On the Vibrations of a Clamped Square Plate Uniformly Compressed in One Direction

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(1) The problem of the vibration of a clamped square plate under tension was solved by A. Weinstein and W. Z. Chien and that of a clamped square plate under shearing stress by the author.

The problem of the vibrations of a clamped square plate uniformly compressed in one direction is one of the important and interesting problems. We shall now here consider this problem.

(2) Let us take coordinate axes (ox, oy) in the middle plane of the undisturbed as in Fig. 1, o being its center and axis oz perpendicular to the plate. Theedges $x=\pm 1$, are assumed to be subjected to compressive forces, of uniform intensity N per unit length of edges. The differential equation for the transverse displacement w of a point on the middle surface is

$$D\left(\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\,\partial y^2}+\frac{\partial^4 w}{\partial y^4}+N\frac{\partial^2 w}{\partial x_2}\right)-\rho h\frac{\partial^2 w}{\partial t^2}=0 \qquad (1)$$

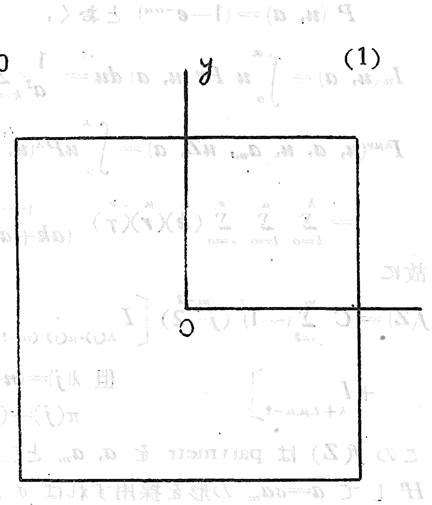

Fig 1

in which

$h=$ *thickness of the plate*

$t=$ *time,*

$\rho=$ *density of the plate,*

$D=\dfrac{Eh^3}{12(1-\sigma^2)}$, *the flexural rigidity,*

$\sigma=$ *poisson's ratio,*

$E=$ *Young's modulus.*

The boundary conditions are

$$\left.\begin{aligned} w=0 \text{ and } \frac{\partial w}{\partial y}=0 \quad at\ x=\pm 1,\\ w=0 \text{ and } \frac{\partial w}{\partial y}=0 \quad at\ y=\pm 1.\end{aligned}\right\} \qquad (2)$$

As the problem cannot be obtained explicitly in terms of elementary functions, we will use the Ritz method. According this method, the mode of the displacement is assumed. If we assume the following mode

$$w=W(x,\,y)\,cos pt, \qquad (3)$$

Eq. (1) is transformed into

$$\frac{\partial^4 W}{\partial x^4}+2\frac{\partial^4 W}{\partial x^2 \partial y^2}+\frac{\partial^4 W}{\partial y^4}+\lambda\frac{\partial^2 W}{\partial x^2}-KW=0 \tag{4}$$

with the boundary conditions

$$\left.\begin{aligned} W=0 \quad and\ \frac{\partial W}{\partial x}=0 \qquad at \quad x=\pm 1,\\ W=0 \quad and\ \frac{\partial W}{\partial y}=0 \qquad at \quad y=\pm 1,\end{aligned}\right\} \tag{5}$$

where

$$\lambda=\frac{N}{D}, \qquad K=\frac{\rho h p^2}{D}. \tag{6}$$

Then, by the method, we put

$$U=\int_{-1}^{1}\int_{-1}^{1}\left\{(\Delta W)^2-\lambda\left(\frac{\partial W}{\partial x}\right)^2-KW^2\right\}dxdy \tag{7}$$

We assume for the expression of W which satisfy the boundary conditions (5), that

$$W=(1-x^2)^2(1-y^2)^2\{c_0+c_1x^2+c_2y^2\} \tag{8}$$

in which c_0, c_1 and c_2 arbitrary parameters. Eq. (8) is introduced into U and then we obtain the following equation

$$\begin{aligned} U=&\frac{1}{1225}+\frac{2}{13475}(c_1+c_2)+\frac{251}{1576575}(c_1^2+c_2^2)+\frac{2}{121275}c_1c_2\\ &-\lambda\left[\frac{1}{33075}+\frac{2}{363825}c_2+\frac{1}{363825}c_1^2+\frac{1}{1576575}c_2^2\right]\\ &-K\left[\frac{1}{99225}+\frac{2}{1091475}(c_1+c_2)+\frac{1}{4729725}(c_1^2+c_2^2)+\frac{2}{12006225}c_1c_2\right]\end{aligned} \tag{9}$$

