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# Some Notes on Algebraic Groups with Affine Root Structures (1)

Yasuhiro Asoo
Department of Mathematics, Shimane University, Matsue, Japan
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**In the present note, we summarize some results needed to develop representation theory of the titled groups about its structures.**

## § 0. Introduction

Natural geometrical objects attached to noncompact real semisimple Lie groups are the symmetric spaces and we have many deep results on representation theory of these groups and analysis on these spaces, which are intimately related. In 1955 C. Chevalley, in his fundamental paper "Sur certains groupes simples" (Tohoku math. J., 7, pp. 14–66), develops fundamental theory of Chevalley groups over any field.

At present we have a natural geometrical object attached to algebraic groups over locally compact non-archimedean field, that is, Tits affine building. In the present paper, we consider groups with "the discrete-valuated generic root-data system" in the sense of Bruhat-Tits. Algebraic groups defined over a complete local field with perfect residue field whose neutral component is reductive are contained in this class. To develop representation theory of these groups which is explained in a forthcoming paper, we summarize here some results of structures of these groups. We do not endow our groups with any further topological conditions for the present. I. G. Macdonald [3] and H. Matsumoto [4] get more deep results in the harmonic analysis related to this class under some topological restrictions.

In § 1, we describe some fundamental definitions and properties. In § 2, we construct a saturated affine Tits system associated with our groups, and in § 3 we construct its affine building. In § 4, we give the natural bornologic structure in our groups, and in § 5, we give Iwasawa, and Cartan decompositions.

Our main results are these decompositions, Bruhat decomposition and specification of actions of our group $G$ on our building stated in § 3. Some parts are informally presented at 1979-Japan and France colloquy on "Unitary representations of groups" at Strasbourg. The author acknowledge Professor J. Tits, Professor H. Matsumoto, Professor T. Hirai and Professor N. Tatsuuma for their valuable comments and encouragements.

## § 1. Some fundamental definitions and properties

At first we will recall some definitions. Let $V$ be a real vector space, $V^*$ be its dual and $\Phi$ be a root system in $V^*$ with the Weyl group ${}^vW$. We assume that there is given a ${}^vW$-invariant scalar product in $V$. Fix a vector chamber $D$ in $V$ and $\Pi=\Pi(D)$ be the system of simple roots of $\Phi$ associated with the chamber $D$, and $\Phi^+$ (resp. $\Phi^-$) be the set of positive roots (resp. negative roots) in $\Phi$.

DEFINITION 1. A system $(T, (U_a, M_a)_{a\in\Phi})$ associated with a group $G$ satisfying the following conditions is called a "system of generic root data" of type $\Phi$ in a group $G$:

(DR1) $T$ is a subgroup of $G$ and for each $a\in\Phi$, $U_a$ is a subgroup of $G$ which does not reduce to $\{e\}$, $e$ is the unit element of $G$.

(DR2) For any $a, b\in\Phi$, the commutator subgroup $(U_a, U_b)$ is contained in the group generated by $U_{pa+qb}$, where $p, q\in N^*$ and $pa+qb$ belongs to $\Phi$, which we denote by $U_{(a,b)}$.

(DR3) If $a, 2a\in\Phi$, we have $U_{2a}\subsetneqq U_a$.

(DR4) For each $a\in\Phi$, $M_a$ is a right coset in $G$ with respect to $T$ and we have

$$U^*_{-a}=U_{-a}-\{e\}\subset U_aM_aU_a.$$

(DR5) For $a, b\in\Phi$ and $n\in M_a$, we have $nU_bn^{-1}=U_{r_a(b)}$, where $r_a(\cdot)$ is the reflection associated to $a\in\Phi$.

(DR6) Denoting $U^+$ (resp. $U^-$) the group generated by $U_a$, $a\in\Phi^+$ (resp. $\Phi^-$), we have $TU^+\cap U^-=\{e\}$.

(DR7) $T$ and $U_a$, $a\in\Phi$, generate the group $G$.

As a typical example satisfying Def. 1, we mention Chevalley group over any field. Moreover, let $G$ be an algebraic $K$-group, with char$(K)=0$. Then it is known that $G$ is the semi-direct product of Levi $K$-subgroup and its unipotent radical $Ru(G)$. The group $G/Ru(G)$ is reductive and we can associate with it a "system of generic root data" taking its maximal semi-simple subgroup; that is, let $G$ be a reductive connected group: $G$ is an almost direct product of a torus and its derived group $\mathscr{D}G$: $\mathscr{D}G$ is the maximal semi-simple subgroup of $G$ and contains all unipotent subgroups of $G$: the group $G$ satisfies Def. 1.

