On the New Method for Obtaining Solution of Boundary-value Problems of Elasticity

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1. Introduction

When we treat stability and vibration problems of an elastic thin plate, we shall frequently encounter the difficulties of finding their solutions. Therefore, the methods of Rayleigh-Ritz, Weinstein, Taylor, Iguchi and Collatz were found for finding their approximate solutions.

In Weinstein method, the problem is solved as a modified variational problem with a slightly different boundary conditions. In Taylor's method, the complete integral of the differential equation is firstly found and then expanded, for example, in $\cos nx$ series with even integers n in order to satisfy the boundary conditions, at $x=\pm\frac{\pi}{2}$, if it is an even function. In Rayleigh-Ritz's method, the problem is usually solved without much labour and serious loss of accuracy.

It is well known that the eigenvalue obtained by Rayleigh's principle is always larger, while the value by Wenstein's method smaller than its true value. Therefore, the values given by these two methods are considered as the upper und lower limits of its true eigenvalue.

Among these methods, Weinstein's one is the best one from the mathematical standpoint, but there is the defect that its range of application is very limited. On the contrary, Rayleigh-Ritz's method gives a useful process to obtain an approximate solution but its accuracy is bad in comparison with the solutions obtained by the methods of Weinstein and Taylor.

In the following section, we shall propose the new method which has not the defects above mentioned and is applicable in many fields of mathematical physics. This method contains as a special case Taylor's one and that of Weinstein, if its solution exists.

2. The new proposal

Let us consider to solve the following linear partial differential equation of w:

$$L(w)=f(x, y), \tag{1}$$

under the boundary conditions of w and its derivatives:

$$M_i(w)=0, \quad (i=1, 2, \cdots\cdots p) \tag{2}$$

where $f(x, y)$ is the given function of x and y.

If we can find the integral of (1), the arbitrary constants involved into it must be determined so as to satisfy the boundary conditions (2). But, as this is very difficult in general, we shall consider here that the boundary conditions (2), which do not simply determine the above constants, are

$$M_i(w)=0 \ (i=1,\ 2,\cdots\cdots k). \tag{3}$$

As the conditions (3) hold always on the all points of the boundary, the following relations will be obtained for any arbitrary function $h(x,\ y)$

$$\int_c M_i(w)h(x,\ y)ds=0 \ \ (i=1,\ 2,\cdots\cdots k) \tag{3'}$$

where the path of integration being a part of the boundary. Therefore, we can determine the values of the arbitrary constants involved into the integral so as to satisfy the following boundary conditions

$$\int_c M_i(w)h(x,\ y)ds=0, \ (i=1,\ 2,\cdots\cdots k) \tag{3'}$$

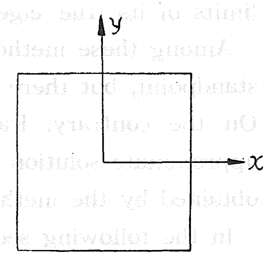
and $$M_i(w)=0 \qquad (i=k+1,\cdots\cdots p) \tag{2'}$$

If we consider a eigenvalue problem, the values of ratios of every constants and the eigenvalue are determind from (2′) and (3′). In practice, we may use the adequate expressions for $h(x,\ y)$, if possible, the system of orthogonal normalized functions. We shall solve some examples by this method in the following section.

3. Procedure

By the above mentioned method, we shall now solve the following three problems, the first two of them were solved already by the methods of Weinstein and Taylor.

A) Let us assume that a square plate with clamped edges is compressed in its middle plane by forces uniformly distributed along the all edges.



Let us take the coordinate axes $(x,\ y)$ in the middle plane of a square plate of uniform small thickness such that the origin coincides with it center, the x, y-axes are parallel to the sides, and z-axis perpendicular to its plane, and denote the side length and the thickness of the plate by $2a$ and h respectively.

Let the density, Young's modulus, Poisson's ratio and the flexural rigidity of the material of the plate, which is assumed to be uniform and isotropic be denoted by ρ, E, σ, and $D=\dfrac{Eh^3}{12(1-\sigma^2)}$, respectively. If we denote the magnitude of the compressive force per unit edge by P, the vertical displacement w, that is the displacement in z direction, must satisfy the following partial differential equation:

$$D\left(\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}\right)+P\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)=0, \tag{4}$$

and the boundary conditions

$$w=\frac{\partial w}{\partial x}=0 \qquad \text{at } x=\pm a,$$

$$w=\frac{\partial w}{\partial y}=0 \qquad \text{at } y=\pm a. \tag{5}$$

If we introduce new variables such that

$$\xi=\frac{\pi}{2a}x, \qquad \eta=\frac{\pi}{2a}y, \tag{6}$$

the equations (4) and (5) are transformed into

$$\frac{\partial^4 w}{\partial\xi^4}+2\frac{\partial^4 w}{\partial\xi^2\partial\eta^2}+\frac{\partial^4 w}{\partial\eta^4}+\mu\left(\frac{\partial^2 w}{\partial\xi^2}+\frac{\partial^2 w}{\partial\eta^3}\right)=0, \tag{4'}$$

