroup, and $G\sim\Sigma\{G_\lambda: \lambda\in\Lambda\}$ a kernel normal system of S. If

$$E(G_\lambda)E(G_\mu)\subset E(G_\lambda) \quad \text{for} \quad \lambda<\mu \tag{3.4}$$

$$[E(G_\mu)E(G_\lambda)\subset E(G_\lambda) \quad \text{for} \quad \lambda<\mu]$$

then S is said to be *upwards directed* [*downwards directed*].

It is easy to see from Lemma 3.1 that this concept is independent from the selection of a kernel normal system of S. A natural regular semigroup introduced by Warne [12] and a completely regular semigroup are quasi-orthodox semigroups, and an orthodox semigroup and a completely simple semigroup are both an upwards directed quasi-orthodox semigroup and a downwards directed quasi-orthodox semigroup. Hereafter, we shall investigate the construction of upwards directed quasi-orthodox semigroups.

Now, let $\Gamma(\Lambda)$ be an inverse semigroup with basic semilattice Λ. Let $L[\Lambda]=\Sigma\{L_\lambda:\lambda\in\Lambda\}$ be a lower partial chain Λ of left groups $\{L_\lambda:\lambda\in\Lambda\}$, and $R[\Lambda]=\Sigma\{R_\lambda:\lambda\in\Lambda\}$ an upper partial chain Λ of right zero semigroups $\{R_\lambda:\lambda\in\Lambda\}$. Further, assume that L_λ, R_λ have an idempotent u_λ as their common element for each $\lambda\in\Lambda$; that is, $L_\lambda\cap R_\lambda=\{u_\lambda\}$, and $u_\lambda^2=u_\lambda$ in both L_λ and R_λ for each $\lambda\in\Lambda$.

For each pair of γ, $\delta\in\Gamma(\Lambda)$, let

$$f_{\langle\gamma,\delta\rangle}:\ R_{\gamma^{-1}\gamma}\times L_{\delta\delta^{-1}}\longrightarrow L_{\gamma\delta(\gamma\delta)^{-1}} \quad \text{and}$$

$$g_{\langle\gamma,\delta\rangle}:\ R_{\gamma^{-1}\gamma}\times L_{\delta\delta^{-1}}\longrightarrow R_{(\gamma\delta)^{-1}\gamma\delta}$$

be mappings. Let us introduce two conditions I and II for the set $\Delta=\{f_{\langle\gamma,\delta\rangle}:\gamma,\delta\in\Gamma(\Lambda)\}\cup\{g_{\langle\gamma,\delta\rangle}:\gamma,\delta\in\Gamma(\Lambda)\}$ as follows:

(3.4)
(I) For $a\in L_{\gamma\gamma^{-1}}$, $e\in R_{\gamma^{-1}\gamma}$, $b\in L_{\delta\delta^{-1}}$, $f\in R_{\delta^{-1}\delta}$, $c\in L_{\tau\tau^{-1}}$ and $h\in R_{\tau^{-1}\tau}$,

$$a(e,b((f,c)f_{\langle\delta,\tau\rangle}))f_{\langle\gamma,\delta\tau\rangle}=a((e,b)f_{\langle\gamma,\delta\rangle})((e,b)g_{\langle\gamma,\delta\rangle}f,c)f_{\langle\gamma\delta,\tau\rangle}.$$

(II) For $a\in L_{\gamma\gamma^{-1}}$, $e\in R_{\gamma^{-1}\gamma}$, there exist $b\in L_{\gamma^{-1}\gamma}$ and $f\in R_{\gamma\gamma^{-1}}$ such that $(e,b)f_{\langle\gamma,\gamma^{-1}\rangle}(f,a)f_{\langle\gamma\gamma^{-1},\gamma\rangle}\in E(L_{\gamma\gamma^{-1}})$.

