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Smooth Circle Group Actions on Complex Surfaces

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Complex surfaces have been classified by Kodaira in I of [12] up to birational isomorphism. In this note we study effective smooth circle and torus actions on complex surfaces. In § 1 we consider actions on ruled surfaces, Enriques surfaces and the elliptic modular surface $B_{\Gamma(8)}$. The series of elliptic modular surfaces $B_{\Gamma(N)}$ ($N \geqq 5$) belong to the class IV_0 and some family of elliptic surfaces derived from basic ones are in the class IV_0 or VI_0 ([11]). Actions on these surfaces are considered in § 2. Hopf surfaces and Inoue surfaces [8] are in the class VII_0. In § 3 we study actions on these surfaces. In this note we mean by an action an effective action.

§ 1. Surfaces of class I_0

For a complex surface, the following formulas are known:

(i) $12(p_g-q+1)=c_1^2+c_2$ (Noether's formula e.g. (3) in I of [12]),

(ii) $3b^+-3b^-=c_1^2-2c_2$ (Hirzebruch signature theorem),

(iii) if b_1 is even, then $2q=b_1$, $2p_g=b^+-1$,
if b_1 is odd, then $2q=b_1+1$, $2p_g=b^+$ (Theorem 3 in I of [12]),

(iv) $c_2=2-2b_1+b_2$.

First we have

LEMMA 1. *If a complex surface admits a holomorphic torus action, then its Chern numbers are all zero.*

PROOF. Since the action is effective and holomorphic, its isotropy groups are all finite. Denote by C the additive group of complex numbers. Then we have a holomorphic C-action on the surface and the action is fixed point free. Thus on the surface there exists a non vanishing holomorphic vector field, and it follows from Corollary 1 in [2] that its Chern numbers c_1^2, c_2 are zero.

Now we consider some rational surfaces. The projective plane admits a series of inequivalent holomorphic S^1-actions which are given by $[z_0, z_1, z_2] \to [\rho^m z_0, \rho z_1, z_2]$, where $\rho \in S^1$ and for all integers m. By Lemma 1, it does not yield any holomorphic torus action, because its Chern numbers are given by $c_1^2=9x^2$, $c_2=3x^2$ for a generator x of the second cohomology group with integer coefficient. However it has an infinite number of inequivalent smooth torus actions, e.g. [17]. Then we have

PROPOSITION 1. *On the projective plane, there exists an infinite number of* *inequivalent holomorphic circle and smooth torus actions, but it does not yield any holomorphic torus action.*

Consider the series of Hirzebruch manifolds $\sum_n$ in [6], where n in a non negative integers. These surfaces are ruled surfaces of genus zero. Then we have

PROPOSITION 2. *Each surface* $\sum_n$ $(n=0, 1, 2,...)$ *has an infinite number of inequivalent holomorphic circle and smooth torus actions, but it does not yield any holomorphic torus action.*

PROOF. The surface is given by

$$P^2(x_0, x_1, x_2)\times P^1(y_1, y_2)\supset \textstyle\sum_n\colon x_1y_1^n-x_2y_2^n=0.$$

It admits a holomorphic circle action which is given by

$$(x_0, x_1, x_2)=(\rho x_0, \rho^k x_1, \rho^k x_2) \qquad \text{for each} \quad \rho\in S^1 \quad \text{and} \quad k=0, 2, 3,\dots.$$

For an even integer n, the surface is diffeomorphic to the product of 2-spheres $S^2\times S^2$, while for an odd integer n, it is diffeomorphic to a connected sum $P\# Q$ about a fixed point, where P is the projective plane and Q is the one with the reversed orientation. Then by Proposition 1 there exists an infinite number of inequivalent smooth torus actions on the surface. By (i), (ii) above we have $c_2=4$, $c_1^2=8$. By Lemma 1 we complete the proof of the proposition.

