On a Large Deflection of Clamped Square Plate under Actions of Four Concentrated Loads.

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I. Introduction

The problem to be treated here is that of initially flat square plate under actions of four concentrated loads, P as shown in Fig. 1, with deflections permitted of the same order of magnitude as the thickness. All four edges assumed to be rigidly clamped and the loads are assumed to have the same magnitude P and situate at points $P_1\left(\frac{a}{2}, \frac{a}{2}, -\frac{h}{2}\right)$, $P_2\left(-\frac{a}{2}, \frac{a}{2}, -\frac{h}{2}\right)$, $P_3\left(-\frac{a}{2}, -\frac{a}{2}, -\frac{h}{2}\right)$ and $P_4\left(\frac{a}{2}, -\frac{a}{2}, -\frac{h}{2}\right)$.

Problems closely allied this problem has been solved approximately by S. Way,(1) S. Levy(2) and the author(3) etc.

The course of attack used in the present paper is the Ritz energy method. Expressions are assumed for the three displacements in the form algebraic polynomials satisfying the boundary conditions, and containing eight unknown constants in all.

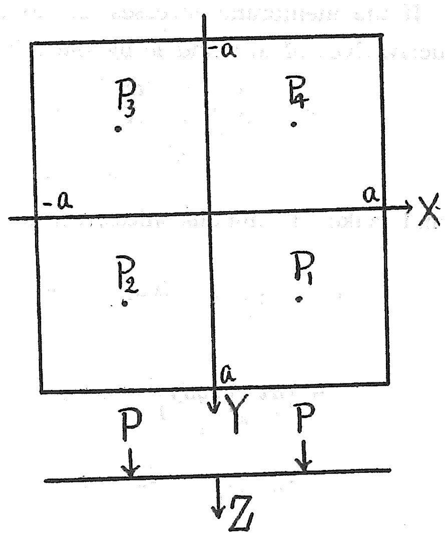

Fig. 1

II. Notations

Let the plate be situated with respect to the coordinates axes as shown in Fig. 1.

The following notations will be used:

X, Y, Z=coordinates of a point

$2a$=length of side of the plate.

h=thickness of the plate.

u, v, w=displacements of points in the middle surface in directions X, Y, Z.

E, μ=Young's modulus and Poisson's ratio

G=shear modulus of elasticity.

$D=\frac{Eh^3}{12(1-\mu^2)}$=the flexural rigidity.

P=lateral concentrated load situated at P_1, P_2, P_3 and P_4.

p=lateral load per unit area.

U=total energy of the plate and loads.

$\sigma'_X, \sigma'_Y, \tau''_{XY}$=stresses in the middle surface.

$\sigma''_X, \sigma''_Y, \tau''_{XY}$=bending stresses and shearing stress at the surface $Z=-\frac{h}{2}$.

e'_X, e'_Y, e'_{XY}=extensions and shear in the middle surface.

III. Calculations

In applying Ritz method, we need the expression for the total potential energy of the system consisting of the plate and the loads upon it. This will be

$$U=\frac{D}{2}\iint(\Delta w)^2 dXdY+V+\frac{h}{2}\iint(\sigma'_X e'_X+\sigma'_Y e'_Y+\tau'_{XY}\gamma'_{XY})dXdY. \tag{1}$$

for a clamped plate. The first and third integrals are taken over the entire plate. The first integral represents the strain energy of bending and twisting, and the third the strain energy of stretching. The second integral represents the potential energy of the concentrated loads and is expressed by the following limiting values