If Δ satisfies (3.4), then Δ is called a *factor set of* $\{L[\Lambda], R[\Lambda]\}$ *belonging to* $\Gamma(\Lambda)$. Assume that Δ above satisfies (3.4). Then,

LEMMA 3.2. *$S=\{(x,\gamma,e): x\in L_{\gamma\gamma^{-1}}, e\in R_{\gamma^{-1}\gamma}, \gamma\in\Gamma(\Lambda)\}$ is a quasi-orthodox semigroup under the multiplication defined by*

$$(x,\gamma,e)(y,\delta,f)=(x(e,y)f_{\langle\gamma,\delta\rangle},\gamma\delta,(e,y)g_{\langle\gamma,\delta\rangle}f). \tag{3.5}$$

PROOF. By using the condition (I), it is easily verified that S is a semigroup. By the condition (II), for any (x,γ,e) of S there exist $y\in L_{\gamma^{-1}\gamma}$ and $h\in R_{\gamma\gamma^{-1}}$ such that $(e,y)f_{\langle\gamma,\gamma^{-1}\rangle}(h,x)f_{\langle\gamma\gamma^{-1},\gamma\rangle}\in E(L_{\gamma\gamma^{-1}})$. Now, $(x,\gamma,e)(y,\gamma^{-1},h)(x,\gamma,e)=$

$(x(e, y)f_{\langle\gamma,\gamma^{-1}\rangle}((e, y)g_{\langle\gamma,\gamma^{-1}\rangle}h, x)f_{\langle\gamma\gamma^{-1},\gamma\rangle}, \gamma, ((e, y)g_{\langle\gamma,\gamma^{-1}\rangle}h, x)g_{\langle\gamma\gamma^{-1},\gamma\rangle}e)$. Since $(e, y)g_{\langle\gamma,\gamma^{-1}\rangle}\in R_{\gamma\gamma^{-1}}$ and $h\in R_{\gamma\gamma^{-1}}$, $(e, y)g_{\langle\gamma,\gamma^{-1}\rangle}h=h$. Since $x\in L_{\gamma\gamma^{-1}}$, we have $x(e, y)f_{\langle\gamma,\gamma^{-1}\rangle}((e, y)g_{\langle\gamma,\gamma^{-1}\rangle}h, x)f_{\langle\gamma\gamma^{-1},\gamma\rangle}=x$. On the other hand, $(e, y)g_{\langle\gamma,\gamma^{-1}\rangle}\in R_{\gamma\gamma^{-1}}$, $(h, x)g_{\langle\gamma\gamma^{-1},\gamma\rangle}\in R_{\gamma^{-1}\gamma}$ and $e\in R_{\gamma^{-1}\gamma}$. Hence, we obtain $((e, y)g_{\langle\gamma,\gamma^{-1}\rangle}h, x)g_{\langle\gamma\gamma^{-1},\gamma\rangle}e=e$. Consequently, $(x, \gamma, e)(y, \gamma^{-1}, h)(x, \gamma, e)=(x, \gamma, e)$. This implies that S is regular. Next, the mapping ϕ: $S\to\Gamma(\Lambda)$ defined by $(x, \gamma, e)\phi=\gamma$ is clearly a surjective homomorphism. Put $\{(x, \lambda, u): x\in L_\lambda, u\in R_\lambda\}=S_\lambda$ for each $\lambda\in\Lambda$. It is obvious that S_λ is a regular subsemigroup of S. Let $(z, \lambda, u)\in E(S_\lambda)$. $(z, \lambda, u)^2=(z, \lambda, u)$ implies $(z(u, z)f_{\langle\lambda,\lambda\rangle}, \lambda, u)=(z, \lambda, u)$. Hence, $z(u, z)f_{\langle\lambda,\lambda\rangle}=z$. Since L_λ is a left group, $(u, z)f_{\langle\lambda,\lambda\rangle}$ is an idempotent. Therefore, $(z, \lambda, u)\in E(S_\lambda)$ if and only if $(u, z)f_{\langle\lambda,\lambda\rangle}\in E(L_\lambda)$. For $(z, \lambda, u), (w, \lambda, v)\in E(S_\lambda)$, $(z, \lambda, u)(w, \lambda, v)=(w, \lambda, v)(z, \lambda, u)=(z, \lambda, u)$ implies $(z(u, w)f_{\langle\lambda,\lambda\rangle}, \lambda, v)=(w(v, z)f_{\langle\lambda,\lambda\rangle}, \lambda, u)=(z, \lambda, u)$. Hence, $u=v$ and $z(u, w)f_{\langle\lambda,\lambda\rangle}=w(v, z)f_{\langle\lambda,\lambda\rangle}=z$. Since $u=v$, $(v, z)f_{\langle\lambda,\lambda\rangle}=(u, z)f_{\langle\lambda,\lambda\rangle}$, and hence $(v, z)f_{\langle\lambda,\lambda\rangle}\in E(L_\lambda)$. Therefore $w(v, z)f_{\langle\lambda,\lambda\rangle}=w$, and hence $z=w$. Consequently $(z, \lambda, u)=(w, \lambda, v)$. Thus, S_λ is a regular semigroup in which every idempotent is primitive. That is, S_λ is a completely simple subsemigroup of S. Accordingly, S is quasiorthodox.

