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Right Self-Injective Semigroups are Absolutely Closed

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Hinkle [3] has shown that the direct product of column monomial matrix semigroups over groups is right self-injective. The author [12] has shown that the full transformation semigroup on a non-empty set (written on the left) is right self-injective and so every semigroup is embedded in a right self-injective regular semigroup. While absolutely closed semigroups has been first studied in Isbell [7]. In Howie and Isbell [5] and Scheiblich and Moore [8] it has been shown that inverse semigroups, totally division-ordered semigroups, right [left] simple semigroups, finite cyclic semigroups and full transformation semigroups are absolutely closed. In § 1 we show that every right [left] self-injective semigroup is absolutely closed. This gives alternative proofs that right [left] simple semigroups, finite cyclic semigroups and full transformation semigroups are absolutely closed. By using a result of [5] we show that the class of right [left] self-injective [regular] semigroups has the special amalgamation property. In § 2 we show that a commutative separative semigroup is absolutely closed if and only if it is a semilattice of abelian groups. By using a characterization of self-injective inverse semigroups [9] we give a structure theorem for self-injective commutative separative semigroups.

§ 1. Right self-injective semigroups

Throughout this paper we freely use the terms "right S-system", "S-homomorphism", "right self-injective" and so on, which are referred to [12].

Let A, B be semigroups such that A is a subsemigroup of B. Then by Isbell [7] the set $\{b \in B \mid f(b)=g(b)$ for all semigroups C and for all homomorphisms $f, g: B \to C$ such that $f \mid A = g \mid A\}$ is called the *dominion* of A in B and is denoted by $\mathrm{Dom}_B(A)$. A semigroup S is called *absolutely closed* if $\mathrm{Dom}_T(S)=S$ for all semigroups T containing S as a subsemigroup.

The following result is due to J. R. Isbell [7].

THEOREM 1. ([4, Isbell's zigzag theorem]) *Let T be a semigroup and S a subsemigroup of T. Then for each $d \in T$, $d \in \mathrm{Dom}_T(S)$ if and only if $d \in S$ or there exist $s_0, s_1, \dots, s_{2m} \in S$ and $x_1, \dots, x_m, y_1, \dots, y_m \in T$ such that $d = s_0 y_1$, $s_0 = x_1 s_1$, $s_{2i-1} y_i = s_{2i} y_{i+1}$, $x_i s_{2i} = x_{i+1} s_{2i+1}$ $(1 \leqq i \leqq m-1)$, $s_{2m-1} y_m = s_{2m}$ and $x_m s_{2m} = d$.*

By using Isbell's zigzag theorem we prove the following

THEOREM 2. *Every right [left] self-injective semigroup is absolutely closed.*

PROOF. Let S be a right self-injective semigroup and T a semigroup containing S as a subsemigroup. Suppose that there is $d \in \mathrm{Dom}_T(S) \backslash S$. Then by Isbell's zigzag theorem there exist $s_0, s_1, \dots, s_{2m} \in S$ and $x_1, \dots, x_m, y_1, \dots, y_m \in T$ such that $d = s_0 y_1$, $s_0 = x_1 s_1$, $s_{2i-1} y_i = s_{2i} y_{i+1}$, $x_i s_{2i} = x_{i+1} s_{2i+1}$ $(1 \leqq i \leqq m-1)$, $s_{2m-1} y_m = s_{2m}$ and $x_m s_{2m} = d$. Consider S, T as right S-systems. Let ι_T, ι_S denote the inclusion mappings $\iota_T: S \to T$, $\iota_S: S \to S$, respectively. Since S is right self-injective, there exists an S-homomorphism $\xi: T \to S$ such that $\xi \iota_T = \iota_S$. Hence, $s_0 = \xi(s_0) = \xi(x_1) s_1$, $\xi(x_i) s_{2i} = \xi(x_{i+1}) s_{2i+1}$ $(1 \leqq i \leqq m-1)$ and hence, $d = s_0 y_1 = \xi(x_1) s_1 y_1 = \xi(x_1) s_2 y_2 = \xi(x_2) s_3 y_2 = \cdots = \xi(x_{m-1}) s_{2m-2} y_m = \xi(x_m) s_{2m-1} y_m = \xi(x_m) s_{2m} \in S$. This is a contradiction. Then it follows that $\mathrm{Dom}_T(S) = S$. Therefore we have that S is absolutely closed. The theorem holds.

The next result follows from Theorem 2 and Corollary 1, 2 of [12].

