# On a Large Deflection of Clamped Square Plate under a Lateral Load.

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## Introduction.

1. It is known that two fundamental assumptions are made usually in the theory of elasticity : small deflections and linear stress-strain relations. Under these assumptions the problem of stress distribution in elastic systems is governed by linear partial differential equations. However, if we give up one or both of these assumptions, we obtain non-linear equations. The Kirchhoff theory of the bending of elastic plates is also based on the above same assumptions and that thickness is small compared to the area dimensions. The first assumption allows one to neglect the extensions of the middle surface which accompanies bending, while the third assumption permits one to consider that the straight lines initially normal to the middle surface remain straight and normal to that surface after bending.

When the deflection of the plate is of the same order as the thickness, the effect of the middle surface extension becomes important and must be in considerations. And then the differential equations governing the deformation of the plate becomes non-linear and exact solutions are feasible only in a few simple cases. We must then adopt a method of systematic successive approximations such as the method of finite differences or the Ritz method.

Hencky solved approximately by using the method of finite differences the problem of the bending of the uniformly laterally loaded rectangular plate which has negligible flexural rigidity, no initial tension and the clamped four edges. The problem of a square plate with simply supported edges and no edge traction is solved by Kaiser by the same method. S. Way solved, by the Ritz energy method, the problem of a uniformly laterally loaded clamped rectangular plate with large deflection.

The problem to be treated here is such that the distribution of lateral load over a square plate is symmetrical to its center and the plate is clamped. And Ritz method will be applied. It is very interesting and important both theoretically and practically to solve these problems.

2. Fundamental Equations and Notations.

Let the plate be situated with respect to the coordinates axes as shown in Fig 1. The following notations will be used :

$X, Y, Z$ = coordinates of a point

$2a$ = length of side of the plate

$2\delta$ = length of side of the square part on which the load situates

$h$ = thickness of the plate

u, v, w = displacements of points in the middle surface in directions X, Y, Z

E, $\mu$ = Young's modulus and Poission's ratio

G = shear modulus of elasticity

$D=\frac{Eh^3}{12(1-\mu^2)}$, the flexural rigidity

P = lateral load per unit of area

U = total energy of the plate and load

$\sigma'_x$, $\sigma'_y$, $\tau'_{xy}$ = stresses in the middle surface

$\sigma''_x$, $\sigma''_y$, $\tau''_{xy}$ = bending stresses and shearing stress at the surface $Z=-\frac{h}{2}$

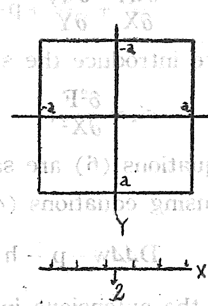


Fig. 1

$Q_x$, $Q_y$ = vertical shearing forces per unit of length on plate X = const, Y = const

$e_x'$, $e_y'$, $\gamma'_{xy}$ = extensions and shear in the middle surface

$m_x$, $m_y$, $m_{xy}$ = bending and twisting moments per unit of length.

Then the following relations exists:

$$\left.\begin{aligned} m_x=-D\left(\frac{\partial^2 w}{\partial X^2}+\mu\frac{\partial^2 w}{\partial Y^2}\right),\quad m_y=-D\left(\frac{\partial^2 w}{\partial Y^2}+\mu\frac{\partial^2 w}{\partial X^2}\right),\\ m_{xy}=-(1-\mu)D\frac{\partial^2 w}{\partial X\partial Y},\end{aligned}\right\}\tag{1}$$

while the bending stresses on the upper surface $Z=-\frac{h}{2}$ are give in terms of the moments by:

$$\sigma''_x=-\frac{6}{h^2}m_x;\ \sigma''_y=-\frac{6}{h^2}m_y;\ \tau''_{xy}=-\frac{6}{h^2}m_{xy}\tag{2}$$

The vertical shearing forces are

$$Q_x=-D\frac{\partial}{\partial X}(\Delta w);\ Q_y=-D\frac{\partial}{\partial y}(\Delta w)\tag{3}$$

in which $\Delta w=\frac{\partial^2 w}{\partial X^2}+\frac{\partial^2 w}{\partial Y^2}$