REMARK. Let $G$ be a $K$-group which is algebraic and connected whose any proper closed invariant subgroup is finite, i.e., $G$ is an almost simple group. Then we have $(U_a, U_b)=U_{(a,b)}$ except for the following cases; Char$(K)=2$, $G=B_n, C_n, G_2, F_4$; char$(K)=3$, $G=G_2$.

Let $N$ be the group generated by the union of $M_a$, $a\in\Phi$. We have a unique epimorphism ${}^vv\colon N\to{}^vW$ such that for each $a\in\Phi$ and $n\in N$, $nU_an^{-1}=U_b$ with $b={}^vv(n)(a)$. Moreover, for each $a\in\Phi$, ${}^vv(M_a)=\{r_a\}$. We have the following

PROPOSITION 1. (cf. [1], p. 115) *Let* $R=\{M_a/a\in\Pi\}$. *Then the quadruplet* $(G, TU^+, N, R)$ *is a saturated Tits system with Weyl group* $N/T$ *isomorphic to* ${}^vW$.

The conjugates of $B=TU^+$ are Borel subgroups. We note that $N$ is generated by the union of $M_a$, $a\in\Pi$ and $({}^vv)^{-1}(e)=T=N\cap TU^+$. We have also $\bigcap_{n\in N} nTU^+n^{-1}=T$. The injection of $N$ into $G$ defines a bijection of $N$ (resp. ${}^vW$) on the set of double cosets $U^+\backslash G/U^+$ (resp. $TU^+\backslash G/TU^+$). (Bruhat decomposition)

Let $X$ be a subset of $\Pi$, ${}^vW_X$ be the subgroup of ${}^vW$ generated by $r_a$, $a\in X$. Put $G_X=U^+T^vW_XU^+$ and $N_X=({}^vv)^{-1}(T^vW_X)$. Then, we have $G_X=U^+TN_XU^+$ and it is a parabolic subgroup of $G$ containing $B$, and the application $X\mapsto G_X$ is a bijection of $2^\Pi$ on the set of parabolic subgroups of $G$ containing $B$, and $\{G_X, X\in 2^\Pi\}$ is a filtration of $G$.

PROPOSITION 2. *For any* $w\in{}^vW$, *let* $E_w$ *be the set* $\{a\in\Phi^+;\ w(a)\in\Phi^-\}$ *and* $U_w$ *be the group generated by* $\{U_a;\ a\in E_w\}$. *The group* $G$ *is the union of sets* $U^+T^vv^{-1}(w)U_w$, $w\in{}^vW$. *The correspondence* $w\mapsto U^+T^vv^{-1}(w)U_w$ *is bijective. Let* $\pi$ *be the canonical projection* $G\to G/T$; $G/T$ *is the disjoint union of* $\pi(U^{+v}v^{-1}(w)U_w)$, $w\in{}^vW$ *and given* $w\in{}^vW$, *the correspondence* $(u, u')\mapsto(u^vv^{-1}(w)u')$ *of* $U^+\times U_w$ *into* $G/T$ *is bijective.*

PROOF. Let $E'_w$ be the set $\{a\in\Phi^+;\ w(a)\in\Phi^+\}$ and $U'_w$ be the group generated by $\{U_a;\ a\in E'_w\}$. Then, we have $U^+=U_wU'_w=U'_wU_w$. By the Bruhat decomposition, $G=U^+TNU'_wU_w$: $G$ is the union of sets $U^+T^vv^{-1}(w)U_w$.

Suppose that $ut^vv^{-1}(w)u'=u_1t^vv^{-1}(w_1)u'_1$, $u, u_1\in U^+$, $t, t_1\in T$, $w, w_1\in{}^vW$ and $u'\in U_w$, $u'_1\in U_{w_1}$. Put $x=ut^vv^{-1}(w)u'$. Let $u''\in U^+$ be such that $xu''x^{-1}\in U^+$. Then we must have $u''\in u'^{-1}U'_wu'$ and we have $U_w=U_{w_1}$ and $E_w=E_{w_1}$, $E'_w=E'_{w_1}$. Let $a\in\Phi^+$. When $w^{-1}(a)\in\Phi^+$, we have $w^{-1}(a)\in E'_w=E'_{w_1}$ and $(w_1w^{-1})(a)\in\Phi^+$, and when $w^{-1}(a)\in\Phi^+$, we have $-w^{-1}(a)\in E_w=E_{w_1}$ and $(w_1w^{-1})(a)\in\Phi^+$, so that $w_1=w$. Put $y={}^vv^{-1}(w)u'({}^vv^{-1}(w))^{-1}$ and $y_1={}^vv^{-1}(w)u'_1({}^vv^{-1}(w))^{-1}$, then we have $y=y_1$ and $u'=u'_1$, and also $u=u_1$, $t=t_1$. c.q.f.d.