The semigroup S in Lemma 3.2 is called *the regular product of* $L[\Lambda]$, $\Gamma(\Lambda)$ *and* $R[\Lambda]$ *determined by* Δ, and denoted by $S=R(L[\Lambda]\times\Gamma(\Lambda)\times R[\Lambda]; \Delta)$. This concept is a generalization of the concept of a quasi-direct product introduced by [13], an H.D.-product introduced by [15] and a regular product introduced by [17] for orthodox semigroups.

Further, let us introduce the following conditions for the factor set Δ:

(3.6)

(A) $(u, z)f_{\langle\lambda,\lambda\rangle}\in E(L_\lambda)$, $(v, w)f_{\langle\mu,\mu\rangle}\in E(L_\mu)$ and $\lambda<\mu$ imply $((u, w)g_{\langle\lambda,\mu\rangle}v, z(u, w)f_{\langle\lambda,\mu\rangle})f_{\langle\lambda,\lambda\rangle}\in E(L_\lambda)$.

(Upwards directed condition)

(B) For any $\gamma, \delta\in\Gamma(\Lambda)$,

$(u_{\gamma^{-1}\gamma}, u_{\delta\delta^{-1}})f_{\langle\gamma,\delta\rangle}=u_{\gamma\delta(\gamma\delta)^{-1}}$ and $(u_{\gamma^{-1}\gamma}, u_{\delta\delta^{-1}})g_{\langle\gamma,\delta\rangle}=u_{(\gamma\delta)^{-1}\gamma\delta}$.

(Splitting condition)

LEMMA 3.3. *If the factor set* Δ *satisfies* (A), *then* S *in Lemma* 3.2 *is upwards directed.*

PROOF. Suppose that $\lambda<\mu$, $\lambda, \mu\in\Lambda$. If (z, λ, u) and (w, μ, v) are idempotents, then $(u, z)f_{\langle\lambda,\lambda\rangle}\in E(L_\lambda)$ and $(v, w)f_{\langle\lambda,\lambda\rangle}\in E(L_\lambda)$. Hence, $(z, \lambda, u)(w, \mu, v)=(z(u, w)f_{\langle\lambda,\mu\rangle}, \lambda, (u, w)g_{\langle\lambda,\mu\rangle}v)$, and $((u, w)g_{\langle\lambda,\mu\rangle}v, z(u, w)f_{\langle\lambda,\mu\rangle})f_{\langle\lambda,\lambda\rangle}\in E(L_\lambda)$ follows from the condition (A). Therefore, $E(S_\lambda)E(S_\mu)\subset E(S_\lambda)$. That is, S is upwards directed.

By using Lemmas 3.2 and 3.3, we obtain the following theorem:

THEOREM 3.4. *A regular semigroup S is an upwards directed quasi-orthodox semigroup if and only if S is isomorphic to some $R(L[\Lambda]\times\Gamma(\Lambda)\times R[\Lambda];\Delta)$, where $L[\Lambda]$ is a lower partial chain Λ of left groups $\{L_\lambda: \lambda\in\Lambda\}$, $\Gamma(\Lambda)$ is an inverse semigroup, $R[\Lambda]$ is an upper partial chain Λ of right zero semigroups $\{R_\lambda: \lambda\in\Lambda\}$, and Δ is a factor set of $\{L[\Lambda], R[\Lambda]\}$ belonging to $\Gamma(\Lambda)$ which satisfies the condition (A).*