$e'_x$, $e'_y$ and $\gamma'_{xy}$ are

$$e'_x=\frac{\partial u}{\partial X}+\frac{1}{2}\left(\frac{\partial w}{\partial X}\right)^2;\ e'_y=\frac{\partial v}{\partial Y}+\frac{1}{2}\left(\frac{\partial w}{\partial Y}\right)^2;\ \gamma'_{xy}=\frac{\partial u}{\partial Y}+\frac{\partial v}{\partial X}+\frac{\partial w}{\partial X}\frac{\partial w}{\partial Y}\tag{4}$$

By Hooke's law,

$$\sigma'_x=\frac{E}{1-\mu^2}('e_x+\mu e'_y);\ \sigma'_y=\frac{E}{1-\mu^2}(e'_y+\mu e'_x);\ \tau_{xy}=G\gamma'_{xy}\tag{5}$$

From the above equations, we can calculate the stresses and moments everywhere in the plate when the displacement u, v, and w are known. Three other equations which are necessary to determine the displacements are obtained by considering the equilibrium of an element hdXdY:

$$\left.\begin{aligned}\frac{\partial\sigma'_x}{\partial X}+\frac{\partial\tau'_{xy}}{\partial Y}=0,\\ \frac{\partial\sigma'_y}{\partial Y}+\frac{\partial\tau'_{xy}}{\partial X}=0,\end{aligned}\right\}\tag{6}$$

$$\frac{\partial Q_x}{\partial X}+\frac{\partial Q_y}{\partial Y}+p+h\left\{\sigma'_x\frac{\partial^2 w}{\partial X^2}+\sigma'_y\frac{\partial w^2}{\partial Y^2}+2\tau'_{xy}\frac{\partial^2 w}{\partial X\partial Y}\right\}=0. \tag{7}$$

If we introduce the stress function F such as

$$\sigma'_x=\frac{\partial^2 F}{\partial X^2};\qquad \sigma'_y=\frac{\partial^2 F}{\partial Y^2};\qquad \tau'_{xy}=-\frac{\partial^2 F}{\partial X\partial Y}, \tag{8}$$

the equations (6) are satisfied identically.

By using equations (4), (5) and (8) the equation (7) becomes

$$D\Delta\Delta w-p-h\left\{\frac{\partial^2 F}{\partial Y^2}\cdot\frac{\partial^2 w}{\partial X^2}+\frac{\partial^2 F}{\partial X^2}\cdot\frac{\partial^2 w}{\partial Y^2}-2\frac{\partial^2 F}{\partial X\partial Y}\cdot\frac{\partial^2 w}{\partial X\partial Y}\right\}=0. \tag{9}$$

When the extensions in equations (5) are introduced into the equations (6)and from the three resulting equations u and v are eliminated, we obtain the following equation:

$$\Delta\Delta F=E\left\{\left(\frac{\partial^2 w}{\partial X\partial Y}\right)^2-\frac{\partial^2 w}{\partial X^2}\cdot\frac{\partial^2 w}{\partial Y^2}\right\}. \tag{10}$$

These equations were given by Kàrmàn. The two equations (9) and (10) in the unknown w and F are the non-linear differential equations governing the general problem of the bending of a plate with large deflection. Since these equations are virtually impossible to solve in most cases, we must be had to approximate methods of treatment. To this problem, we apply the Ritz method. In applying the energy method, we need the expression for the total potential energy of the system consisting of the plate and the loads upon it. This will be,

$$U=\frac{D}{2}\iint(\Delta w)^2\,dXdY-\iint pw\,dX\,dY+\frac{h}{2}\iint(\sigma'_x e'_x+\sigma'_y e'_y+\tau'_{xy}\gamma'_{xy})dXdY \tag{11}$$

for a clamped plate. The first and third integrals are taken over the entire plate. The second integral is taken the part of the plate upon which the loads situate. The first integral represents the strain energy of bending and twisting, the second the potential energy of the lateral load, and the third the strain energy of stretching.

If the membrane stresses and extensions in (11) are expressed in terms of the derivatives of u, v and w and make (11) dimmensionless, the energy expression becomes:

$$A=\frac{1}{2}\iint(\Delta\zeta)^2 dxdy-\int_{-\delta}^{\delta}\int_{-\delta}^{\delta}q\zeta\,dxdy+6\iint\Big[\xi^2_x+\zeta^2_x\xi_x+\eta^2_y+\zeta^2_y\eta_y+\left(\frac{\zeta^2_x}{2}+\frac{\zeta^2_y}{2}\right)^2$$
$$+2\mu\left(\xi_x\eta_y+\eta_y\frac{\zeta^2_x}{2}+\xi_x\frac{\zeta_y^2}{2}\right)+\frac{1-\mu}{2}\left(\xi^2_y+\eta^2_x+2\xi_y\eta_x+2\xi_y\zeta_x\zeta_y+2\eta_x\zeta_x\zeta_y\right)\Big]dxdy. \tag{12}$$