DEFINITION 2. We call a family $\phi=\{\phi_a\colon a\in\Phi\}$, where $\phi_a$ is an application of $U_a$ into $\boldsymbol{R}\cup\{\infty\}$, a *valuation* of the generic root-data $(T, (U_a)_{a\in\Phi})$ when it has the following properties:

(V1) For each $a\in\Phi$, the image of $\phi_a$ consists of at leats three elements.

(V2) For each $a\in\Phi$, and $k\in\boldsymbol{R}$, $U_{a,k}=\phi_a^{-1}([k,\infty))$ is a subgroup of $U_a$ and we have $U_{a,\infty}=\{e\}$.

(V3) For each $a\in\Phi$ and each $n\in M_a$, the function

$$u\longmapsto\phi_{-a}(u)-\phi_a(nun^{-1})$$

is constant on $U^*_{-a}$.

(V4) Let $a, b\in\Phi$ and $k, m\in\boldsymbol{R}$. Then if $b\notin-\boldsymbol{R}^+a$, the commutator group $(U_{a,k}, U_{b,m})$ is contained in the group generated by the groups $U_{pa+qb,pk+qm}$ where $p, q\in\boldsymbol{N}^*$ and $pa+qb\in\Phi$.

(V5) When $a$, $2a$ belong to $\Phi$, $\phi_{2a}$ is the restriction of $2\phi_a$ on $U_{2a}$.

(V6) For any $a\in\Phi$, $u\in U_a$ and $u'$, $u''\in U_{-a}$, when $u'uu''\in M_a$, we have $\phi_{-a}(u') = -\phi_a(u)$.

We remark that for any $a\in\Phi$ and $u\in U^*_{-a}$, there exits a unique element $n=n(u)\in M_a$ such that $u=u'nu''$, $nU_an^{-1}=U_{-a}$, $nU_{-a}n^{-1}=U_a$ and $u'$, $u''\in U^*_a$. The family $\{U_{a,k};\ k\in\boldsymbol{R}\}$ gives us a filter base of neighborhood of the unit element in the group $U_a$. Let $\phi$ be a valuation of the generic root-data $(T, (U_a)_{a\in\Phi})$ and put $\Gamma_a=\Gamma^\phi_a=\phi_a(U^*_a)$ for $a\in\Phi$. The valuation $\phi$ is called *discrete* when $\Gamma_a$ is a discrete subset of $\boldsymbol{R}$ for each $a\in\Phi$. Let $\lambda$: $\Phi\to\boldsymbol{R}_+$ be a function and $v\in V$. Define $\psi_a=\lambda(a)\phi_a(u)+a(v)$ for each $a\in\Phi$ and $u\in U_a$. Then $\Psi=\{\psi_a\colon a\in\Phi\}$ is also a valuation, which is written as $\Psi=\lambda\phi+v$. For each $n\in N$ and $w={}^v\nu(n)$, we define $(n\phi)_a=\phi_{w^{-1}(a)}(n^{-1}un)$, $u\in U_a$. Then we have a formula $n(\lambda\phi+v)=\lambda(n\phi)+{}^v\nu(n)(v)$. The valuations $\phi$ and $\Psi=\phi+v$ $(v\in V)$ are called *equipollent*.

EXAMPLE. Chevalley groups over $\mathscr{P}$-adic field

Let $K$ be a $\mathscr{P}$-adic field with discrete valuation $\phi$ and $G$ be a Chevalley group over $K$. Let $U_{a,k}$ be a group such that $U_{a,k}=\{u(x)\in U_a;\ x\in K$ and $\phi(x)\geqq k\}$, $a\in\Phi$ and $k\in\boldsymbol{R}$. Then these satisfy our Def. 2.