$$V=-\lim_{\varepsilon\to 0}\left\{\int_{\frac{a}{2}-\varepsilon}^{\frac{a}{2}+\varepsilon} dX\int_{\frac{a}{2}-\varepsilon}^{\frac{a}{2}+\varepsilon} pw\,dY+\int_{-\frac{a}{2}-\varepsilon}^{-\frac{a}{2}+\varepsilon} dX\int_{\frac{a}{2}-\varepsilon}^{\frac{a}{2}+\varepsilon} pw\,dY+\int_{-\frac{a}{2}-\varepsilon}^{-\frac{a}{2}+\varepsilon} dX\int_{-\frac{a}{2}-\varepsilon}^{-\frac{a}{2}+\varepsilon} pw\,dY+\int_{\frac{a}{2}-\varepsilon}^{\frac{a}{2}+\varepsilon} dX\int_{-\frac{a}{2}-\varepsilon}^{-\frac{a}{2}+\varepsilon} pw\,dY\right\} \tag{2}$$

provieded that $4\lim\limits_{\varepsilon\to 0} p\varepsilon^2=P$. (3)

If the membrane stresses and extensions in (1) are expressed in terms of the derivatives of u, v and w by the relations

$$e'_X=\frac{\partial u}{\partial X}+\frac{1}{2}\left(\frac{\partial w}{\partial X}\right)^2;\ e'_Y=\frac{\partial v}{\partial Y}+\frac{1}{2}\left(\frac{\partial w}{\partial Y}\right)^2;\ \gamma'_{XY}=\frac{\partial u}{\partial Y}+\frac{\partial v}{\partial X}+\frac{\partial w}{\partial X}\frac{\partial w}{\partial Y}, \tag{4}$$

$$\sigma'_X=\frac{E}{1-\mu^2}(e'_X+\mu e'_Y);\ \sigma'_Y=\frac{E}{1-\mu^2}(e'_Y+\mu e'_X);\ \tau'_{XY}=G\gamma'_{XY}. \tag{5}$$

and make (1) dimensionless, the energy expression becomes.

$$\begin{aligned}A=&\frac{1}{2}\iint(\Delta\xi)^2\,dx\,dy-\lim_{\delta\to 0}\left\{\int_{\frac{1}{2}-\delta}^{\frac{1}{2}+\delta} dx\int_{\frac{1}{2}-\delta}^{\frac{1}{2}+\delta}\zeta q\,dy+\int_{-\frac{1}{2}-\delta}^{-\frac{1}{2}+\delta} dx\int_{\frac{1}{2}-\delta}^{\frac{1}{2}+\delta}\zeta q\,dy+\int_{-\frac{1}{2}-\delta}^{-\frac{1}{2}+\delta} dx\int_{-\frac{1}{2}-\delta}^{-\frac{1}{2}+\delta}\zeta q\,dy\right.\\&\left.+\int_{\frac{1}{2}-\delta}^{\frac{1}{2}+\delta} dx\int_{-\frac{1}{2}-\delta}^{-\frac{1}{2}+\delta}\zeta q\,dy\right\}+6\iint\left[\xi_x^2+\zeta_x^2\xi_x+\eta_y^2+\zeta_y^2\eta_y+\left(\frac{\zeta_x^2}{2}+\frac{\zeta_y^2}{2}\right)^2\right.\\&\left.+2\mu\left(\xi_x\eta_y+\eta_y\frac{\zeta_x^2}{2}+\xi_x\frac{\zeta_y^2}{2}\right)+\frac{1-\mu}{2}(\xi_y^2+\eta_x^2+2\xi_y\eta_x+2\xi_y\zeta_x\zeta_y+2\eta_x\zeta_x\zeta_y)\right]\\&dxdy.\end{aligned} \tag{6}$$