COROLLARY 1. I. ([8, H. Scheiblich and K. Moore]) *Full transformation semigroups are absolutely closed.*
II. *The direct product of column [row] monomial matrix semigroups over groups is absolutely closed.*

According to [11] a semigroup S with 1 is called *completely right injective* if every right S-system is injective. It is clear that all the homomorphic images of a completely right injective semigroup are completely right injective, of course, right self-injective. It also follows from [12, Theorem 9] that any direct product of completely right injective semigroups is right self-injective. Thus we have

COROLLARY 2. I. *All the homomorphic images of a completely right injective semigroup are absolutely closed.*
II. *The direct product of completely right injective semigroups is absolutely closed.*

REMARK. It easily follows from Isbell's zigzag theorem that a semigroup S is absolutely closed if and only if S_0 $[S^1]$ is absolutely closed, where S_0 $[S^1]$ denotes the semigroup obtained from S by adjoining with zero [identity]. If a semigroup S is right simple, then $S_0^1 (=(S_0)^1)$ is completely right injective. Thus it follows from Corollary 2 that S is absolutely closed. Also, if a semigroup S is finite and cyclic, then we can show that S_0^1 is a self-injective semigroup (see [12]). Then it follows from Theorem 2 that S is absolutely closed. These results have been obtained by Howie and Isbell [5].

Let $\mathfrak{a}$ be any class of algebras. According to Hall [2], if for some index set I, $\{S_i: i \in I\}$ is an indexed set of algebras from $\mathfrak{a}$ having a common subalgebra U also in $\mathfrak{a}$, then the list $(S_i: i \in I: U)$ is called an *amalgam from* $\mathfrak{a}$. If there exist an algebra W and monomorphisms $\phi_i: S_i \to W (i \in I)$ such that $\phi_i | U = \phi_j | U$ and $\phi_i(S_i) \cap \phi_j(S_j) = \phi_i(U)$ for all distinct $i, j \in I$, then the amalgam $(S_i: i \in : U)$ is said to be *strongly embeddable in W.* If an amalgam of the form $(S, S; U)$ from $\mathfrak{a}$ is strongly embeddable in an algebra from $\mathfrak{a}$, then U is said to be *closed in S* (within $\mathfrak{a}$). If U is closed in S

within $\mathfrak{a}$ for all $U, S \in \mathfrak{a}$ with $U \subseteq S$, then $\mathfrak{a}$ is said to *have the special amalgamation property*. If every amalgam from $\mathfrak{a}$ is strongly embeddable in an algebra from $\mathfrak{a}$, then $\mathfrak{a}$ is said to *have the strong amalgamation property*.

The following result is also due to J. R. Isbell.

THEOREM 3. ([4, Theorem 2.3]) *Let U, S be semigroups such that U is a subsemigroup of S. Then U is closed in S (within the class of semigroups) if and only if* $\mathrm{Dom}_S(U) = U$.

The following result follows from Theorems 2 and 3 above and Corollary 3 of [12].

THEOREM 4. *The class of right [left] self-injective [regular] semigroups has the special amalgamation property.*

The following example, which is obtained by modifying an example in Imaoka [6], shows that the class of right [left] self-injective [regular] semigroups does not have the strong amalgamation property.

EXAMPLE. Let $U=\{0, e, f, g, 1\}$, $V=\{0, e, f, g, h, 1\}$ and $W=\{0, e, f, g, x, y, z, 1\}$ be semigroups with the following multiplicative tables:

U	0	e	f	g	1
0	0	0	0	0	0
e	0	e	f	g	e
f	0	e	f	g	f
g	0	e	f	g	g
1	0	e	f	g	1

V	0	e	f	g	h	1
0	0	0	0	0	0	0
e	0	e	f	g	f	e
f	0	e	f	g	f	f
g	0	e	f	g	g	g
h	0	e	f	g	h	h
1	0	e	f	g	h	1

W	0	e	f	g	x	y	z	1
0	0	0	0	0	0	0	0	0
e	0	e	f	g	x	y	z	e
f	0	e	f	g	x	y	z	f
g	0	e	f	g	x	y	z	g
x	0	x	y	x	x	y	z	x
y	0	x	y	x	x	y	z	y
z	0	z	z	z	x	y	z	z
1	0	e	f	g	x	y	z	1

By [11] U, V and W are completely right injective, of course, right self-injective and regular. Suppose now that the amalgam $(V, W\colon U)$ is embedded in a semigroup S. But in S we have $xh=(xe)h=x(eh)=xf=y$ and $xh=(xg)h=x(gh)=xg=x$. This is a contradiction. Hence the amalgam $(V, W\colon U)$ can not be embedded in any semigroup.

§ 2. Commutative separative semigroups

Let S be a commutative separative semigroup. Then by [1, Theorem 4.18] S is uniquely expressible as a semilattice Λ of archimedean cancellative semigroups S_α

$(\alpha \in \Lambda)$ and S can be embedded in a semigroup T which is the same semilattice Λ of groups G_α $(\alpha \in \Lambda)$ where G_α is the quotient group of S_α for each $\alpha \in \Lambda$, i.e., every element of G_α can be expressed in the form ab^{-1} with a and b in S_α.