## §2. Construction of a saturated affine Tits system

Let $\phi$ be a discrete valuation and $A$ be the set of valuations equipollent to $\phi$. For each $a\in V^*$ and $k\in\boldsymbol{R}$, we define $\alpha_{a,k}=\{x=\phi+v\in A/a(v)+k\geqq 0\}$. We call $\alpha_{a,k}$ with $a\in\Phi$ and $k\in\Gamma'_a=\{\phi_a(u)/u\in U^*_a,\ \phi_a(u)=\sup\phi_a(uU_{2a})\}$ *affine roots* of the space $A$. We denote by $\Sigma$ the set of affine roots. Let $\Phi^{red}$ (resp. $\Phi^{nm}$) be the set of indivisible (resp. non-multiable) roots in $\Phi$. For each $\alpha=\alpha_{a,k}\in\Sigma$ with $a\in\Phi^{red}$, we put $U_\alpha=U_{a,k}$. By its construction, we get

PROPOSITION 3. (cf. [1], p. 122) (1) *The space $A$ is stable under the action of $N$ defined in the above. For each $n\in N$, the application $\nu(n)$: $\psi\mapsto n\psi$ of $A$ is an automorphism of the Euclidean space $A$, whose canonical image in $Aut(V)$ is equal to ${}^v\nu(n)$.*

(2) *For each $a\in\Phi$, and $k\in\Gamma_a$, put $M_{a,k}=M_a\cap U_{-a}\phi_a^{-1}(k)U_{-a}$. Then the image of elements in $M_{a,k}$ by $\nu$ is the orthogonal reflection $r_{a,k}$ with the reflecting hyperplane*

$$\partial\alpha_{a,k}=\{x=\phi+v\in A/a(v)+k=0\}.$$

(3) *For each $n\in N$ and $\alpha\in\Sigma$, we have $\nu(n)(\alpha)\in\Sigma$ and $nU_\alpha n^{-1}=U_{\nu(n)(\alpha)}$.*

Let $H=\nu^{-1}(e)$, $\hat{W}$ be the image of $N$ by $\nu$ and $W$ be the subgroup of $\hat{W}$ generated by reflections $r_{a,k}$ with $a\in\Phi$ and $k\in\Gamma_a$, $N'=\nu^{-1}(W)$, $T'=T\cap N'$. Let $G'$ be the subgroup of $G$ generated by $N'$ and $U_a$ with $a \in \Phi$. We call the group $G'$ *the adjoint group of G*. By these definitions, we get

PROPOSITION 4. (cf. [1], pp. 123–128) (1) *The system* $(T', (U_a)_{a\in\Phi})$ *is a generic root-data of type* $\Phi$ *in* $G'$.

(2) *The set* $\Phi' = \{a \in \Phi / \Gamma'_a \neq \phi\}$ *is a root system containing* $\Phi^{nm}$. $W$ *is an affine Weyl group and* $\Sigma$ *is the corresponding affine root system.*

The group $G'$ is a normal subgroup of $G$ and $G/G'$ is isomorphic to $T/T' \cong N/N'$.

A point $\psi$ of $A$ is called *special* if for each $a \in \Phi^{red}$, $\Gamma_a^{\psi}$ contains zero. Taking a special point as the origin, we identify ${}^vA$ with $V$. We see that $W$ is the semidirect product of ${}^vW$ by the invariant vector subgroup $V \cap W$, generated by translations $ka^v$, $a \in \Phi^{red}$ and $a^v$ is the inverse root of $a$, and $k$ are elements of the subgroup of $\boldsymbol{R}$ generated by $\Gamma_a$; under the natural topology of $W$, $W$ is locally compact and ${}^vW$ is a maximal compact subgroup of $W$.

Let $C$ be a $W$-chamber contained in $D$, and for each $a \in \Phi$, put $f_C(a) = \inf\{k \in \boldsymbol{R};\ \alpha_{a,k} \supset C\}$. Let $U_{f_C}$ be the subgroup of $G$ generated by the union of subgroups $U_{a,f_C(a)}$, $a \in \Phi$ and put $B = HU_{f_C}$. Checking conditions of saturated affine Tits system, we get

PROPOSITION 5. *Let S be the set of reflections with respect to walls of the chamber C. Then* $B \cap N' = H$ *and* $N'/H = W$. *The quadruplet* $(G', B, N', S)$ *is a saturated affine Tits system.*

## §3. Construction of an affine building

In the situation described in §2, conjugates of the group $B$ are called *Iwahori subgroups* and each proper subgroup of $G'$ which contains an Iwahori subgroup is called a *paraholic subgroup*. Let $X$ be a subset of $S$ and $W_X$ the subgroup of $W$ generated by elements of $X$ and put $B_X = BW_XB$. When a paraholic subgroup of $G'$ is conjugate to $B_X$, we say that it has *type X*.