PROOF. The "only if" part is obvious. The "if" part: Suppose that S is an upwards directed quasi-orthodox semigroup. There exist an inverse semigroup $\Gamma(\Lambda)$ and a surjective homomorphism $\phi: S\to\Gamma(\Lambda)$ such that $\lambda\phi^{-1}$ is a completely simple semigroup for each $\lambda\in\Lambda$. Put $\gamma\phi^{-1}=S_\gamma$ for $\gamma\in\Gamma(\Lambda)$. Further, $E(S_\gamma)E(S_\mu)\subset E(S_\gamma)$ for $\lambda, \mu\in\Lambda, \lambda<\mu$, since S is upwards directed. Now, let u_γ be a representative of S_γ for each $\gamma\in\Gamma(\Lambda)$ such that u_γ is an idempotent of S_γ for $\lambda\in\Lambda$. Let L_λ and R_λ^* be the L-class and R-class of S_λ containing u_λ respectively for each $\lambda\in\Lambda$. Put $E(R_\lambda^*)=R_\lambda$. Then for $\lambda, \mu\in\Lambda$ such that $\lambda<\mu$, $R_\lambda R_\mu\subset E(S_\lambda)\cap R_\lambda^*=R_\lambda$. Hence, $R_\lambda R_\mu\subset R_\lambda$. Therefore, $L[\Lambda]=\Sigma\{L_\lambda: \lambda\in\Lambda\}$ and $R[\Lambda]=\Sigma\{R_\lambda: \lambda\in\Lambda\}$ are a lower partial chain Λ of the left groups $\{L_\lambda: \lambda\in\Lambda\}$ and an upper partial chain Λ of the right zero semigroups $\{R_\lambda: \lambda\in\Lambda\}$ respectively with respect to the multiplication in S. For any $a\in S_\gamma$ and $b\in S_\delta$, there exist $p\in L_{\gamma\gamma^{-1}}$, $e\in R_{\gamma^{-1}\gamma}$, $p'\in L_{\delta\delta^{-1}}$ and $e'\in R_{\delta^{-1}\delta}$ such that $a=pu_\gamma e$ and $b=p'u_\delta e'$. Now, $ab=p(u_\gamma ep'u_\delta)e'$. Since $u_\gamma ep'u_\delta\in S_{\gamma\delta}$, $u_\gamma ep'u_\delta$ can be uniquely written in the form $u_\gamma ep'u_\delta=p''u_{\gamma\delta}e''$, $p''\in L_{\gamma\delta(\gamma\delta)^{-1}}$, $e''\in R_{(\gamma\delta)^{-1}\gamma\delta}$. Define mappings $f_{\langle\gamma,\delta\rangle}$: $R_{\gamma^{-1}\gamma}\times L_{\delta\delta^{-1}}\to L_{\gamma\delta(\gamma\delta)^{-1}}$ and $g_{\langle\gamma,\delta\rangle}$: $R_{\gamma^{-1}\gamma}\times L_{\delta\delta^{-1}}\to R_{(\gamma\delta)^{-1}\gamma\delta}$ by $(e, p')f_{\langle\gamma,\delta\rangle}=p''$ and $(e, p')g_{\langle\gamma,\delta\rangle}=e''$. Then, it is easily verified by simple calculation that $\Delta=\{f_{\langle\gamma,\delta\rangle}: \gamma, \delta\in\Gamma(\Lambda)\}\cup\{g_{\langle\gamma,\delta\rangle}: \gamma, \delta\in\Gamma(\Lambda)\}$ is a factor set of $\{L[\Lambda], R[\Lambda]\}$ belonging to $\Gamma(\Lambda)$ and satisfies the condition (A). Hence, we can consider the regular product $\underline{S}=R(L[\Lambda]\times\Gamma(\Lambda)\times R[\Lambda];\Delta)$. Define $\psi: \underline{S}\to S$ by

$$(x, \gamma, e)\psi=xu_\gamma e.$$

Of course, it is obvious that $xu_\gamma e\in S_\gamma$ and ψ is surjective. Further, $((x, \gamma, e)(y, \delta, f))\psi=(x(e, y)f_{\langle\gamma,\delta\rangle}, \gamma\delta, (e, y)g_{\langle\gamma,\delta\rangle}f)\psi=x(e, y)f_{\langle\gamma,\delta\rangle}u_{\gamma\delta}(e, y)g_{\langle\gamma,\delta\rangle}f=(xu_\gamma e)(yu_\delta f)=((x, \gamma, e)\psi)((y, \delta, f)\psi)$. Hence, ψ is a homomorphism. Since ψ is clearly injective by Theorem of [16] above, ψ is an isomorphism.