in which the subscripts indicate partial differentiation and in which we put,

$$\left.\begin{aligned}
&x=\frac{X}{a},\quad y=\frac{Y}{a},\quad z=\frac{Z}{a},\quad \delta=\frac{d}{a},\\
&\xi=\frac{ua}{h^2},\quad \eta=\frac{va}{h^2},\quad \zeta=\frac{w}{h},\quad A=\frac{Ua^2}{Dh^2},\quad q=\frac{pa^4}{Dh}\\
&s'_x=\frac{\sigma'_x a^2(1-\mu^2)}{Eh^2},\quad s'_y=\frac{\sigma'_y a^2(1-\mu^2)}{Eh^2},\quad s'_{xy}=\frac{\tau'_{xy}a^2}{Gh^2}\\
&s''_x=\frac{\sigma''_x a^2(1-\mu^2)}{Eh^2},\quad s''_{xy}=\frac{\sigma''_y a^2(1-\mu^2)}{Eh^2},\quad s''_{xy}=\frac{\tau''_{xy}a^2}{Gh^2}
\end{aligned}\right\} \tag{13}$$

As the plate are clamped, the following boundary conditions must be satisfied:

$$(\zeta)_{x=\pm1}=(\zeta)_{y=\pm1}=0$$
$$(\zeta_x)_{x=\pm1}=(\zeta_y)_{y=\pm1}=0 \tag{14}$$
$$(\xi)_{x=\pm1}=(\xi)_{y=\pm1}=0$$
$$(\eta)_{x=\pm1}=(\eta)_{y=\pm1}=0$$

These conditions will be satisfied if we assume for $\xi$, $\eta$, and $\zeta$ the following expressions,

$$\xi=x(1-x^2)(1-y^2)(b_{00}+b_{02}\,y^2+b_{20}\,x^2+b_{22}\,x^2y^2) \tag{15}$$
$$\eta=y(1-x^2)(1-y^2)(b_{00}+b_{02}\,x^2+b_{20}\,y^2+b_{22}\,x^2y^2) \tag{16}$$
$$\zeta=(1-x^2)^2(1-y^2)^2(a_{00}+a_{02}\,x^2+a_{02}\,y^2+a_{22}\,x^2y^2) \tag{17}$$

These expressions satisfy the following symmetry conditions:

$$\xi(x,y)=\xi(x,-y)=-\xi(-x,y)=-\xi(-x,-y)$$
$$\eta(x,y)=\eta(-x,y)=-\eta(x,-y)=-\eta(-x,-y) \tag{18}$$
$$\xi(x,y)=\eta(y,x)$$

In equation (27), $a_{00}$ is the only constant and $a_{02}$, $a_{22}$ provide an additional means of variation of the deflection form.

As the expression of symmetrical lateral load, we make use of the following expression;

$$q=q_{00}+q_{02}(x^2+y^2) \tag{19}$$

By Ritz method, since the constants are to be detemined so as to total energy A a minimum, we must have

$$\frac{\partial A}{\partial b_{00}}=0\ ;\quad \frac{\partial A}{\partial b_{02}}=0\ ;\quad \frac{\partial A}{\partial b_{20}}=0\ ;\quad \frac{\partial A}{\partial b_{22}}=0 \tag{20}$$
$$\frac{\partial A}{\partial a_{00}}=0\ ;\quad \frac{\partial A}{\partial a_{02}}=0\ ;\quad \frac{\partial A}{\partial a_{22}}=0 \tag{21}$$