in which the subscripts indicate partial differentiation and in which we put

$$\left.\begin{aligned}&x=\frac{X}{a},\ y=\frac{Y}{a},\ z=\frac{Z}{a},\ \delta=\frac{\varepsilon}{a}\\&\xi=\frac{ua}{h^2},\ \eta=\frac{va}{h^2},\ \zeta=\frac{w}{h},\ A=\frac{Ua^2}{Dh^2},\ q=\frac{pa^4}{Dh}\\&S'_x=\frac{\sigma'_X a^2(1-\mu^2)}{Eh^2},\ S'_y=\frac{\sigma'_Y a^2(1-\mu^2)}{Eh^2},\ S'_{xy}=\frac{\tau'_{XY}a^2}{Gh^2}\\&S''_x=\frac{\sigma''_X a^2(1-\mu^2)}{Eh^2},\ S''_y=\frac{\sigma''_Y a^2(1-\mu^2)}{Eh^2},\ S''_{xy}=\frac{\tau''_{XY}a^2}{Gh^2}\\&Q=\lim_{\delta\to 0}4q\delta^2=\frac{Pa^2}{Dh}.\end{aligned}\right\} \tag{7}$$

As the plate are clamped, the following boundary conditions must be satisfied:

$$\left.\begin{aligned}&(\zeta)_{x\pm1}=(\zeta)_{y=\pm1}=0,\\&(\zeta_x)_{x=\pm1}=(\zeta_y)_{y=\pm1}=0,\\&(\xi)_{x=\pm1}=(\xi)_{y=\pm1}=0,\\&(\eta)_{x=\pm1}=(\eta)_{y=\pm1}=0.\end{aligned}\right\}\qquad(8)$$

These conditions will be satisfied if we assume for ξ, η, and ζ the following expressions,

$$\left.\begin{aligned}&\xi=x(1-x^2)(1-y^2)(b_{00}+b_{02}y^2+b_{20}x^2+b_{22}x^2y^2),\\&\eta=y(1-x^2)(1-y^2)(b_{00}+b_{02}x^2+b_{20}y^2+b_{22}x^2y^2),\\&\zeta=(1-x^2)^2(1-y^2)^2(a_{00}+a_{02}x^2+a_{02}y^2+a_{22}x^2y^2).\end{aligned}\right\}\qquad(9)$$

These expressions satisfy the following symmetry conditions:

$$\begin{aligned}&\xi(x,y)=\xi(x,-y)=-\xi(-x,y)=-\xi(-x,-y),\\&\eta(x,y)=\eta(-x,y)=-\eta(x,-y)=-\eta(-x,-y),\\&\xi(x,y)=\eta(y,x).\end{aligned}\qquad(10)$$

By Ritz method, since the constants are to be determined so as to total energy A a minimum, we must have

$$\frac{\partial A}{\partial b_{00}}=0;\ \frac{\partial A}{\partial b_{02}}=0;\ \frac{\partial A}{\partial b_{20}}=0;\ \frac{\partial A}{\partial b_{22}}=0,\qquad(11)$$

$$\frac{\partial A}{\partial a_{00}}=0;\ \frac{\partial A}{\partial a_{02}}=0;\ \frac{\partial A}{\partial a_{22}}=0.\qquad(12)$$

The four equations (11) are linear in the b's and found to be:

$$\begin{aligned}&(0.513016-0.015238\mu)b_{00}+(0.066032-0.015238\mu)b_{02}\\&+(0.215026-0.001693\mu)b_{20}+(0.027332-0.005563\mu)b_{22}\\&+(-0.030022+0.150109\mu)a_{00}^2+(-0.089645-0.004829\mu)a_{00}a_{02}\\&+(-0.004409-0.002729\mu)a_{00}a_{22}+(-0.015536+0.011337\mu)a_{02}\\&+(-0.002277+0.000759\mu)a_{02}a_{22}-0.000097a_{22}^2=0\end{aligned}\qquad(13.1)$$

$$\begin{aligned}&(0.036032-0.015238\mu)b_{00}+(0.037007-0.015238\mu)b_{02}\\&+(0.031202-0.001693\mu)b_{20}+(0.013787-0.005563\mu)b_{22}\\&+(0.013646+0.019105\mu)a_{00}^2+(-0.007348+0.008608\mu)a_{00}a_{02}\\&+(-0.002471+0.001405\mu)a_{00}a_{22}+(-0.001324+0.002293\mu)a_{02}^2\\&+(-0.000613+0.000242\mu)a_{02}a_{22}-0.000029a_{22}^2=0\end{aligned}\qquad(13.2)$$