Let $G\sim\Sigma\{G_\lambda: \lambda\in\Lambda\}$ be an upwards directed completely regular semigroup, and $\Gamma(\Lambda)$ an inverse semigroup. Let u_λ be an idempotent of G_λ, L_λ the L-class (of G_λ) containing u_λ, and R_λ the right zero semigroup of idempotents of the R-class (of G_λ) containing u_λ. Let $L[\Lambda]=\Sigma\{L_\lambda: \lambda\in\Lambda\}$ and $R[\Lambda]=\Sigma\{R_\lambda: \lambda\in\Lambda\}$ be the lower partial chain Λ of the left groups $\{L_\lambda: \lambda\in\Lambda\}$ and the upper partial chain Λ of the right zero semigroups $\{R_\lambda: \lambda\in\Lambda\}$ with respect to the multiplication in G. For a factor set $\Delta=\{f_{\langle\gamma,\delta\rangle}: \gamma, \delta\in\Gamma(\Lambda)\}\cup\{g_{\langle\gamma,\delta\rangle}: \gamma, \delta\in\Gamma(\Lambda)\}$ of $\{L[\Lambda], R[\Lambda]\}$ belonging to $\Gamma(\Lambda)$, let us consider the following condition:

(C) For $\lambda, \tau\in\Lambda$, $x\in R_\lambda$ and $y\in L_\tau$, $xy=(x, y)f_{\langle\lambda,\tau\rangle}u_{\lambda\tau}(x, y)g_{\langle\lambda,\tau\rangle}$.
(G-embedding condition)

In this case, $S=R(L[\Lambda]\times\Gamma(\Lambda)\times R[\Lambda];\Delta)$ is a regular extension of $G\sim\Sigma\{G_\lambda:\lambda\in\Lambda\}$ by $\Gamma(\Lambda)$. In fact: Consider the homomorphism $\phi: S\to\Gamma(\Lambda)$ defined by $(x,\gamma,e)\phi=\gamma$ for $(x,\gamma,e)\in S$. Let $\lambda\phi^{-1}=\underline{G}_\lambda$ and $\Lambda\phi^{-1}=\underline{G}$. We need only to show that there exists an isomorphism $\psi:\underline{G}\to G$ such that $\underline{G}_\lambda\psi=G_\lambda$ for $\lambda\in\Lambda$. Define ψ by $(t,\lambda,u)\psi=tu_\lambda u$. It is obvious that ψ is surjective and injective. For $(t,\lambda,u),(s,\tau,h)\in\underline{G}$, $((t,\lambda,u)(s,\tau,h))\psi=(t(u,s)f_{\langle\lambda,\tau\rangle},\lambda\tau,(u,s)g_{\langle\lambda,\tau\rangle}h)\psi=t((u,s)f_{\langle\lambda,\tau\rangle}u_{\lambda\tau}(u,s)g_{\langle\lambda,\tau\rangle})h=tush$ (by (C)) $=(tu_\lambda u)(su_\tau h)=((t,\lambda,u)\psi)((s,\tau,h)\psi)$. Further, $\underline{G}_\lambda\overset{\psi}{\cong}G_\lambda$ obviously holds.

Now, we have the following theorem:

THEOREM 3.5. *Let $G\sim\Sigma\{G_\lambda:\lambda\in\Lambda\}$ be an upwards directed completely regular semigroup, and u_λ an idempotent of G_λ for $\lambda\in\Lambda$. Let L_λ and R_λ be the L-class of G_λ and the right zero semigroup of idempotents of the R-class of G_λ such that L_λ and R_λ contain u_λ. Let $L[\Lambda]$, $R[\Lambda]$ be the lower partial chain Λ of $\{L_\lambda:\lambda\in\Lambda\}$ and the upper partial chain Λ of $\{R_\lambda:\lambda\in\Lambda\}$ with respect to the multiplication in G. Let $\Gamma(\Lambda)$ be an inverse semigroup. If $\Delta=\{f_{\langle\gamma,\delta\rangle}:\gamma,\delta\in\Gamma(\Lambda)\}\cup\{g_{\langle\gamma,\delta\rangle}:\gamma,\delta\in\Gamma(\Lambda)\}$ is a factor set of $\{L[\Lambda],R[\Lambda]\}$ belonging to $\Gamma(\Lambda)$ and if Δ satisfies (C), then $S=R(L[\Lambda]\times\Gamma(\Lambda)\times R[\Lambda];\Delta)$ is a regular extension of $G\sim\Sigma\{G_\lambda:\lambda\in\Lambda\}$ by $\Gamma(\Lambda)$. Conversely, every regular extension of $G\sim\Sigma\{G_\lambda:\lambda\in\Lambda\}$ by $\Gamma(\Lambda)$ can be constructed in this way.*

PROOF. The first half was already proved above. The latter half is obvious.