We will associate to the group $G'$ the "affine building" and endow it with the "bornologic structure". Let $\mathcal{V}$ be the set of parabolic subgroups of $G'$. Each paraholic subgroup $P$ has type $\tau(P)$, and the latter defines a facet of the chamber $C$, which we denote by $C_{\tau(P)}$. Put $I = \{(P, x)/P \in \mathcal{V}, x \in C_{\tau(P)}\}$. For each $P \in \mathcal{V}$, the set $F = F(P) = \{(P, x);\ x \in C_{\tau(P)}\}$ is called a facet of $I$ with type $\tau(F) = \tau(P)$ and codimension Card$(\tau(P))$. An application $(P, x) \mapsto x$ is called the application of $I$ into $\bar{C}$. Let $\bar{F}$ be the union of facets of the facet $F$ of $I$. We define chambers of $I$ to be facets of type $\phi$, and the facets corresponding to maximal paraholic subgroups are vertices of $I$. When rank$(\Phi) = l$, $C$ is a $l$-simplex. The group $G'$ acts on $I$ by the action $g(P, x) = (gPg^{-1}, x)$ for $P \in \mathcal{V}$ and $x \in C_{\tau(P)}$. By these constructions, we get the following two propositions:

PROPOSITION 6. (1) *For any type X, the group* $G'$ *permutes transitively facets* *of type $X$; let $P_X$ be a paraholic subgroup of $G'$ of type $X$, then $G'/P_X$ is naturally identified with the set of facets of type $X$. In particular, $G'/B$ is identified with the set of chambers in $I$.*

(2) *The closed facet $\overline{F(P)}$ is the $P$-stable point set of $I$ and $\bar{C}$ is a fundamental domain of $G'$ in $I$.*

PROPOSITION 7. *The set $I$ equipped with the family of facets, the incidence relation among facets and the affine structures on each of closed facet $\overline{F(P)}$, $P \in \mathcal{V}$, is a (poly)simplicial complex, and $G'$ operates on $I$ by automorphism of (poly)simplicial complex. In particular, when $l=1$, $I$ is a tree.*

The next two propositions due to Bruhat-Tits are fundamental.

PROPOSITION 8. *There exists a unique mapping $j: A \to I$ having the following properties;*

(1) *the restriction of $j$ to $\bar{C}$ is the bijection of $\bar{C}$ onto $\overline{F(B)}$,*
(2) *for any $n \in N'$ and $x \in A$, we have*

$$j(\nu(n)) = nj(x).$$

PROOF. The unicity follows from the fact $\bar{C}$ being fundamental domain. Let $j_0: \bar{C} \to \overline{F(B)}$ be such that $j_0(x)$ is facets containing $x \in \bar{C}$. Let $x \in C_{\tau(P)}$, $x' \in C_{\tau(P')}$, $n, n' \in N$ be such that

$$(1) \quad nj_0(x) = n'j_0(x'), \qquad (2) \quad \nu(n)x = \nu(n')x'.$$

The condition (2) is equivalent to $x = x'$ and $\nu(n^{-1}n') \in W_{\tau(P)}$; the condition (1) is equivalent to $x = x'$ and $n^{-1}n' \in B_{\tau(P)}$; and we have $\nu^{-1}(W_{\tau(P)}) = B_{\tau(P)} \cap N$. Thus, the condition (1) and (2) are equivalent and $j$ exists and injective. c.q.f.d.

Note that $j(F)$ is a facet of $j(A)$ having the same type with $F$, $j(\bar{F}) = \overline{j(F)}$ and the restriction of $j$ to $\bar{F}$ is a bijective affine map of $\bar{F}$ onto $\overline{j(F)}$, in particular the restriction of $j$ to $\bar{C}$ is a bijective affine map of $\bar{C}$ onto $\overline{F(B)}$, thus $\overline{F(B)}$ has a natural $l$-simplex structure, and $j(A)$ has a natural affine structure.