$$\begin{aligned}&(0.215026-0.001693\mu)b_{00}+(0.031202-0.001693\mu)b_{02}\\&+(0.169994-0.001165\mu)b_{20}+(0.024517-0.000813\mu)b_{22}\\&+(0.047657+0.034640\mu)a_{00}^2+(-0.001566-0.001050\mu)a_{00}a_{22}\\&+(-0.018265+0.007768\mu)a_{00}a_{02}+(-0.004083+0.003426\mu)a_{02}^2\\&+(-0.000776+0.000611\mu)a_{02}a_{22}+(-0.000038+0.000089\mu)a_{22}^2=0\end{aligned}\qquad(13.3)$$

$$\begin{aligned}&(0.027332-0.005563\mu)b_{00}+(0.013787-0.005563\mu)b_{02}\\&+(0.024517-0.000814\mu)b_{20}+(0.009917-0.002338\mu)b_{22}\\&+(0.007348+0.004409\mu)a_{00}^2+(0.000468+0.003117\mu)a_{00}a_{02}\\&+(-0.000662+0.000501\mu)a_{00}a_{22}+(-0.000146+0.000948\mu)a_{02}^2\\&+(-0.000236+0.000307\mu)a_{02}a_{22}+(-0.000015+0.000028\mu)a_{22}^2=0\end{aligned}\qquad(13.4)$$

The equations (12) become:

$$1.114558a_{00} + 0.202647a_{02} + 0.011258a_{22}$$
$$+ a_{00}(-0.120087b_{00} + 0.05485b_{02} + 0.190628b_{20} + 0.029392b_{22})$$
$$+ a_{02}(-0.179291b_{00} - 0.014696b_{02} - 0.036530b_{20} + 0.000937b_{22})$$
$$+ a_{22}(-0.008818b_{00} - 0.004942b_{02} - 0.003133b_{20} - 0.001324b_{22})$$
$$+ \mu a_{00}(0.600435b_{00} + 0.076419b_{02} + 0.138562b_{20} + 0.017635b_{22})$$
$$+ \mu a_{02}(-0.009657b_{00} + 0.017215b_{02} + 0.015536b_{20} + 0.006234b_{22})$$
$$+ \mu a_{22}(-0.005458b_{00} + 0.002810b_{02} - 0.002099b_{20} + 0.001001b_{22})$$
$$+ 0.785994a_{00}^3 + 0.079869a_{00}^2a_{02} - 0.016480a_{00}^2a_{22} + 0.150197a_{00}a_{02}^2$$
$$+ 0.001510a_{00}a_{22}^2 + 0.014022a_{00}a_{02}a_{22} + 0.010995a_{02}^3 + 0.001940a_{02}^2a_{22}$$
$$+ 0.000291a_{02}a_{22}^2 + 0.000007a_{22}^3 - 0.00659180Q = 0 \tag{14.1}$$

$$0.202647a_{00} + 0.457255a_{02} + 0.036372a_{22}$$
$$+ a_{00}(-0.179291b_{00} - 0.014696b_{02} - 0.036530b_{20} + 0.000937b_{22})$$
$$+ a_{02}(-0.062143b_{00} - 0.005297b_{02} - 0.016332b_{20} - 0.000585b_{22})$$
$$+ a_{22}(-0.004554b_{00} - 0.001226b_{02} - 0.001552b_{20} - 0.000473b_{22})$$
$$+ \mu a_{00}(-0.009657b_{00} + 0.017215b_{02} + 0.015536b_{20} + 0.006234b_{22})$$
$$+ \mu a_{02}(0.045348b_{00} + 0.009173b_{02} + 0.013702b_{20} + 0.003792b_{22})$$
$$+ \mu a_{22}(0.001518b_{00} + 0.000484b_{02} + 0.001222b_{20} + 0.000614b_{22})$$
$$+ 0.026623a_{00}^3 + 0.150197a_{00}^2a_{02} + 0.007011a_{00}^2a_{22} + 0.032985a_{00}a_{02}^2$$
$$+ 0.003880a_{00}a_{02}a_{22} + 0.009870a_{02}^3 + 0.000291a_{00}a_{22}^2 + 0.001945a_{02}^2a_{22}$$
$$+ 0.000269a_{02}a_{22}^2 + 0.000014a_{22}^3 - 0.0032959Q = 0 \tag{14.2}$$