REMARK. For a given (fundamental) inverse semigroup $\Gamma(\Lambda)$ and for a given upwards directed completely regular semigroup $G\sim\Sigma\{G_\lambda:\lambda\in\Lambda\}$, there is not necessarily a regular extension of $G\sim\Sigma\{G_\lambda:\lambda\in\Lambda\}$ by $\Gamma(\Lambda)$. This can be seen from an example given by Hall [5] for the orthodox case.

II. The construction of split quasi-orthodox semigroups

A completely regular semigroup $G\sim\Sigma\{G_\lambda:\lambda\in\Lambda\}$ is said to be *split* if there exist homomorphisms $\phi: G\to\Lambda$ and $\psi:\Lambda\to G$ such that $\psi\phi=\iota_\Lambda$ (where ι_Λ denotes the identity mapping on Λ). Similarly, a quasi-orthodox semigroup S is said to be *split* if there exist an inverse semigroup $\Gamma(\Lambda)$, a surjective homomorphism $\phi: S\to\Gamma(\Lambda)$ and a homomorphism $\psi:\Gamma(\Lambda)\to S$ such that

(1) $\lambda\phi^{-1}$ is a completely simple subsemigroup of S for all $\lambda\in\Lambda$, and

(2) $\psi\phi=\iota_{\Gamma(\Lambda)}$ (the identity mapping on $\Gamma(\Lambda)$).

It is easily verified that if a completely regular semigroup $G\sim\Sigma\{G_\lambda:\lambda\in\Lambda\}$ is split as a completely regular semigroup then G is also split as a quasi-orthodox semigroup. Conversely, if a completely regular semigroup $G\sim\Sigma\{G_\lambda:\lambda\in\Lambda\}$ is split as a quasi-orthodox semigroup then $G\sim\Sigma\{G_\lambda:\lambda\in\Lambda\}$ is split as a completely regular semigroup. Therefore the concept of split for quasi-orthodox semigroups can be considered as a generalization of that for completely regular semigroups. Let S be a regular extension of a completely regular semigroup $G\sim\Sigma\{G_\lambda:\lambda\in\Lambda\}$ by an inverse semigroup $\Gamma(\Lambda)$.

Let $\phi: S \to \Gamma(\Lambda)$ be the homomorphism determined by $G \sim \Sigma\{G_\lambda: \lambda \in \Lambda\}$ (that is, $\lambda\phi^{-1} = G_\lambda$ for all $\lambda \in \Lambda$). If there exist $\psi: \Gamma(\Lambda) \to S$ such that $\psi\phi = \iota_{\Gamma(\Lambda)}$, then S is said to be *a split extension* of $G \sim \Sigma\{G_\lambda: \lambda \in \Lambda\}$ by $\Gamma(\Lambda)$. Of course, in this case $(\psi \mid \Lambda)(\phi \mid G)$ (where $\psi \mid \Lambda$ and $\phi \mid G$ denote the restrictions of ψ, ϕ to Λ, G respectively)$=\iota_\Lambda$, and hence $G \sim \Sigma\{G_\lambda: \lambda \in \Lambda\}$ is a split completely regular semigroup. Further, let $\gamma\psi = u_\gamma$ for $\gamma \in \Gamma(\Lambda)$. Then, ψ gives an isomorphism of $\Gamma(\Lambda)$ onto $U_{\Gamma(\Lambda)} = \{u_\gamma: \gamma \in \Gamma(\Lambda)\}$, and $U_\Lambda = \{u_\lambda: \lambda \in \Lambda\}$ is the basic semilattice of the inverse subsemigroup $U_{\Gamma(\Lambda)}$ of S. This $U_{\Gamma(\Lambda)}$ is called *a skeleton* of S with respect to $\Gamma(\Lambda)$.

Now, let S be a split quasi-orthodox semigroup. Then, there exist an inverse semigroup $\Gamma(\Lambda)$, a surjective homomorphism $\phi: S \to \Gamma(\Lambda)$ and a homomorphism $\psi: \Gamma(\Lambda) \to S$ such that

(1) $\lambda\phi^{-1}$ is a completely simple subsemigroup of S for each $\lambda \in \Lambda$, and

(2) $\psi\phi = \iota_{\Gamma(\Lambda)}$.