Ruled surfaces of genus 1 have been completely classified by Atiyah and Suwa [21]. If an algebraic surface admits a holomorphic torus action then by [7] it is a principal Seifert fibre space over an algebraic curve. Using Folgerung in p. 122 of [19], the curve is non singular. Thus the surface is an elliptic surface. Elliptic ruled surfaces are given by Theorem 5 in [21]. Each ruled surface of genus 1 is diffeomorphic to S_0 or A_{-1}, which are elliptic ruled surfaces. Thus we have

PROPOSITION 3. *Any ruled surface of genus* 1 *admits a smooth torus action.*

REMARK. In [21], Suwa has constructed complex analytic families for the surfaces. We may see that some of ruled surfaces admit holomorphic circle actions.

Next let us consider ruled surfaces of genus $g\geqq 2$. Such a surface is a projective line bundle over a non singular algebraic curve X_g of genus $g\geqq 2$. We prove

THEOREM 1. *Any ruled surface of genus* $g\geqq 2$ *admits a smooth circle action, but these surfaces do not yield any smooth torus action.*

PROOF. For any projective line bundle there is an associated holomorphic plane bundle $E\to X_g$. In the smooth category, its structure group $GL(2, C)$ can be reduced into the unitary group $U(2)$. Denote by $E_0\to X_g$ the associated principal $U(2)$ bundle. Then the projective line bundle is given by $\pi\colon P(E)=E_0/U(1)\times U(1)\to X_g$. Since a fibre is a 2-sphere, it admits a cross section and the bundle E_0 is reducible to a $U(1)$

$\times U(1)$-bundle i.e. $E_0 = F_0 \times_{U(1)\times U(1)} U(2)$. Then we have

$$P(E) = (F_0 \times_{U(1)\times U(1)} U(2)) \times_{U(2)} (U(2)/U(1)\times U(1))$$
$$= F_0 \times_{U(1)\times U(1)} (U(2)/U(1)\times U(1)).$$

We can define a $U(1)\times(e)$-action on $P(E)$ by $x(f, gH) = (f, xgH)$, $H = U(1)\times U(1)$ and $x \in U(1)\times(e)$, where $f \in F_0$, $gH \in U(2)/U(1)\times U(1)$ and e is the unit element of $U(1)$. Now the Euler characteristic of the surface $P(E)$ is given by

$$\chi(P(E)) = c_2(P(E)) = \chi(S^2)\chi(X_g) = 2(2-2g) < 0.$$

Since a torus is arcwise connected, the induced homomorphism $f_*: H_*(P(E)) \to H_*(P(E))$ is the identity mapping for a torus action $T^2 \times P(E) \to P(E)$ and any $f \in T^2$, and its Lefschetz number Λ_f is non zero. Choose f to be a generator of the torus group. Then we see that the fixed point set of the action is non empty. If the action is effective, then by VI of [16] the fundamental group $\pi_1(P(E))$ must be a free product of infinite and finite cyclic groups. On the other hand, we have $\pi_1(P(E)) = \pi_1(X_g)$, which can not be a free product. This is a contradiction.

Next we prove

THEOREM 2. *Enriques surfaces may not yield any smooth circle action.*

PROOF. Since $c_2 = 12 \neq 0$, any circle action has the non empty fixed point set. Any Enriques surface admits a K3-surface as an unramified double covering (§ 5, Chap. X in [18]). If an Enriques surface admits a circle action, then by 9. Chap. II in [3], we may lift the action to an action over the K3-surface, but a K3-surface may not yield any circle action except for a trivial action, which is a contradiction.

Let $\Gamma(N)$ be the principal congruence subgroup of level N in the group $SL(2, Z)$ and $B_{\Gamma(N)}$ be the elliptic modular surface attached to the group $\Gamma(N)$ as in [20]. Then we have

LEMMA 2. *We have an equality $c_2(B_{\Gamma(N)}) = \mu(N)$, where right hand side denotes the index of the subgroup $\Gamma(N)\cdot\{\pm 1\}$ in the group $SL(2, Z)$.*

PROOF. By (5.5), (5.3) in [20] and the relation $q = g(\Delta_{\Gamma(N)})$, the genus of the base curve $\Delta_{\Gamma(N)}$, we have

$$p_g - q + 1 = \frac{(N-3)\mu(N)}{6N} - 1 - \frac{(N-6)\mu(N)}{12N} + 1 = \frac{(2N-6)-(N-6)}{12N}\mu(N) = \frac{\mu(N)}{12}.$$

Since $B_{\Gamma(N)}$ is an elliptic surface for each N, $c_1^2 = 0$. Then by the Noether's formula (i), we obtain the lemma.