$$0.011258a_{00} + 0.036372a_{02} + 0.009841a_{22}$$
$$+ a_{00}(-0.00818b_{00} - 0.004942b_{02} - 0.003133b_{20} - 0.001324b_{22})$$
$$+ a_{02}(-0.004554b_{00} - 0.001226b_{02} - 0.001552b_{20} - 0.000473b_{22})$$
$$+ a_{22}(-0.000388b_{00} - 0.000114b_{02} - 0.000152b_{20} - 0.000061b_{22})$$
$$+ \mu a_{00}(-0.005458b_{00} + 0.002810b_{02} - 0.002099b_{20} + 0.001001b_{22})$$
$$+ \mu a_{02}(0.001518b_{00} + 0.000484b_{02} + 0.001222b_{20} + 0.000614b_{22})$$
$$+ \mu a_{22}(0.000357b_{20} + 0.000114b_{22})$$
$$- 0.005493a_{00}^3 + 0.007011a_{00}^2a_{02} + 0.001510a_{00}^2a_{22} + 0.001940a_{00}a_{02}^2$$
$$+ 0.000582a_{00}a_{02}a_{22} + 0.000020a_{00}a_{22}^2 + 0.000648a_{02}^3 + 0.000269a_{02}^2a_{22}$$
$$+ 0.000043a_{02}a_{22}^2 + 0.000003a_{22}^3 - 0.000411987Q = 0 \tag{14.3}$$

The values of b_{00}, b_{02}, b_{20}, and b_{22} can be obtained from equation (13); for μ=0.3, the values are:

$$b_{00} = 0.405052a_{00}^2 + 0.307930a_{00}a_{02} + 0.00310a_{00}a_{22} + 0.034605a_{02}^2 + 0.003999a_{02}a_{22} + 0.000292a_{22}^2$$

$$b_{02} = -1.231092a_{00}^2 - 0.134661a_{00}a_{02} + 0.077909a_{00}a_{22} - 0.009763a_{02}^2 + 0.014283a_{02}a_{22} + 0.000674a_{22}^2$$

$$b_{20} = -0.910273a_{00}^2 - 0.243842a_{00}a_{02} + 0.001338a_{00}a_{22} - 0.015872a_{02}^2 - 0.003371a_{02}a_{22} - 0.000469a_{22}^2$$

$$b_{22} = 1.946906a_{00}^2 - 0.190484a_{00}a_{02} - 0.059594a_{00}a_{22} - 0.56702a_{02}^2 - 0.005404a_{02}a_{22} + 0.000294a_{22}^2$$

When the expression (15) for b's are introduced in (14.1) to (14.3), we have the three equations containing a_{00}, a_{02}, a_{22}, and Q. If we put a_{00} equal to certain value, the remaining constants a_{02}, a_{22} and Q will be determined. The values b's will be found from the equations (15). Thus, the calculated values of a_{02}, a_{22}, Q, b_{00}, b_{02}, b_{20} and b_{22} for a_{00}=0.4, 0.8, 1.0, 1.2, 1.6, and 2 are given in Table I.