$$\left.\begin{aligned} w=\frac{\partial w}{\partial\xi}=0 \quad & \text{at } \xi=\pm\frac{\pi}{2}, \\ w=\frac{\partial w}{\partial\eta}=0 \quad & \text{at } \eta=\pm\frac{\pi}{2}, \end{aligned}\right\} \tag{5'}$$

where

$$\mu=\frac{4Pa^2}{D\pi^2}. \tag{7}$$

We shall adopt the following expression as the solution of (4′) which satisfies the boundary condition $w=0$ at the edges and the symmetry conditions $w(\xi, \eta)=w(\eta, \xi)=w(-\xi, \eta)=w(-\xi, -\eta)=w(\xi, -\eta)=w(-\eta, -\xi)$

$$w=\sum_n{}' A_n(\cosh\alpha_n\eta\cosh\frac{1}{2}\beta_n\pi-\cosh\beta_n\eta\cosh\frac{1}{2}\alpha_n\pi)\cos n\xi$$

$$+\sum_n{}' A_n(\cosh\alpha_n\xi\cosh\frac{1}{2}\beta_n\pi-\cosh\beta_n\xi\cosh\frac{1}{2}\alpha_n\pi)\cos n\eta$$

where

$$\alpha_n{}^2=n^2, \ \beta_n{}^2=n^2-\mu,$$

and Σ' means to sum up with odd integers of n from 1 to ∞.

As it is known that the value of μ is smaller than 5.33 form Rayleigh's principle, the above solution is transformed into

$$w=A_1\left(\cosh\eta\cos\frac{\pi}{2}\sqrt{\mu-1}-\cos\sqrt{\mu-1}\,\eta\cosh\frac{\pi}{2}\right)\cos\xi$$

$$+\sum_{n\geqq3}{}' A_n\left(\cosh n\eta\cosh\frac{\pi}{2}\sqrt{n^2-\mu}-\cosh\sqrt{n^2-\mu}\,\eta\cosh\frac{n\pi}{2}\right)\cos n\xi$$

$$+A_1\left(\cosh\xi\cos\frac{\pi}{2}\sqrt{\mu-1}-\cos\sqrt{\mu-1}\,\xi\cosh\frac{\pi}{2}\right)\cos\eta$$

$$+\sum_{n\geqq3}{}' A_n\left(\cosh n\xi\cosh\frac{\pi}{2}\sqrt{n^2-\mu}-\cosh\sqrt{n^2-\mu}\,\xi\cosh\frac{n\pi}{2}\right)\cos n\eta \tag{8}$$

The remaining boundary conditions are now substituded by the following conditions,

$$\left.\begin{aligned} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\partial w}{\partial\xi}\cos j\eta d\eta=0 \quad & \text{at } \xi=\pm\frac{\pi}{2}, \\ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\partial w}{\partial\eta}\cos j\xi d\xi=0 \quad & \text{at } \eta=\pm\frac{\pi}{2}. \end{aligned}\right\} \tag{9}$$

That is to say, we adopt as the functions $h(x, y)$ in § 2, the following functions

$$h(\xi, \eta)=\cos j\eta \quad \text{at } \xi=\pm\frac{\pi}{2},$$

$$h(\xi, \eta)=\cos j\xi \quad \text{at } \eta=\pm\frac{\pi}{2},$$

in which j denotes 0 and positive integers.

From the symmetry conditions of the solution, it is clear that the solution (8) which satisfies the boundary conditions at $\xi=\frac{\pi}{2}$ also satisfies the remaining boundary conditions at $\xi=-\frac{\pi}{2}$ and $\eta=\pm\frac{\pi}{2}$. The expression of (8) must satisfy the following conditions

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\partial w}{\partial \xi} \cos j\eta d\eta=0 \qquad \text{at } \xi=\frac{\pi}{2} \tag{9'}$$

When we insert the expression for w of (8) in (9'), we obtain the following equations

$$\Sigma' A_n b_{nj}=0, \tag{10}$$

where

$$\begin{aligned} b_{1j}=&-\Big[2\cos\frac{\pi}{2}\sqrt{\mu-1}\Big\{\frac{1}{1+j^2}\sinh\frac{\pi}{2}\cos j\frac{\pi}{2}+\frac{j}{1+j^2}\cosh\frac{\pi}{2}\sin j\frac{\pi}{2}\Big\} \\ &-\cosh\frac{\pi}{2}\Big\{\frac{\sin(\sqrt{\mu-1}-j)\frac{\pi}{2}}{\sqrt{\mu-1}-j}+\frac{\sin(\sqrt{\mu-1}+j)\frac{\pi}{2}}{\sqrt{\mu-1}+j}\Big\}\Big] \\ &+\Big\{\sinh\frac{\pi}{2}\cos\frac{\pi}{2}\sqrt{\mu-1}+\sqrt{\mu-1}\sin\sqrt{\mu-1}\frac{\pi}{2}\cosh\frac{\pi}{2}\Big\} \\ &\times\Big\{\frac{\sin(1-j)\frac{\pi}{2}}{1-j}+\frac{\sin(1+j)\frac{\pi}{2}}{1+j}\Big\}, \end{aligned} \tag{11}$$