Now, let $S_\gamma = \lambda\phi^{-1}$ for $\gamma \in \Gamma(\Lambda)$ and $G = \Sigma\{S_\lambda: \lambda \in \Lambda\}$. Then, G is a completely regular semigroup $G \sim \Sigma\{S_\lambda: \lambda \in \Lambda\}$. Further, let $u_\lambda = \lambda\psi$ for each $\lambda \in \Lambda$. Then, ψ becomes an isomorphism of Λ to $U_\Lambda = \{u_\lambda: \lambda \in \Lambda\}$. That is, U_Λ is a skeleton of G with respect to Λ. Hence, of course G is a split completely regular semigroup. For each $\lambda \in \Lambda$, let L_λ be the L-class (of G) containing u_λ and R_λ the right zero semigroup of idempotents of the R-class (of G) containing u_λ. In this case, $L[\Lambda] = \Sigma\{L_\lambda: \lambda \in \Lambda\}$ is a lower partial chain Λ of $\{L_\lambda: \lambda \in \Lambda\}$ with respect to the multiplication in G. Further,

LEMMA 3.6.

(1) *If* $\lambda < \mu$, $e \in R_\mu$ *and* $f \in R_\lambda$, *then* $ef = f$.

(2) *If* $\lambda < \mu$, $e \in R_\mu$ *and* $f \in R_\lambda$, *then* $fe \in R_\lambda$.

(3) *Accordingly,* $R[\Lambda] = \Sigma\{R_\lambda: \lambda \in \Lambda\}$ *is an upper partial chain* Λ *of* $\{R_\lambda: \lambda \in \Lambda\}$ *with respect to the multiplication in* G.

PROOF. (1) We have $ef = eu_\lambda f = eu_\mu u_\lambda f = u_\mu u_\lambda f = u_\lambda f = f$.

(2) Since $fefe = ffe = fe$, it follows that $fe \in R_\lambda$.

(3) It is obvious from (1), (2) above that $R[\Lambda]$ is an upper partial chain Λ of $\{R_\lambda: \lambda \in \Lambda\}$ with respect to the multiplication in G.

For $\lambda, \tau \in \Lambda$ above, define $f_{\langle\lambda,\tau\rangle}: R_\lambda \times L_\tau \to L_{\lambda\tau}$ and $g_{\langle\lambda,\tau\rangle}: R_\lambda \times L_\tau \to R_{\lambda\tau}$ by

$$(e, x)f_{\langle\lambda,\tau\rangle} = x_1 \quad \text{and} \quad (e, x)g_{\langle\lambda,\tau\rangle} = e_1$$

if $u_\lambda exu_\tau = ex = x_1 u_{\lambda\tau} e_1$ (in G). Then $\Delta = \{f_{\langle\lambda,\tau\rangle}: \lambda, \tau \in \Lambda\} \cup \{g_{\langle\lambda,\tau\rangle}: \lambda, \tau \in \Lambda\}$ satisfies (I), (II) of (3.4). Further since $u_\lambda u_\tau = u_{\lambda\tau}$ for $\lambda, \tau \in \Lambda$, it follows that $(u_\lambda, u_\tau)f_{\langle\lambda,\tau\rangle} = u_{\lambda\tau}$ and $(u_\lambda, u_\tau)g_{\langle\lambda,\tau\rangle} = u_{\lambda\tau}$ for $\lambda, \tau \in \Lambda$. It is easily proved that $\xi: R(L[\Lambda] \times \Lambda \times R[\Lambda]; \Delta) \to G$ defined by $(x, \lambda, e)\xi = xu_\lambda e$ induces an isomorphism. Now, we have the following theorem:

THEOREM 3.7. *Let* Λ *be a semilattice,* $L[\Lambda] = \Sigma\{L_\lambda: \lambda \in \Lambda\}$ *a lower partial chain* Λ *of left groups* $\{L_\lambda: \lambda \in \Lambda\}$ *and* $R[\Lambda] = \Sigma\{R_\lambda: \lambda \in \Lambda\}$ *an upper partial chain* Λ *of right* *zero semigroups* $\{R_\lambda: \lambda\in\Lambda\}$. *Further assume that* $L_\lambda\cap R_\lambda=\{$*an idempotent* $u_\lambda\}$ *for all* $\lambda\in\Lambda$ *and* $u_\lambda u_\tau=u_\lambda$ *in* $R[\Lambda]$ *and* $u_\tau u_\lambda=u_\lambda$ *in* $L[\Lambda]$ *for* $\lambda\leq\tau$. *If a family* $\Delta=\{f_{\langle\lambda,\tau\rangle}: \lambda, \tau\in\Lambda\}\cup\{g_{\langle\lambda,\tau\rangle}: \lambda, \tau\in\Lambda\}$ *of mappings* $f_{\langle\lambda,\tau\rangle}: R_\lambda\times L_\tau\to L_{\lambda\tau}$ *and* $g_{\langle\lambda,\tau\rangle}: R_\lambda\times L_\tau\to R_{\lambda\tau}$ *is a factor set of* $\{L[\Lambda], R[\Lambda]\}$ *belonging to* Λ *and satisfies*