Table I.

a_{00}	a_{02}	a_{22}	Q	b_{00}	b_{02}	b_{20}	b_{22}
0.4	0 25173	2.1109	84.809	0.10423	−0.13478	−0.16619	0.23685
0.8	0.62138	5.0849	215.99	0.45932	−0.47912	−0.72724	0.87754
1.0	0.86996	7.1242	317.25	0.76232	−0.67786	−1.1696	1.2952
1.2	1.1670	9.6399	457.36	1.1722	−0.85010	−1.7399	1.7367
1.6	1.8677	15.885	911.16	2.3543	−1.0141	−3.2986	2.6164
2.0	2.7213	23.237	1671.8	4.1163	−0.84172	−5.4898	3.3786

The ratio w_{max}/h, that is the ratio of the maximum deflection to the thickness is identically equal to a_{00}. The values of Q corresponding to various values of a_{00} are

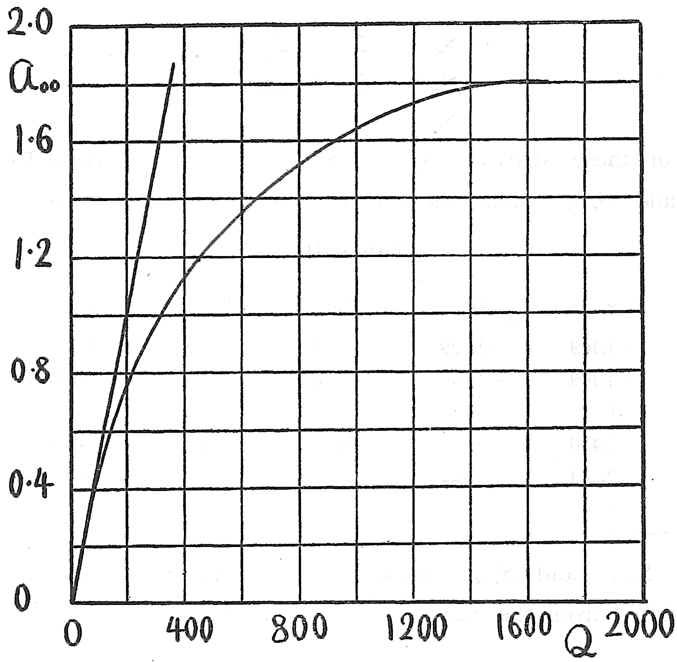

Fig. 2

given in Table I. and Fig. 2 shows a curve for w_{max}/h as a function of Q. In Fig. 2 the straight line Q=195.06a_{00} obtained by the linear theory is plotted for comparison.

We will now calculate the dimensionless stresses. The membrane stresses at any point are given by

$$\left.\begin{aligned} S'_x &= \left(\xi_x + \frac{\zeta_x^2}{2}\right) + \mu\left(\eta_y + \frac{\zeta_y^2}{2}\right) \\ S'_y &= \left(\eta_y + \frac{\zeta_y^2}{2}\right) + \mu\left(\xi_x + \frac{\zeta_x^2}{2}\right) \\ S'_{xy} &= \eta_x + \xi_y + \zeta_x \zeta_y \end{aligned}\right\} \tag{16}$$

The membrane stresses $(S'_x)_{x=0}$, $(S'_y)_{x=0}$ and $(S'_{xy})_{x=0}$ along the y axis and $(S'_x)_{x=1}$, $(S'_y)_{x=1}$ and $(S'_{xy})_{x=1}$ along the side $x=1$ are given in Table II for $a_{00}=1$.

Table II.

y	0	0.2	0.4	0.6	0.8	1.0
$(S'_x)_{x=0}$	0.9808	0.9014	0.7069	0.5572	0.3428	0.2443
$(S'_y)_{x=0}$	0.9568	0.8639	0.6903	0.8511	0.7841	0.8145
$(S'_{xy})_{x=0}$	0	0	0	0	0	0
$(S'_x)_{x=1}$	0.8145	0.7345	0.5182	0.2368	0.0088	0
$(S'_y)_{x=1}$	0.2443	0.2203	0.1555	0.0710	0.0026	0
$(S'_{xy})_{x=1}$	0	−0.03436	−0.07026	−0.09959	−0.09495	0