for $n\geqq3$,

$$\begin{aligned} b_{nj}=&(-1)^{\frac{n+1}{2}}n\Big[2\cosh\frac{\pi}{2}\sqrt{n^2-\mu}\Big(\frac{n}{n^2+j^2}\sinh\frac{n\pi}{2}\cos\frac{j\pi}{2}+\frac{j}{n^2+j^2}\cosh\frac{n\pi}{2}\sin\frac{j\pi}{2}\Big) \\ &-2\cosh\frac{n\pi}{2}\Big\{\frac{\sqrt{n^2-\mu}}{n^2-\mu+j^2}\sinh\sqrt{n^2-\mu}\frac{\pi}{2}\cos j\frac{\pi}{2}+\frac{j}{n^2-\mu+j^2}\cosh\sqrt{n^2-\mu}\frac{\pi}{2}\sin\frac{j\pi}{2}\Big\}\Big] \\ &+\Big\{n\sinh\frac{n\pi}{2}\cosh\frac{\pi}{2}\sqrt{n^2-\mu}-\sqrt{n^2-\mu}\sinh\sqrt{n^2-\mu}\frac{\pi}{2}\cos\frac{n\pi}{2}\Big\} \\ &\times\Big\{\frac{\sin(n-j)\frac{\pi}{2}}{n-j}+\frac{\sin(n+j)\frac{\pi}{2}}{n+j}\Big\}. \end{aligned} \tag{12}$$

As it is well known from the theorem of linear homogeneous equations that all the values of A_n which satisfy (10) are zero unless the value of μ is suitable values, that is, the values which make the value of the determinant Δ being formed by the coefficients of A_n in (10), zero. When μ takes the smallest root μ_c of $\Delta=0$, the plate is in equilibrium in the buckling state and therefore the corresponding value of p is said the critical compressive force.

If we adopt 0, 1, 2,······as the value of j, the above determinantal equation $\Delta=0$ is expressed by the following expression:

$$\Delta=\begin{vmatrix} b_{10} & b_{30} & b_{50}\cdots\cdots\cdots \\ b_{11} & b_{31} & b_{51}\cdots\cdots\cdots \\ b_{12} & b_{32} & b_{52}\cdots\cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots\cdots \end{vmatrix}=0. \tag{13}$$

If we put

$$b_{1j}=2\cos\frac{\pi}{2}\sqrt{\mu-1}\quad\cosh\frac{\pi}{2}\cdot c_{1j},$$

$$b_{nj}=(-1)^{\frac{n+1}{2}}2n\cosh\frac{\pi}{2}\sqrt{n^2-\mu}\cosh\frac{n\pi}{2}\cdot c_{nj},\ \text{for } n\geqq 3, \tag{14}$$

the determinantal equation (13) is transformed into

$$D=\begin{vmatrix} c_{10} & c_{30}\cdots\cdots\cdots \\ c_{11} & c_{31}\cdots\cdots\cdots \\ c_{12} & c_{32}\cdots\cdots\cdots \\ \cdots\cdots & \cdots\cdots\cdots \end{vmatrix}=0. \tag{15}$$

For obtain the smallest root of $D=0$, we put

$$D_m=\begin{vmatrix} c_{10}, & c_{30}\cdots\cdots & c_{2m-1,\,0} \\ c_{11}, & c_{31}\cdots\cdots & c_{2m-1,\,1} \\ \vdots & & \\ c_{1,m-1} & c_{3,m-1}\cdots\cdots & c_{2m-1,m-1} \end{vmatrix}=0. \tag{16}$$

Then, we calculate the series μ_1, $\mu_2\cdots\cdots$ of the smallest root μ_m of $\Delta_m=0$ for $m=1, 2, 3,\cdots\cdots$

The values of μ_m will converge to a definite value μ_c as m increases. This limiting value μ_c is our required root of the determinantal equation $D=0$. Table I shows the calculated values of μ_m for $m=1$, 2, 3 and 4

Table I. The values of μ_m

	D_m	D_m (even)	D_m (odd)
1	5.0	5.0	5.155
2	5.308	5.304	5.2565
3	5.3034	5.3034	5.3034
4	5.3043	5.3031	5.30358

If we adopt zero and even integers as the values of j, we have the corresponding determinantal equation D_m (even)=0. In this case, this solution is identical with that of Taylor. The calculated values of μ_m are also shown in Table I.

If we adopt odd integers as the values of j, we have the corresponding determinantal equation D_m (odd)=0. In this case, this solution is identical with that of Weinstein. The calculated values of μ_m are also shown in Table I.

It is known from the theorem of variation $\mu_c\geqq 5.3058$

B) As the second example, let us consider a transverse vibration of a square plate clamped at all four edges. If we use the same notations in the preceding example, the differential equation for the transverse vibration of the plate is given

$$\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}+ph\frac{\partial^2 w}{\partial t^2}=0, \tag{17}$$

in which t represents time.