Let ξ, ψ be homomorphisms of T to any semigroup W such that $\xi | S = \psi | S$. Then for each $\alpha \in \Lambda$, $\xi(G_\alpha)$ and $\psi(G_\alpha)$ are contained in a subgroup H of W. Hence $\xi(a^{-1}) = \psi(a^{-1})$ for all $a \in S_\alpha$. Because that both $\xi(a^{-1})$ and $\psi(a^{-1})$ are inverses of $\xi(a)$ in the group H. Then it is clear that $\xi | G_\alpha = \psi | G_\alpha$. Therefore we have $\xi = \psi$. This implies that $\mathrm{Dom}_T(S) = T$. Therefore if a commutative separative semigroup S is absolutely closed, then S is a semilattice of abelian groups. Conversely, by [4, Theorem 2.3], a semilattice of abelian groups is absolutely closed. Thus we have

THEOREM 5. *Let S be a commutative separative semigroup. Then S is absolutely closed if and only if S is a semilattice of abelian groups*

In [10] we studied self-injective non-singular semigroups and showed that every self-injective non-singular semigroup is a semilattice of groups and that every commutative non-singular semigroup is separative. Furthermore, by Theorems 1 and 3 we have

THEOREM 6. *Every self-injective separative commutative semigroup is a semilattice of abelian groups.*

In [9] B. Schein characterized self-injective inverse semigroups as follows:

Let S be an inverse semigroup and E_S the set of idempotents of S. A subset B of S is *compatible* if for each $b \in B$ there is $e_b \in E_S$ with $be_b = b$ and $be_c = ce_b$ for all $b, c \in B$. Define an order $\leqq$ on S by $a \leqq b$ $(a, b \in S)$ if and only if $a \in bE_S$. Then (1) S is *complete* if every compatible set B of S has the least upper bount $\vee B$ relatively to $\leqq$, (2) S is *infinitely distributive* if $(\vee B)a = \vee Ba$ for any compatible set B of S and for any $a \in S$, and (3) S is *E_S-reflexive* if $st \in E_S$ $(s, t \in S)$ implies $ts \in E_S$.

THEOREM 7. ([9, Theorem 2.3]) *Let S be an inverse semigroup and E_S the set of idempotents of S. Then S is self-injective if and only if S is complete, infinitely distributive and E_S-reflexive.*

Here we can obtain the following

THEOREM 8. *Let S be a commutative semigroup. Then S is self-injective and separative if and only if S is a semilattice Λ of abelian groups G_α $(\alpha \in \Lambda)$ satisfying the following conditions: (1) $\Lambda (\cong E_S)$ is self-injective, (2) for any set $\{g_\alpha\}_{\alpha \in X}$ such that $g_\alpha e_\beta = g_\beta e_\alpha$ $(\alpha, \beta \in X$, $g_\alpha \in G_\alpha$, $g_\beta \in G_\beta$, e_α, e_β are identities of G_α, G_β, respectively$)$, there exists a unique $g \in G_\gamma$ such that $ge_\alpha = g_\alpha$ for all $\alpha \in X$, where $\gamma = \vee X$.*

PROOF. The "only if" part: Let S be a self-injective separative commutative semigroup. By Theorem 6, S is a semilattice Λ of abelian groups G_α $(\alpha \in \Lambda)$. Let E_S

be a semilattice of all idempotents of S. Then E_S is isomorphic to Λ. On the other hand, it easily follows from Theorem 7 that E_S is complete, infinitely distributive, equivalently, E_S is self-injective. Therefore the condition (1) is satisfied. Let $B = \{g_\alpha\}_{\alpha \in X}$ be as given in the condition (2). It is clear that B is a compatible set. By Theorem 7, there exists $g \in G_\gamma$ such that $ge_\alpha = g_\alpha$ for all $\alpha \in X$ ($\gamma = \vee X$). The condition (2) is satisfied. The "if" part: We first show that S is complete. Let A be a compatible set of S. If A is empty, then the least upper bound of A in E_S is the one of A in S, since E_S is complete. On the other hand, if A is non-empty, say, $A = \{g_\alpha\}_{\alpha \in X}$, then by the condition (2), there exists $g \in G_\gamma$ such that $ge_\alpha = g_\alpha$ ($\alpha \in X$), where $\gamma = \vee X$. It is easy to check that g is the least upper bound of A. Therefore, S is complete. Since E_S is infinitely distributive, so is S. It is also clear that S is E_S-reflexive. Then the theorem follows from Theorem 7.

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