We call the application $j$ *canonical application* of $A$ into $I$. An application $\psi$ of $A$ into $I$ such that $\psi(x) = gj(x)$ with $g \in G'$ is called a *structural application* of $I$ and a subset of $I$ which is an image of $A$ by a structural application is called *an appartment of $I$*. $j(A)$ is an appartment which is identified with $A$ and $G'$ acts on $A$; $gx = gj(x)$, $x \in A$ and $g \in G'$. Under this action, $H$ is the fixer in $G'$ of $A$ and $N'$ is the stabilizer in $G'$ of $A$. Let $\psi$ be a structural application of $I$, and $x \in \bar{C}$, then $\psi(x) = gj(x)$ which belongs to $g\overline{F(B)} = \overline{F(gBg^{-1})}$, $gBg^{-1}$ is a Iwahori subgroup fixing chamber $\overline{gC}$, and for any $n \in N'$ and $x \in A$, we have $\psi(nx) = gn \cdot x = g \cdot nx$ which belongs to $gA$, thus $\psi$ maps $A$ to $gA$. $G'/N'$ is naturally identified with the set of appartments. In some cases, $\psi$ may be continuous.

The group $G$ acts also on the building associated to $G'$. We will specify this action. The group $G$ is generated by $T$ and $G'$ and we have $G=T\cdot G'=N\cdot G'$.

When $n\in N$, we have $nN'n^{-1}=N'$ and $nBn^{-1}$ is the stabilizer of the chamber $v(n)C$ so that there exists an $n'\in N'$ such that $nBn^{-1}=n'Bn'^{-1}$. Moreover, let $\hat{N}$ be the stabilizer of the appartment $A$ in $G$. Then we have $N\subset\hat{N}$ and $\hat{N}=N(\hat{N}\cap G')=NN'=N$. Let $g\in G$, then there exists $g'$ of $G'$ such that $g'N'g'^{-1}=gN'g^{-1}$, so that there exists the unique permutation $\xi(g)$ of the affine Weyl group $W$ such that for any $w\in W$,

$$B\xi(g)(w)B=g'^{-1}gBwBg^{-1}g'.$$

We see that the application $\xi\colon G\to S_W$, where $S_W$ is the permutation group of $W$, is a homomorphism. Let $g\in G$ be an element of Ker$(\xi)$, then we have $BwB=g'^{-1}gBwBg^{-1}g'$, $w\in W$. This shows that $G'\subset$Ker$(\xi)$ and for any $g=tg'$, $g\in G$, $g'\in G'$ and $t\in T$, we have $B\xi(g)(w)B=B\xi(t)(w)B$, $w\in W$.

When $P$ is a paraholic subgroup of type $X$ in $G'$, ${}^gP=gPg^{-1}$ is a paraholic subgroup of type $\xi(g)(X)$ in $G'$. Thus, if $y=(P, x)\in I$, the couple $gy=({}^gP, \xi(g)x)$ is also a point of $I$, and in this way, we define the action of $G$ on the building $I$. Under this action, the stabilizer of a facet $F(P)$ in $G$ is the subgroup $P$ and $B$ is the fixer of chamber $F(B)$.

PROPOSITION 9. *Let $A$ be an appartment and $C$ a chamber contained in $A$. Then there exists a unique application $\rho=\rho_{A,C}$ of $I$ into $A$, called the retraction of $I$ into $A$ with center $C$, such that* (1) $\rho(C)=C$; (2) *for each appartment $A'$ containing $C$, there exists $g\in G'$ such that $\rho(x)=gx$, $x\in A'$.*

By a structural application, we transport the affine-space structure of the space $A$ into an appartment $gA$, in particular we could define a metric $d_A$ on the appartment $A$.

There exists a unique function $d\colon I\times I\to\boldsymbol{R}_+$ such that

(1) its restriction on $A\times A$ is the metric $d_A$,

(2) for each $x, y\in I$, we have $d(\rho(x), \rho(y))\leqq d(x, y)$ and when $x\in\bar{C}$, we have $d(\rho(x), \rho(y))=d(x, y)$,

(3) $d$ is a complete metric on $I$ and the metric space $I$ is contractible.

Now we define the affine building of our Tits system $(G', B, N', S)$.

DEFINITION 3. The affine building $I$ associated to the saturated affine Tits system $(G', B, N', S)$ is the set $I$ equipped with the (poly)simplicial complex structure, the family of structural applications and the metric.

The application $y\mapsto gy$ $(g\in G)$ of the building $I$ is an isometric automorphism of the (poly)simplicial complex $I$ and permutes appartments, quarters and walls in $I$.

REMARK. (Oral communication by Tits) When the affine rank$\geqq 4$, the group $G'$ is algebraic and $Aut(G')=Aut(I)$.