When the plate vibrates in a normal mode, the displacement w is of the form

$$w=W\cos(pt+\varepsilon), \tag{18}$$

where W is a function of x, y or ξ, η. If we put (18) in (17), and transform x and y into ξ and η, we get the partial differential equation for W:

$$\frac{\partial^4 W}{\partial \xi^4}+2\frac{\partial^4 W}{\partial \xi^2\partial \eta^2}+\frac{\partial^4 W}{\partial \eta^4}-k^2 W=0, \tag{19}$$

where

$$k^2\equiv\frac{\rho h a^4 p^2}{D\pi^4}. \tag{20}$$

As the four edges of the plate are clamped, the boundary conditions at the edges are

$$\left.\begin{aligned}W&=\frac{\partial W}{\partial \xi}=0 \qquad \xi=\pm\frac{\pi}{2},\\ W&=\frac{\partial W}{\partial \eta}=0 \qquad \eta=\pm\frac{\pi}{2}.\end{aligned}\right\} \tag{21}$$

Now we adopt the following series as the solution of (19) which satisfies $W=0$ at edges and the symmetry conditions, from all particular solutions of (19)

$$W=\Sigma' A_i v_i, \tag{22}$$

where Σ' means to sum up with odd integers of i from 1 to ∞, and

$$\left.\begin{aligned}v_1=&\left(\cosh\frac{\pi}{2}\sqrt{k+1}\cos\sqrt{k-1}\,\eta-\cos\frac{\pi}{2}\sqrt{k-1}\cosh\sqrt{k+1}\,\eta\right)\cos\xi\\ &+\left(\cosh\frac{\pi}{2}\sqrt{k+1}\cos\sqrt{k-1}\,\xi-\cos\frac{\pi}{2}\sqrt{k-1}\cosh\sqrt{k+1}\,\xi\right)\cos\eta\end{aligned}\right\} \tag{23}$$

for $i\geqq 3$

$$\left.\begin{aligned}v_i=&\left(\cosh\frac{\pi}{2}\sqrt{i^2+k}\cosh\sqrt{i^2-k}\,\eta-\cosh\frac{\pi}{2}\sqrt{i^2-k}\cosh\sqrt{i^2+k}\,\eta\right)\cos i\xi\\ &+\left(\cosh\frac{\pi}{2}\sqrt{i^2+k}\cosh\sqrt{i^2-k}\,\xi-\cosh\frac{\pi}{2}\sqrt{i^2-k}\cosh\sqrt{i^2+k}\,\xi\right)\cos i\eta\end{aligned}\right\} \tag{24}$$

The expression W of (22) must now satisfy the boundary conditions

$$\left.\begin{aligned}&\frac{\partial W}{\partial \xi}=0 \qquad \text{at } \xi=\pm\frac{\pi}{2},\\ \text{and}\quad &\frac{\partial W}{\partial \eta}=0 \qquad \text{at } \eta=\pm\frac{\pi}{2}.\end{aligned}\right\} \tag{21$'$}$$

As discussed in the preceding, we consider the following relations as the boundary conditions:

$$\left.\begin{aligned}&\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\partial W}{\partial \xi}\cos j\eta\, d\eta=0 \qquad \text{at } \xi=\pm\frac{\pi}{2}\\ \text{and}\quad &\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\partial W}{\partial \eta}\cos j\xi\, d\xi=0 \qquad \text{at } \eta=\pm\frac{\pi}{2}\end{aligned}\right\} \tag{21$''$}$$

in place of (21′). Let us take 0 and integers as the values of j.

When we insert (22) in the equations of (21″) at $\xi=\frac{\pi}{2}$, we get the following equations

$$\sum_i{}' A_i b_{ij}=0, \tag{24}$$

where

$$b_{ij}=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\frac{\partial v_i}{\partial \xi}\right)_{\xi=\frac{\pi}{2}}\cos j\eta d\eta. \tag{25}$$

It is easily proved that if the coefficients A_i satisfy the relations (24), the remaining conditions of (21″) are also satisfied by A_i.

Eliminating the A_i's between (24), we obtain the determinantal equation

$$\Delta=\begin{vmatrix} b_{10} & b_{30} & \cdots\cdots \\ b_{11} & b_{31} & \cdots\cdots \\ b_{12} & b_{32} & \cdots\cdots \\ \cdots & \cdots & \cdots\cdots \end{vmatrix}=0. \tag{26}$$

When we put

$$\left.\begin{aligned} & b_{1j}=\cos\frac{\pi}{2}\sqrt{k-1}\cosh\frac{\pi}{2}\sqrt{k+1}\,c_{1j}, \\ \text{for}\quad & i\geqq 3 \\ & b_{ij}=\cosh\frac{\pi}{2}\sqrt{i^2-k}\cosh\frac{\pi}{2}\sqrt{i^2+k}\,c_{ij}, \end{aligned}\right\} \tag{27}$$

the determinantal equation becomes

$$D=\begin{vmatrix} c_{10} & c_{30} & \cdots\cdots \\ c_{11} & c_{31} & \cdots\cdots \\ c_{12} & c_{32} & \cdots\cdots \\ \cdots & \cdots & \cdots\cdots \end{vmatrix}=0. \tag{28}$$