The dimensionless bending stresses on the upper surface $Z=-\dfrac{h}{2}$ are given by;

$$\left.\begin{aligned} S''_x &= \tfrac{1}{2}\left(\zeta_{xx} + \mu\zeta_{yy}\right) \\ S''_y &= \tfrac{1}{2}\left(\zeta_{yy} + \mu\zeta_{xx}\right) \\ S''_{xy} &= \zeta_{xy} \end{aligned}\right\} \tag{17}$$

The values of these stresses $(S''_x)_{x=0}$, $(S''_y)_{x=0}$ and $(S''_{xy})_{x=0}$ along the y axis and $(S''_x)_{x=1}$, $(S''_y)_{x=1}$ and $(S''_{xy})_{x=1}$ along the side $x=1$ are given in Tabe III.

Table III.

y	0	0.2	0.4	0.6	0.8	1.0
$(S''_x)_{x=0}$	−1.469	−1.229	−0.641	0.020	1.104	2.244
$(S''_y)_{x=0}$	−1.469	−1.540	−1.563	−0.938	2.769	7.480
$(S''_{xy})_{x=0}$	0	0	0	0	0	0
$(S''_x)_{x=1}$	7.480	8.072	8.883	7.779	3.622	0
$(S''_y)_{x=1}$	2.244	2.422	2.666	2.334	1.087	0
$(S''_{xy})_{x=1}$	0	0	0	0	0	0

The stresses $(S''_x)_{x=0}$ and $(S''_y)_{x=0}$ along the y axis and $(S''_x)_{x=1}$ and $(S''_y)_{x=1}$ along the side $x=1$ calculated from the linear theory are given in Table IV for comparison.

Table IV.

y	0	0.2	0.4	0.6	0.8	1.0
$(S''_x)_{x=0}$	−1.856	−1.611	−0.987	−0.213	0.845	1.887
$(S''_y)_{x=0}$	−1.856	−1.802	−1.522	−0.637	2.430	6.290
$(S''_x)_{x=1}$	6.290	6.605	6.913	5.808	2.633	0
$(S''_y)_{x=1}$	1.887	1.982	2.074	1.742	0.780	0

The total stresses S_x, S_y, and S_{xy} at top surface are given by

$$\left.\begin{aligned} S_x &= S'_x + S''_x, \\ S_y &= S'_y + S''_y, \\ S_{xy} &= S'_{xy} + S''_{xy}. \end{aligned}\right\} \tag{18}$$

For instance, the stress $(S_x)_{x=1,\, y=0}$ at middle edge has been plotted as function of a_{00} in Fig. 3.

From above Tables and Figures, we will see that the moderate difference between the linear and the present theory for large $a_{00} = \frac{w_{\max}}{h}$.

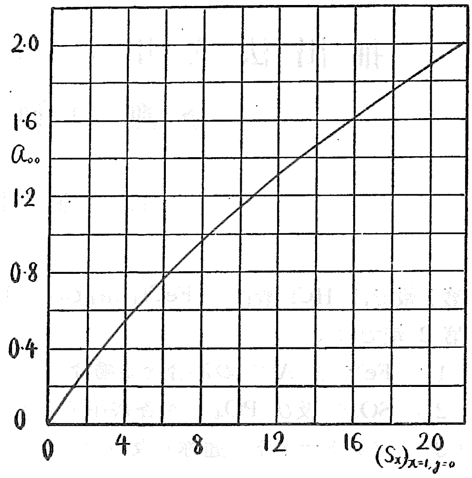

Fig. 3

IV. Summary

By Ritz method, we have solved approximately the problem of an initially flat clamped square plate under actions of four concentrated loads, and have seen that there is the moderate difference between the linear and the present theory when the deflections is of the same order of magnitude as the thickness. This research has been maintained by the grant in Aid for Fundamental Scientific Research of the Department of Education.

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