Now we consider the elliptic modular surface $B_{\Gamma(3)}$ especially. Then we have

THEOREM 3. *The elliptic modular surface $B_{\Gamma(3)}$ admits a smooth torus action.*

PROOF. The Lefschetz pencil in § 1, Chap. VII of [18] is an elliptic surface over the projective line and by definition it is diffeomorphic to the connected sum $P\#9Q$ and the second Chern class $c_2=12$ by the formula (iv) above. On the other hand the surface $B_{\Gamma(3)}$ is an elliptic surface over the projective line with $c_2=12$. Taking an analogue to the proof of Theorem 8 using Lemma 4 which appeared in § 1 of Part II of [15], we obtain the following: $B_{\Gamma(3)}$ is deformable into a regular Lefschetz fibration with singular fibres of type $I_1\times 12$ by Lemma 6 in p. 155 of [15]. Then we can apply Theorem 9 in p. 175 of [15]. Hence the Lefschetz pencil is diffeomorphic to the surface $B_{\Gamma(3)}$. The surface $B_{\Gamma(3)}$ has the invariants $q=0$, $p_g=0$, $c_1^2=0$ and it is a basic member in the sense II of [11], then it is a surface of class I_0. The surface $P\#9Q$ admits an infinite number of smooth torus actions ([17]).

§ 2. Surfaces of classes IV_0 and VI_0

In this section, first we have

PROPOSITION 4. *In the classes* IV_0 and VI_0, *there exist infinitely many surfaces with holomorphic torus actions, while in the class* IV_0, *there exist infinitely many surfaces without smooth circle actions.*

PROOF. Let Δ be a non singular algebraic curve with genus p, and $C_0=C/G$ a torus. Consider the product surface $B_0=\Delta\times C_0$. The invariants are given by

$$b_1=2p+2,\quad q=p+1,\quad b_2=4p+2,\quad c_2=2-2b_1+b_2=0,\quad p_g=q-1=p.$$

Then if $p=0$, then the surface is of class I_0; and if $p\geqq 1$, then it is of class IV_0. Clearly the surface admits a holomorphic torus action. Let f be the normal bundle of the embedding $\Delta\subset B$. Then f is the product bundle and we have the exact sequence of cohomology groups ((11.7) in III of [11]),

$$\longrightarrow H^1(\Delta,\,\Omega(f))\longrightarrow H^1(\Delta,\,\Omega(B))\xrightarrow{\delta^*} H^2(\Delta,\,G)\longrightarrow 0.$$

In this case, it follows that $G\approx Z\oplus Z$. Further, we have

$$b_1(B^\eta)=\begin{cases} 2p+2 & \text{if}\quad c(\eta)=\delta^*(\eta)=0 \qquad \text{for}\quad \eta\in H^1(\Delta,\,\Omega(B)),\\ 2p+1 & \text{otherwise.}\end{cases}$$

Then the surface B^η is of class IV_0 if $c(\eta)=0$ and of class VI_0 if $c(\eta)\neq 0$. By the construction 9 of II in [11], we have

$$\dim H^1(\Delta,\,\Omega(f))=p_g=p.$$

Now we consider the elliptic modular surface $B_{\Gamma(N)}\xrightarrow{\Psi}\Delta_{\Gamma(N)}$ for each $N\geqq 5$. Let f be a holomorphic line bundle over the curve $\Delta_{\Gamma(N)}$ with Chern class $c_1=-p_g+q-1$.

Then the canonical line bundle of the surface $B_{\Gamma(N)}$ is given by $K=\Psi^*(k\cdot f^{-1})$, where k is the canonical line bundle of the curve $\Delta_{\Gamma(N)}$ (III of [11]). We have $c_1(k\cdot f^{-1})=2(g-1)+p_g-q+1$, where g is the genus of the curve $\Delta_{\Gamma(N)}$. By $\mu(N)=\frac{1}{2}N^3\prod_{p|N}(1-1/p^2)$ and the proof of the lemma 2 in §1 it follows that $c_1(K)$ is an even multiple of some class in $H^2(B_{\Gamma(N)}, Z)$ if $N\equiv 0 \bmod 8$. Thus in this case the surface $B_{\Gamma(N)}$ is a spin manifold and $c_2\neq 0$. Since the surface is an elliptic surface, $c_1^2=0$ and we have

$$\hat{A}=\frac{1}{24}(2c_2-c_1^2)=\frac{1}{12}c_2\neq 0\,.$$

Hence by the criterion in [1], the surface may not yield any smooth circle action. Since $b_1=2g$ and $p_g>0$ by (5.5) in [20], the surface is of class IV_0. Thus we have proved the proposition.