If we find the smallest root k_c of (28), the value of k_c corresponds the frequency of the fundamental mode of vibration which we aim. As in the preceding example, we put

$$D_m=\begin{vmatrix} c_{10}, & c_{30}, \cdots\cdots & c_{2m-1,\,0} \\ c_{11}, & \cdots\cdots\cdots & c_{2m-1,\,1} \\ \vdots & & \\ c_{1,\,m-1}, & \cdots\cdots\cdots & c_{2m-1,\,m-1} \end{vmatrix}=0. \tag{29}$$

and calculate the smallest root k_m of $D_m=0$. The value k_m converges to a definite limit as m increases and this limiting value is the required value k_c. The calculated value of μ_m for $m=4$ is shown in Table II.

Table II. The values of k_m

	Δ_4	Δ_4 (even)	Δ_4 (odd)
k_m	3.651	3.6462	3.6461

If we put $j=0, 2, 4\cdots\cdots$ (0 and even integers), we get the corresponding determinantal equation $D_m\,(\text{even})=0$. The smallest root k_m of $D_m\,(\text{even})=0$ is calculated and shown in Table II. This solution is identical with that of Tomotika obtained by the method of Taylor.

If we put $j=1, 3, 5\cdots\cdots$ (odd integers), we get the corresponding determinantal equation D_m (odd)=0. The smallest root k_m of D_m (odd)=0 is calculated and shown also in Table II. This solution is identical with that of Tomotika obtained by the method of Weinstein.

C) As the third example, we shall treat the problem of a laterally loaded clamped square plate. This is not the eigenvalue problem, but the method is also applicable to this case. However, this problem has also discussed by Hencky, Way etc., we shall solve it.

When we denote the lateral load per unit area of the plate by p, the vertical displacement w must satisfy the following partial differential equation

$$\frac{\partial^4 w}{\partial \xi^4}+2\frac{\partial^4 w}{\partial \xi^2 \partial \eta^2}+\frac{\partial^4 w}{\partial \eta^4}=P, \tag{30}$$

where

$$P=\frac{16a^4 p}{\pi^4 D}. \tag{31}$$

As the four edges are clamped, the boundary conditions of w are

$$\left.\begin{aligned} w=\frac{\partial w}{\partial \xi}=0 \qquad & \text{at } \xi=\pm\frac{\pi}{2}, \\ w=\frac{\partial w}{\partial \eta}=0 \qquad & \text{at } \eta=\pm\frac{\pi}{2}. \end{aligned}\right\} \tag{32}$$

We shall first solve the problem by the method of Weinstein. It is easily proved that that the problem of obtaining the solution of (30) which satisfies the boundary conditions (32) is equivalent to a minimum problem of computing the minimum value of the expression:

$$J=\iint_S \left\{\frac{1}{2}(\Delta w)^2-Pw\right\}d\xi d\eta \tag{33}$$

for all functions $w(\xi, \eta)$ which have continuous derivatives up to the fourth order in the square $S: |\xi| \leqq \frac{\pi}{2}$, $|\eta| \leqq \frac{\pi}{2}$ and which also satisfy the boundary conditions (32).

But as it is impossible to obtain such a function in practice, we consider now the modified minimum problem which may be expressed as follows:

The modified problem: It is required to find the minimum value of the expression J of (33) for all function $w(\xi, \eta)$ which satisfy the following boundary conditions on C

$$w=0, \tag{34}$$

and

$$\int_C \frac{\partial w}{\partial n} g_j(\xi, \eta)ds=0, \quad (j=1, 2, 5, \cdots\cdots 2k-1) \tag{35}$$

where n denotes the normal to C, ds a line element along C, and g_j are taken as follows:

$$\left.\begin{aligned} g_j(\xi, \eta)=\cos j\eta \qquad & \text{at } \xi=\pm\frac{\pi}{2}, \\ g_j(\xi, \eta)=\cos j\xi \qquad & \text{at } \eta=\pm\frac{\pi}{2}. \end{aligned}\right\} \tag{36}$$

When we perform the variation of (33) under the conditions (34) and (35), we get the following equations:

$$\Delta^2 w = P \tag{37}$$

and on C

$$w=0, \tag{38}$$

and

$$\Delta w = \sum_{\substack{j=1\\ \text{odd}}}^{2k-1}{}' b_j g_j(\xi, \eta), \tag{39}$$

in which b_j denotes Lagragian indeterminate multipliers, and Σ' means to sum up with odd integers of j from 1 to $2k-1$.

If, therefore, we first obtain the solution of (37) satisfying (38) and (39) and then insert it into (35), the nonvanishing values of constants in the solution will be determined.