By the Ritz method, since the constants are to be determined so as to U a minimum, we have

$$\frac{\partial U}{\partial c_0}=0, \quad \frac{\partial U}{\partial c_1}=0, \quad \frac{\partial U}{\partial c_2}=0. \tag{10}$$

From these conditions, we obtain the following equations determining the parameters c_0, c_1 and c_2 :

$$\left.\begin{aligned} a_0c_1+b_0c_2&=b_3c_0\\ a_1c_1+a_2c_2&=a_0c_0\\ a_2c_1+b_2c_2&=b_0c_0\end{aligned}\right\} \tag{11}$$

in which

$$\left.\begin{aligned} a_1&=\frac{251}{1576575}-\frac{\lambda}{363825}-\frac{K}{4729725},\\ a_2&=\frac{1}{121275}-\frac{K}{12006225},\\ a_0&=-\frac{1}{13475}+\frac{K}{1091475},\\ b_2&=\frac{251}{1576575}-\frac{\lambda}{1576575}-\frac{K}{4729725},\end{aligned}\right\} \tag{12}$$

$$b_0 = -\frac{1}{13475} + \frac{\lambda}{363825} + \frac{K}{1091475},$$

$$b_3 = \frac{1}{1225} - \frac{\lambda}{33075} - \frac{K}{99225},$$

If we elliminate c_0, c_1 and c_2 from these equations (11), we have the following determinantal equation

$$\Delta = \begin{vmatrix} a_0 & b_0 & b_3 \\ a_1 & b_1 & a_0 \\ a_2 & b_2 & b_0 \end{vmatrix} = 0. \tag{13}$$

This equation (13) is the relation between λ and K that is, the compressive force and the frequency of the vibration. We then find smallest values of λ for various values of K, and values of c_1/c_0 and c_2/c_0 are calculated from Eqs (11). These values are given in Table 1 and Fig. 2 shows the curve λ against K.

Table 1

K	0	2.0	4.0	6.0	81
λ	24.94	19.32	13.29	6.913	0
$\frac{c_1}{c_0}$	−0.8193	−0.5490	−0.3039	−0.1358	0
$\frac{c_2}{c_0}$	7.63×10^{-3}	5.75×10^{-3}	3.76×10^{-3}	1.60×10^{-3}	0

From Table 1, it will be found the minimum value of λ for which the stability can become neutral is

$$\lambda_0 = 24.94\,. \tag{14}$$

Cox[3] found for this case

$$\lambda_c = \frac{9.36\pi^2}{4} = 23.1.$$

This value of λ_c is smaller than λ_0.

In the case $\lambda=0$, no compressive forces exert and Prof Tomotika found for this case

$$\frac{16K}{\pi^4} = 13.2948 \text{ or } K = 80.9394.$$

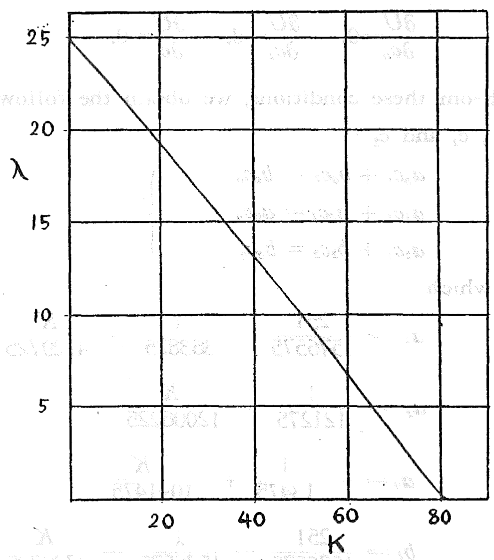

Fig. 2

In the present case we find $K=81$.

From above two cases we may conclude that the values of λ in Table I express fair approximate values for their true values.

(3) Résumé. We have treat the problems of transverse vibration of a clamped square plate uniformly compressive forces at the edges $x=\pm1$.

The critical values of compressive forces are calculated for given frequencies of vibrations and these are given in Table I.

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