Concerning to the proposition 4, here we prove two propositions.

PROPOSITION 5. *The elliptic modular surface $B_{\Gamma(N)}$ may not yield any smooth torus action for each $n\geqq 6$.*

PROOF. The elliptic modular surface $B_{\Gamma(N)}$ has a cross section, then the fundamental group $\pi_1(B_{\Gamma(N)})$ has a subgroup which is isomorphic to the fundamental group $\pi_1(\Delta_{\Gamma(N)})$ and it is not a free product. Then by VI of [16], the surface may not yield any torus action.

PROPOSITION 6. *Let S be an algebraic surface with invariants $b_1=0$, $p_g>0$. Then the surface S may not yield any holomorphic circle action.*

PROOF. Suppose that the surface S admits a holomorphic circle action. The group of all holomorphic automorphisms of the surface S is a complex Lie group. By the assumption its Lie algebra has a positive dimension over the complex number field and there exists a non zero holomorphic vector field X on S. Since $b_1=0$, it follows from Matsushima's theorem (e.g. Theorem 9.8 in III of [10]) that the vector field X has a non empty zero set, i.e. $\mathrm{Zero}(X)\neq\phi$. Further

$$0\leqq \dim \mathrm{Zero}(X)<2, \quad \text{then} \quad \dim H^0(S, \Omega^2)=p_g=0,$$

by Theorem 11.1 (Howard) in [10]. It is a contradiction.

REMARK. The elliptic modular surface $B_{\Gamma(5)}$ has invariants $b_1=0$ and $p_g>0$. Hence, it can not yield any holomorphic circle action. In the projective 3-space CP^3, the surface: $z_0^5+z_1^5+z_2^5+z_3^5=0$ is of class V_0, since its invariants are given by $q=0$, $p_g=4$, $c_1^2=5$, $c_2=55$, $b_1=0$, $b_2=53$. The surface does not admit any holomorphic circle action.

§ 3. Surfaces of class VII_0

First we have

PROPOSITION 7. *Any elliptic surface of class* VII_0 *admits a holomorphic torus action.*

PROOF. By making use of the construction 9 in II of [12], we have a torus action which is given by

$$D_v \times C \ni (\sigma_v, [\zeta_v]) \longrightarrow (\sigma_v, [\zeta_v + \zeta]) \in D_v \times C \qquad \text{for} \quad \zeta \in C.$$

The action is compatible with the action of the group $\mathfrak{H}_v$ (see 9 [12]) and with the identification (78) in II of [12].

REMARK. The generalized Hopf surface $H(a)$ in [4] has the invariants $b_1 = 1$ and $q = 1$, if $\sum(a)$ is a homology sphere and such a surface admits a holomorphic torus action $T_a \times H(a) \to H(a)$ in [4]. Then by the lemma in § 1, we have $c_1^2 = c_2 = 0$ and $p_g = 0$. Hence the surface $H(a)$ is of class VII_0 and is an elliptic surface. Concerning with non elliptic surfaces, first we have

PROPOSITION 8. *Any non elliptic Hopf surface admits a smooth torus action with finite isotropy groups.*

PROOF. By Theorem 32 in II of [12], the surface is given as a quotient manifold W/G, where $W = C^2 - (0)$ and $G = Z \times Z_l$ together with the actions

$$f: (z_1, z_2) \longrightarrow (\alpha_1 z_1 + \lambda z_1^m, \alpha_2 z_2), \quad \text{where } (\alpha_1 - \alpha_2^m) = 0 \text{ for a generator } f \text{ of } Z,$$

and

$$e: (z_1, z_2) \longrightarrow (\varepsilon_1 z_1, \varepsilon_2 z_2), \quad \text{where } (\varepsilon_1 - \varepsilon_2^m) = 0 \text{ for a generator } e \text{ of } Z_l.$$