To get the solution, we expand P in Fourier series in $|\xi| \leqq \frac{\pi}{2}$, $|\eta| \leqq \frac{\pi}{2}$:

$$P = \sum_m{}' \sum_n{}' P_{mn} \cos m\xi \cos n\eta, \tag{40}$$

where

$$P_{mn} = (-1)^{\frac{m+n-1}{2}} \frac{16P}{\pi^2 mn}, \tag{41}$$

and $\sum\limits_m{}'$ and $\sum\limits_n{}'$ mean to sum up with odd integers of m and n from 1 to ∞, respectively.

The solution which satisfies (37), (38) and (39) is as follows:

$$w = \sum_m{}' \sum_n{}' w_{mn} \cos m\xi \cos n\eta + \sum_{n=1}^{2k-1}{}' f_n(\xi) \cos n\eta + \sum_n^{2k-1}{}' f_n(\eta) \cos n\xi, \tag{42}$$

$$w_{mn} = \frac{16P}{\pi^2 mn(m^2+n^2)}, \tag{43}$$

$$f_n(\xi) = a_n\left(\cosh n\xi - \frac{2}{\pi}\xi \coth\frac{n\pi}{2} \sinh n\xi\right), \tag{44}$$

and a_n are indeterminate constants.

When we insert w of (42) into the boundary conditions at $\xi=\frac{\pi}{2}$ of (35), we get the following equations

$$\sum_{1=u}^{2k-1}{}' a_n B_{nj} = A_{jj}, \quad (j=1, 3, \cdots\cdots, 2k-1) \tag{45}$$

in which

$$B_{nj} = \delta_{nj}\left(n \sinh\frac{n\pi}{2} - \frac{2}{\pi}\cosh\frac{n\pi}{2} - \frac{n\cosh^2\frac{n\pi}{2}}{\sinh\frac{n\pi}{2}}\right) - \frac{16n^2 j \sin\frac{n\pi}{2} \sin j\frac{\pi}{2}}{\pi^2(n^2+j^2)^2} \cdot \frac{\cosh^2\frac{n\pi}{2}}{\sinh\frac{n\pi}{2}}, \tag{46}$$

$$A_{nn} = \frac{16P\sin n\frac{\pi}{2}}{n\pi^2}\left\{\frac{\pi}{8n^3}\tanh\frac{n\pi}{2} - \frac{\pi^2}{16n^2}\operatorname{sech}^2\frac{n\pi}{2}\right\}, \tag{47}$$

and
$$\delta_{nj}=\begin{cases}1, & n=j\\ 0, & n\neq j.\end{cases}$$

It is easily proved that the remaining boundary conditions are satisfied when the equations (45) hold.

We can determine the values of a_n from equations (45). When we take $k=4$, the values of a_1, a_3, a_5 and a_7 are determined as follows

$$\left.\begin{aligned} a_1&=-0.13548P,\\ a_3&=-6.0727\times10^{-4}P,\\ a_5&=-1.9468\times10^{-5}P,\\ a_7&=-1.2938\times10^{-8}P. \end{aligned}\right\} \tag{48}$$

Therefore, the deflection at the center of the plate is

$$w_{\xi=\eta=0}=0.020051\frac{pa^4}{D} \tag{49}$$

Now, we shall apply our method to this problem. As the expression (42) satisfies (37) and (38) but does not $\frac{\partial w}{\partial n}=0$, at boundary C, we impose the fol'owing boundary conditions

$$\left.\begin{aligned} &\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\partial w}{\partial\xi}\cos j\eta d\eta=0 \qquad \text{at } \xi=\pm\frac{\pi}{2},\\ \text{and}\qquad &\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\partial w}{\partial\eta}\cos j\xi d\xi=0 \qquad \text{at } \eta=\pm\frac{\pi}{2}, \end{aligned}\right\} \tag{50}$$

in place of $\frac{\partial w}{\partial n}=0$.

When we adopt $j=0, 2,\cdots\cdots 2(k-1)$ and insert (42) into (50), we get the following equations

$$\sum_{n=1}^{2k-1}{}' a_n B'_{nj}=\sum_{n=1}^{2k-1}{}' A'_{nj} \tag{51}$$

where

$$B'_{nj}=\left(n\sinh\frac{n\pi}{2}-\frac{2}{\pi}\cosh\frac{n\pi}{2}-\frac{n\cosh^2 n\pi}{\sinh\frac{n\pi}{2}}\right)\left\{\frac{\sin(n-j)\frac{\pi}{2}}{n-j}+\frac{\sin(n+j)\frac{\pi}{2}}{n+j}\right\}$$
$$-n\sin n\frac{\pi}{2}\left\{\frac{2n}{n^2+j^2}\cos\frac{j\pi}{2}\sinh\frac{n\pi}{2}-\frac{2n}{n^2+j^2}\cos\frac{j\pi}{2}\frac{\cosh^2\frac{n\pi}{2}}{\sinh\frac{n\pi}{2}}\right.$$
$$\left.+\frac{4(n^2-j^2)}{\pi(n^2+j^2)^2}\cos\frac{j\pi}{2}\cosh\frac{n\pi}{2}\right\}, \tag{52}$$

and

$$A'_{nj}=\sum_m{}'\frac{16P\sin\frac{n\pi}{2}}{\pi^2 n(m^2+n^2)}\left\{\frac{\sin(n-j)\frac{\pi}{2}}{n-j}+\frac{\sin(n+j)\frac{\pi}{2}}{n+j}\right\}. \tag{53}$$