$$(3.7)\qquad (u_\lambda, u_\tau)f_{\langle\lambda,\tau\rangle}=u_{\lambda\tau};\quad \textit{and}\quad (u_\lambda, u_\tau)g_{\langle\lambda,\tau\rangle}=u_{\lambda\tau},$$

then $S=R(L[\Lambda]\times\Lambda\times R[\Lambda]; \Delta)$ *is a split completely regular semigroup. Conversely, every split completely regular semigroup can be constructed in this way.*

Proof. We prove that the completely regular semigroup $S=R(L[\Lambda]\times\Lambda\times R[\Lambda]; \Delta)$ constructed as above is split. For any $\lambda\in\Lambda$, let $S_\lambda=\{(x, \lambda, e): x\in L_\lambda, e\in R_\lambda\}$. Of course, $(u_\lambda, \lambda, u_\lambda)\in S_\lambda$. Define $\psi: \Lambda\to S$ and $\phi: S\to\Lambda$ by $\lambda\psi=(u_\lambda, \lambda, u_\lambda)$ and $(x, \lambda, e)\phi=\lambda$. Then, ϕ and ψ are homomorphisms and $\psi\phi=\iota_\Lambda$. Hence, S is split. The converse was already proved above.

Similarly, we obtain the following result:

Theorem 3.8. *Let* Λ *be a semilattice,* $\Gamma(\Lambda)$ *an inverse semigroup,* $L[\Lambda]=\Sigma\{L_\lambda: \lambda\in\Lambda\}$ *a lower partial chain* Λ *of left groups* $\{L_\lambda: \lambda\in\Lambda\}$ *and* $R[\Lambda]=\Sigma\{R_\lambda: \lambda\in\Lambda\}$ *an upper partial chain* Λ *of right zero semigroups* $\{R_\lambda: \lambda\in\Lambda\}$. *Further, assume that* L_λ, R_λ *have a common idempotent* u_λ *as their intersection, i.e.* $L_\lambda\cap R_\lambda=\{u_\lambda\}$, *for all* $\lambda\in\Lambda$, *and* $u_\lambda u_\tau=u_\lambda$ *in* $R[\Lambda]$ *and* $u_\tau u_\lambda=u_\lambda$ *in* $L[\Lambda]$ *for* $\lambda\leq\tau$. *If a family* $\Delta=\{f_{\langle\gamma,\delta\rangle}: \gamma, \delta\in\Gamma(\Lambda)\}\cup\{g_{\langle\gamma,\delta\rangle}: \gamma, \delta\in\Gamma(\Lambda)\}$ *is a factor set of* $\{L[\Lambda], R[\Lambda])$ *belonging to* $\Gamma(\Lambda)$ *and satisfies*

$$(\mathrm{B})\quad (u_{\gamma^{-1}\gamma}, u_{\delta\delta^{-1}})f_{\langle\gamma,\delta\rangle}=u_{\gamma\delta(\gamma\delta)^{-1}}\quad \textit{and}\quad (u_{\gamma^{-1}\gamma}, u_{\delta\delta^{-1}})g_{\langle\gamma,\delta\rangle}=u_{(\gamma\delta)^{-1}\gamma\delta}$$
$$\textit{for all}\quad \gamma, \delta\in\Gamma(\Lambda)$$

(*Split condition*),
then $S=R(L[\Lambda]\times\Gamma(\Lambda)\times R[\Lambda]; \Delta)$ *is a split quasi-orthodox semigroup. Conversely, every split quasi-orthodox semigroup can be constructed in this way.*

Proof. Since this can be proved by slightly modifying the proof of Theorem 3.7, we omit a proof.

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