## § 4. Bornologic structures

In this section, we will give the "bornologic" structure on the groups $G'$ and $G$.

DEFINITION 4. A bornology on a group $G$ is a family $\Delta$ of subsets of $G$ having the following properties;

(1) $\Delta$ is stable under finite union and it contains all finite subsets.

(2) If $M \in \Delta$ and $M' \subset M$, then $M' \in \Delta$.

(3) If $M, M' \in \Delta$, then $M^{-1}M' \in \Delta$.

We call a set $M$ in $\Delta$ *a bounded set*. We see that $\{e\} \in \Delta$ and when $M \in \Delta$, $M^{-1}$ also belongs to $\Delta$.

*The bornology defined by Tits system* $(G, B, N)$ is the set $\Delta = \{X \subset G$; canonical image of $X$ in $B\backslash G/B$ is finite$\}$. Thus, $G$ itself is not bounded.

Now consider our Tits system $(G', B, N', S)$ and its associated building $I$. The group $G'$ acts on $I$ as isometry. The group of isometries of $I$, Isom$(I)$, has the natural bornology structure defined by sets $M$ such that

there exists a point $x \in I$ such that $\{g \cdot x; g \in M\}$ is bounded in $I$.

By these two definitions, we get the following propositions.

PROPOSITION 10. *The bornology in $G'$ defined by Tits system $(G', B, N', S)$ is the inverse image of the natural bornology in* Isom$(I)$ *by the canonical homomorphism of $G'$ into* Isom$(I)$.

Since the group $G$ also acts as an isometry, we define a bounded set in $G$ as inverse image of a bounded set in Isom$(I)$.

PROPOSITION 11. *Suppose $\Phi$ irreducible. Then, a subset of $G'$ is bounded if and only if it is a paraholic subgroup.*

REMARK. When the group $G'$ is defined over some locally compact, non-discrete, local field with discrete valuation, the subgroup $B$ is an open and compact subgroup of $G'$, and bounded subsets of $G'$ are relatively compact subsets, maximal bounded subgroups are maximal compact subgroups of $G'$.

Each appartment has its natural Euclidean space structure and we may endow the building $I$ with (1) a topology invariant by $G'$ which is naturally defined by the metric $d$, or (2) the quotient topology of the natural topology of the disjoint union of all appartments (CW-topology). When the group $G'$ is $G(K)$, the group of $K$-rational points of an algebraic group over complete local field with finite residue field, these two topologies coincide and the building $I$ is locally compact and the group $G'$ is also locally compact.

OPEN PROBLEM. What are conditions on the Tits system $(G', B, N', S)$ to make the building $I$ a locally compact space under its bornologic structure?

## § 5. Iwasawa, and Cartan decompositions

In this section, at first we will show that the Tits system $(G', B, N', S)$ is a *double Tits system* and will give Iwasawa, and Cartan decompositions of $G'$ and $G$.

Let us fix a vector chamber $D$ in $A$ and $E(D)$ be the set of quarters in $A$ with the direction $D$. For any subset $\Omega$ of $I$, put $P_\Omega = \mathrm{Fix}_{G'}(\Omega)$ and $P_\Omega^\dagger = \mathrm{Stab}_{G'}(\Omega)$. We know that $N' = P_A^\dagger$ and $B = P_C^\dagger = P_C$, where $C$ is the fixed chamber in $D$. Put $Q_D^0 = \cup \{P_E;\ E \in E(D)\}$. For any $g \in \nu^{-1}(V)$ and $E \in E(D)$, we have $\nu(g)E \in E(D)$ and $gP_E g^{-1} = P_{\nu(g)E}$. Thus, $\nu^{-1}(V) = \mathcal{N}_{G'}(Q_D^0)$ and $Q = Q_D = \nu^{-1}(V) Q_D^0$ is a subgroup of $G'$. The subgroup $Q \cap N' = \nu^{-1}(V)$ is also a normal subgroup of $N'$ and the quotient group $N'/(Q \cap N')$ is canonically identified with the Weyl group ${}^vW$.

Let $R$ be the set of reflections with respect to walls of vector chamber $D$. We will show that the quadruplet $(G', Q, N', R)$ is a Tits system with Weyl group ${}^vW$.

Definition 5. We call an affine Tits system $(G', B, N', S)$ a double Tits system when the quadruplet $(G', Q, N', R)$ is a Tits system with Weyl group ${}^vW$.