Put $\exp \beta_2 = \alpha_2$, $\exp \beta_1 = \alpha_1$, $(\lambda/\alpha_1) \exp(-\beta_1) = \lambda_1$, and define the mapping $F: R \times S^3 \to W$ by

$$F(t, z_1, z_2) = f^t(z_1, z_2) = ((\exp t\beta_1)(z_1 + t\lambda_1 z_2^m), (\exp t\beta_2) z_2) \text{ (see [13])},$$

where R is the real number field. Then it is a diffeomorphism and satisfies the equality $f \cdot f^t = f^{t+1}$, and it induces a diffeomorphism $\hat{F}: S^1 \times S^3 \to W/\{f\}$. Since the mapping $\hat{F}$ commutes with the action e, it induces a diffeomorphism $\tilde{F}: S^1 \times (S^3/\{e\}) \to W/G$. The quotient space $S^3/\{e\}$ admits a torus action which is given by $[z_1, z_2] \to [(\exp 2\pi\varphi) z_1, (\exp 2\pi\theta) z_2]$ for $(\varphi, \theta) \in T^2$. Thus the Hopf surface has the required action.

Next we discuss an existence of a circle action on Inoue surfaces [8]. First we prove

LEMMA 3. *Inoue surfaces are of type $K(\pi, 1)$.*

PROOF. Denote by H, C and R the upper half of the complex, the complex plane and the real line respectively. Define a diffeomorphism $F: H\times C\to R^4$ by $F(X_1+iY_1, X_2+iY_2)=(X_1, \log Y_1, X_2, Y_2)$. Then the group $G_M=\langle g_0, g_1, g_2, g_3\rangle$ determines a subgroup $\bar{G}_M=\langle \bar{g}_0, \bar{g}_1, \bar{g}_2, \bar{g}_3\rangle$ of the affine group $A(4)=R^4\cdot GL(4, R)$ and we have a diffeomorphism $\bar{F}: S_M\to R^4/\bar{G}_M$. By Theorem 3.3 of [9], the quotient surface $\bar{S}_M$ inherits a flat connection with parallel torsion and is compact. Then by Proposition 4.3 of [9], it is of type $K(\pi, 1)$ i.e. aspherical. We can prove quite similarly the statement about surfaces $S^{(-)}$, $S^{(+)}$.

Now we have

THEOREM 4. (1) *The surfaces S_M and $S^{(-)}_{N,p,q,r}$ do not admit any smooth circle action.* (2) *The surfaces $S^{(+)}_{N,p,q,r,t}$ admit a smooth circle action.*

PROOF. If a surface of type $K(\pi, 1)$ admits a circle action, then by Lemma 5.5 [5] the action is injective and we have a central extension $0\to Z\to\pi_1(S_M)\to N\to 1$. By the relation in p. 274 [8], concerning the generators g_0, g_1, g_2, g_3, we can choose an element of the form $g=g_1^a g_1^b g_3^c g_0^d$ as a generator of the normal subgroup Z. Then by a computation of the element $g_0 g g_0^{-1}$, we have a relation ${}^tM^t(a\ b\ c)={}^t(a\ b\ c)$, where the 3×3-matrix $M\in SL(3, Z)$ is the matrix in § 2 of [8] and t denotes the transposed matrix. But the matrix M does not contain 1 as an eigenvalue. This is a contradiction. The situation is quite similar for the surfaces $S^{(-)}_{N,p,q,r}$. Hence we have (1). Now we consider the surfaces $S^{(+)}_{N,p,q,r,t}$. By the proof of Lemma 3 and § 3 in [8], the element $\bar{g}_3$ generates an infinite cyclic subgroup of $\bar{G}^{(+)}_N$, which is a central subgroup and we have an action of the circle $R(X_2)/(\mathrm{mod}\ s)$ on the surface $\bar{S}^{(+)}_{N,p,q,r,t}$, where $s=(b_1a_2-b_2a_1)/r$. Thus we have proved (2).

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