When we take $k=4$, the values of a_1, a_3, a_5 and a_7 are calculated as follows;

$$\left.\begin{aligned} a_1&=-0.13651P,\\ a_3&=-2.058\times10^{-4}P,\\ a_5&=-1.286\times10^{-5}P,\\ a_7&=6.51\times10^{-8}P. \end{aligned}\right\} \tag{54}$$

Therefore, the deflection at the center of the plate is

$$w_{\xi=\eta=0}=0.019946\frac{pa^4}{D}. \tag{55}$$

When we adopt $j=1, 3, \cdots\cdots, (2k-1)$ we obtain the results (48). This case is identical with that of Weinstein.

Now we consider the following expression

$$w=\frac{P}{8}\left(\xi^2-\frac{\pi^2}{4}\right)\left(\eta^2-\frac{\pi^2}{4}\right)+\sum_{n=1}^{2k-1}{}' f_n(\xi)\cos n\eta+\sum_{n=1}^{2k-1}{}' f_n(\eta)\cos n\xi. \tag{56}$$

This expression satisfies (37) and (38), but does not $\frac{\partial w}{\partial n}=0$ at the boundary. We impose to the solution (56) the conditions (50) in place of $\frac{\partial w}{\partial n}=0$.

When we insert (56) into (50), we get the following equations

$$\sum_{n=1}^{2k-1}{}' a_n B''_{nj}=A''_{jj}. \quad \text{for } j=1,\ 3,\ \cdots\cdots,\ 2k-1 \tag{57}$$

where

$$B''_{nj}=\delta_{nj}\left(n\sinh\frac{n\pi}{2}-\frac{2}{\pi}\cosh\frac{n\pi}{2}-\frac{n\cosh^2\frac{n\pi}{2}}{\sinh\frac{n\pi}{2}}\right)$$

$$-\frac{16n^2 j\sin\frac{n\pi}{2}\sin\frac{j\pi}{2}\cosh^2\frac{n\pi}{2}}{\pi^2(n^2+j^2)^2\sinh\frac{n\pi}{2}} \tag{58}$$

and

$$A''_{jj}=\frac{P}{j^3}\sin\frac{j\pi}{2}. \tag{59}$$

When $k=4$, from the equations (57), we get the following result

$$\left.\begin{aligned} a_1&=-0.31931P,\\ a_3&=-8.9109\times10^{-5}P,\\ a_5&=-1.1458\times10^{-5}P,\\ a_7&=6.1297\times10^{-9}P. \end{aligned}\right\} \tag{60}$$

The deflection of the center of the plate is given by

$$w_{\xi=\eta=0}=0.020035\frac{pa^4}{D}. \tag{61}$$

In this case, we may put $s=0, 2, \cdots\cdots, 2(k-1)$ or $s=0, 1, 2\cdots\cdots k-1$ and obtain the solution in the similar way described above. It is noted that the solution (56) does not satisfy the condition (39) and therefore, can not be applicable to the method of Weinstein.

The deflection of the center of the plate given by Hencky is

$$w_{\xi=\eta=0}=0.020243\frac{pa^4}{D}. \tag{62}$$

This value is in good agreement with (49), (55) and (61).

4. Conclusion

As it was seen in the preceding three examples, the new method explained in this paper gives a good method to get the solutions of the problems in applied mathematics, especially in elasticity and its accuracy is also satisfactorily good.

The main feature of this method is to use the boundary conditions

$$\int_c M_i(w)h(x, y)ds=0 \qquad \text{in place of } M_i(w)=0.$$

The problem of buckling of a clamped square submitted to the action of shearing forces uniformly distributed along the edges will be also solved by this method. Its solution will be published in this Bulletin.

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On Buckling of Clamped Square Plate in Shear

Misao Hasegawa

Let us consider a clamped square plate submitted to the action of shearing forces S uniformly distributed along the edges. The purpose of this paper is to calculate the exact critical value of shearing stress at which buckling of the plate occurs. The method we used is as follows.