By Prop. 1 applied to the root-data $(T', (U_a))$ in the adjoint group $G'$, it is sufficient to show that $Q = T'U^+$. It is necessary and sufficient for $g$ to belong to $Q_D^0$ that there exists a $v \in V$ such that $t^{-1}gt \in B$ for all $t \in T$ with $\nu(t) \in v + D$. Since $B \subset U^- H U^+$ and $U^-$, $H$, $U^+$ are normalized by $T$, we have $Q_D^0 \subset U^- H U^+$. Moreover since $\{u \in U^- / tut^{-1} \in B$ for $\nu(t) \in v + D\} = \{e\}$, we have $Q_D^0 = HU^+$ and $Q = T'U^+$.

Thus we get

Proposition 12. (cf. [1], p. 154) *The quadruplet $(G', B, N', S)$ associated with the adjoint group $G'$ is a double Tits system, and $Q_D^0 = HU^+$, $Q_D = T'U^+$.*

We have, then, $G' = QN'Q = U^+T'N'U^+ = BN'Q = BN'U^+$. Let $\hat{V}$ be the subgroup of translations in $\hat{W}$ and for a quarter $E \in E(D)$, put $\hat{P}_E = \mathrm{Fix}_G(E)$, $\hat{Q}_D^0 = \cup \{\hat{P}_E;\ E \in E(D)\}$ and $\hat{Q}_D = \nu^{-1}(\hat{V})\hat{Q}_D^0$. Then the group $G$ has a decomposition

$G = BN\hat{Q}_D$, and $G = B\hat{W}B$ ("*Bruhat decomposition*").

A maximal paraholic subgroup $K$ of $G'$ (resp. a maximal bounded subgroup $K$ of $G$) is called a good subgroup when we have $G' = QK$ (resp. $G = \hat{Q}K$), and these subgroups correspond to *special vertices of the building $I$*. It is shown by Bruhat-Tits that any group $G_1$ with double Tits system has Iwasawa decomposition and Cartan decomposition with respect to good subgroups, and when a homomorphism $\theta: G_1 \to G_2$ is *B-N-adapted*, $G_2$ has also these decompositions. (cf. [1], pp. 71–106)

We will give these decompositions in our case. Since our group $G'$ has a double Tits system, we get

PROPOSITION 13. *Let $K$ be a good subgroup in the adjoint group $G'$ containing $B$. Then the followings are hold*:

(1) *Iwasawa decomposition $G'=Q_D^0VK=U^+VK$ and the canonical application of $V$ into $Q_D^0\backslash G'/K$ is bijective.*

(2) *Cartan decomposition $G'=KV_DK$, where $V_D=V\cap \bar{D}$, and the canonical application of $V_D$ into $K\backslash G'/K$ is bijective. Let $t\in V_D$, $t'\in V$, and $t''\in V_D$.*

(3) *If $KtK\cap Q_D^0t''K\neq\phi$, then for all dominant weights $p$ of with respect to $\Pi(D)$, $p(t-t')\geqq 0$, i.e., $t\geqq t'(D)$.*

(4) *$KtK\cap Q_D^0tK=tK$.*

(5) *If $t'\in V_D$ and $KtKt'K\cap Kt''K\neq\phi$, $t+t'\geqq t''(D)$.*

(6) *$t^{-1}t'^{-1}Kt'Kt\cap t''Kt''^{-1}K=t^{-1}Kt\cap K$.*

Also, taking $G_1$ as our $G'$, $G_2$ as our $G$ and $\theta$ as inclusion we get

PROPOSITION 14. *Let $K$ be a good subgroup of $G$ containing $B$. Then the followings are hold*:

(1) *Iwasawa decomposition $G=Q_D^0\hat{V}K$ and the canonical application of $\hat{V}$ into $\hat{Q}_D\backslash G/K$ is bijective.*

(2) *Cartan decomposition $G=K\hat{V}_DK$ and the canonical application of $\hat{V}_D^0$ into $K\backslash G/K$ is bijective, where $\hat{V}_D=V\cap \bar{D}$.*

*We have also* (3) *to* (6) *in Prop.* 13 *by replacing $V$, $V_D$ by $\hat{V}$, $\hat{V}_D$ respectively.*

Thus, in some cases, we can consider functions (with bounded supports) which are $B$-biinvariant, or $K$-biinvariant. In the forthcoming paper, we will give some results of representations of these groups, and to specify the structure of "Hecke algebra $H(G, P_X)$", where $P_X$ is a paraholic subgroup, may also be an interesting problem.

## References

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