Let the coordinate axes (x, y) be taken in the middle plane of the plate of uniform small thickness in such a way that the origin coincides with the center of the plate and the axes are parallel to the sides, and let the length of the side of the square, the thickness, the density, Young's modulus, and Poisson's ratio of the plate be denoted by $2a$, h, ρ, E and σ respectively. If w be the transversal displacement of a point on the middle plane, the differential equation for w is given by

$$D\left(\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}\right)-2S\frac{\partial^2 w}{\partial x\partial y}=0 \tag{1}$$

where

$$D=\frac{Eh^3}{12(1-\sigma^2)}.$$

The boundary conditions for w are

$$\left.\begin{aligned} w=\frac{\partial w}{\partial x}&=0 \qquad \text{at} \quad x=\pm a,\\ w=\frac{\partial w}{\partial y}&=0 \qquad \text{at} \quad y=\pm a. \end{aligned}\right\} \tag{2}$$

If we introduce the new variables such that

$$\xi=\frac{\pi}{2}\frac{x}{a}, \qquad \eta=\frac{\pi}{2}\frac{y}{a}$$

the equations (1) and (2) are transformed as follows:

$$\frac{\partial^4 w}{\partial \xi^4}+2\frac{\partial^4 w}{\partial \xi^2\partial \eta^2}+\frac{\partial^4 w}{\partial \eta^4}-4\lambda\frac{\partial^2 w}{\partial \xi\partial \eta}=0 \tag{3}$$

$$\left.\begin{aligned} w=\frac{\partial w}{\partial \xi}&=0 \qquad \text{at} \quad \xi=\pm\frac{\pi}{2},\\ w=\frac{\partial w}{\partial \eta}&=0 \qquad \text{at} \quad \eta=\pm\frac{\pi}{2}, \end{aligned}\right\} \tag{4}$$

in which

$$\lambda=\frac{2Sa^2}{D\pi^2}. \tag{5}$$

Combining all possible particular solutions of (3) which satisfy the relations $w(\xi, \eta)=w(\eta, \xi)=w(-\xi, -\eta)=w(-\eta, -\xi)$, we obtain as the solutions of (3)

$$w=A_0\cos\sqrt{\lambda}\,(\xi-\eta)+\sum_{n=1}^{\infty}A_n\cosh\alpha_n(\xi-\eta)\cos\beta_n(\xi-\eta)\cos n(\xi+\eta)$$

$$
\begin{aligned}
&+\sum_{n=1}^{\infty} A'_n \sinh \alpha_n(\xi-\eta) \sin \beta_n(\xi-\eta) \cos n(\xi+\eta) \\
&+\sum_{n=0}^{\infty} B_n \cosh \omega_n(\xi+\eta) \cos n(\xi-\eta) \\
&+\sum_{n=0}^{2} B'_n \cos \nu_n(\xi+\eta) \cos n(\xi-\eta) \\
&+\sum_{n=0}^{\infty} B'_n \cosh \omega'_n(\xi+\eta) \cos n(\xi-\eta) \qquad (6)
\end{aligned}
$$

where

$$\omega_n=\left\{\left(n^2+\frac{\lambda}{2}\right)+\sqrt{\frac{\lambda^2}{4}+2\lambda n^2}\right\}^{\frac{1}{2}}, \quad (n=0,\ 1,\ 2,\cdots\cdots)$$

$$\nu_n=-\left\{\left(n^2+\frac{\lambda}{2}\right)+\sqrt{\frac{\lambda^2}{4}+2\lambda n^2}\right\}^{\frac{1}{2}}, \quad (n=0,\ 1,\ 2)$$

$$\omega'_n=\left\{\left(n^2+\frac{\lambda}{2}\right)-\sqrt{\frac{\lambda^2}{4}+2\lambda n^2}\right\}^{\frac{1}{2}}, \quad (n=3,\ 4,\cdots\cdots)$$

and

$$\alpha_n^2-\beta_n^2=n^2-\frac{\lambda}{2}, \quad (\alpha_n>0,\ \beta_n>0) \quad (n=1,\ 2,\cdots\cdots)$$

$$4\alpha_n^2\beta_n^2=-\frac{\lambda^2}{4}+2\lambda n^2 \quad (n=1,\ 2,\cdots\cdots)$$

Now we determine the ratio's of unknown constants $A_0 : A_1 : \cdots\cdots$ so as to satisfy the modified boundary conditions

$$
\left.
\begin{aligned}
&\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} w \begin{matrix}\sin s\eta \\ \cos s\eta\end{matrix}\, d\eta=0, \\
&\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\partial w}{\partial \xi} \begin{matrix}\sin s\eta \\ \cos s\eta\end{matrix}\, d\eta=0
\end{aligned}
\right\} \text{ at } \xi=\pm\frac{\pi}{2} \qquad (7)
$$

$$
\left.
\begin{aligned}
&\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} w \begin{matrix}\sin s\xi \\ \cos s\xi\end{matrix}\, d\xi=0, \\
&\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\partial w}{\partial \eta} \begin{matrix}\sin s\xi \\ \cos s\xi\end{matrix}\, d\xi=0,
\end{aligned}
\right\} \text{ at } \eta=\pm\frac{\pi}{2},
$$

$(s=0,\ 1,\ 2,\cdots\cdots)$

in place of (4).

The smallest critical value of λ at which buckling of the plate occurs is found from the conditions (7) and is as follows:

$$\lambda=\frac{2a^2S}{D\pi^2}=7.32\cdots\cdots$$

The value of $\lambda=7.32\cdots\cdots$ exists in the range calculated by B. Budiansky and R. W. Connor. They obtained 7. 319 and 7.396 as the lower and upper limits of λ.