The four equations (20) are linear in the b's and found to be:

$$(0.513016-0.015238\,\mu)b_{00}+(0.066032-0.015238\,\mu)b_{02}+(0.215026-0.001693\,\mu)b_{20}$$
$$+(0.027332-0.005563\,\mu)b_{22}+(-0.030022+0.150109\,\mu)\,a^2_{00}+(-0.089645-0.004829\,\mu)\,a_{00}a_{02}$$
$$+(-0.004409-0.002729\,\mu)a_{00}a_{22}+(-0.015536+0.011337\,\mu]a^2_{02}+(-0.002277+0.000759\,\mu)a_{02}a_{22}$$
$$-0.000097\,a^2_{22}=0. \tag{22. 1}$$

$$(0.066032-0.015238\,\mu)b_{00}+(0.037007-0.015238\,\mu)b_{02}+(0.031202-0.001693\,\mu)b_{20}$$
$$+(0.013787-0.005563\,\mu)b_{22}+(0.013646+0.019105\,\mu)a^2_{00}+(-0.007348+0.008608\,\mu)a_{00}a_{02}$$
$$+(-0.002471+0.001405\,\mu)a_{00}a_{22}+(-0.001324+0.002293\,\mu)a^2_{02}+(-0.000613+0.000242\,\mu)\,a_{02}a_{22}$$
$$-0.000029\,a^2_{22}=0. \tag{2. 22}$$

$$(0.215026-0.001693\,\mu)b_{00}+(0.031202-0.001693\,\mu)b_{02}+(0.169994-0.001165\,\mu)b_{20}$$
$$+(0.024517-0.000813\,\mu)b_{22}+(0.047657+0.034640\,\mu)a^2_{00}+(-0.001566-0.001050\,\mu)a_{00}a_{22}$$
$$+(-0.018265+0.007768\,\mu)a_{00}a_{02}+(-0.004083+0.003426\,\mu)a^2_{02}+(-0.000776+0.000611\,\mu)\,a_{02}a_{22}$$
$$+(-0.000038+0.000089\,\mu)a^2_{22}=0. \tag{22. 3}$$

$$(0.027332-0.005563\,\mu)b_{00}+(0.013787-0.005563\,\mu)\,b_{02}+(0.024517-0.000814\,\mu)b_{20}$$
$$+(0.009917-0.002338\,\mu)b_{22}+(0.007348+0.004409\,\mu)a^2_{00}+(0.000468+0.003117\,\mu)a_{00}a_{02}$$
$$+(-0.000662+0.000501\,\mu)a_{00}a_{22}+(-0.000146+0.000948\,\mu)a^2_{02}+(-0.000236+0.000307\,\mu)a_{02}a_{22}$$
$$+(-0.000015+0.000028\,\mu)a^2_{22}=0 \tag{22. 4}$$

The equations (21) become:

$$1.114558\,a_{00}+0.202647\,a_{02}+0.011258\,a_{22}$$
$$+a_{00}[-0.120087\,b_{00}+0.054585\,b_{02}+0.190628\,b_{20}+0.029392\,b_{22}]$$
$$+a_{02}[-0.179291\,b_{00}-0.014696\,b_{02}-0.036530\,b_{20}+0.000937\,b_{22}]$$
$$+a_{22}[-0.008818\,b_{00}-0.004942\,b_{02}-0.003133\,b_{20}-0.001324\,b_{22}]$$
$$+\mu a_{00}[0.600435\,b_{00}+0.076419\,b_{02}+0.138562\,b_{20}+0.017635\,b_{22}]$$
$$+\mu a_{02}[-0.009657\,b_{00}+0.017215\,b_{02}+0.015536\,b_{20}+0.006234\,b_{22}]$$
$$+\mu a_{22}[-0.005458\,b_{00}+0.002810\,b_{02}-0.002099\,b_{20}+0.001001\,b_{22}]$$
$$+0.785994\,a^3_{00}+0.079869\,a^2_{00}a_{02}-0.016480\,a^2_{00}a_{22}+0.150197\,a_{00}a^2_{02}+0.001510\,a_{00}a^2_{22}$$
$$+0.014022\,a_{00}a_{02}a_{22}+0.010995\,a^3_{02}+0.001940\,a^2_{02}a_{22}+0.000291\,a_{02}a^2_{22}+0.000007\,a^3_{22}$$
$$-\frac{1}{12}\left(\delta-\frac{2}{3}\delta^3+\frac{1}{5}\delta^5\right)^2 q_{00}-\frac{1}{6}\left(\delta-\frac{2}{3}\delta^3+\frac{1}{5}\delta^5\right)\left(\frac{1}{3}\delta^3-\frac{2}{5}\delta^5+\frac{1}{7}\delta^7\right)q_{02}=0. \quad (23.\ 1)$$

$$0.202647\,a_{00}+0.457255\,a_{02}+0.036372\,a_{22}$$
$$+a_{00}[-0.179291\,b_{00}-0.014696\,b_{02}-0.036530\,b_{20}+0.000937\,b_{22}]$$
$$+a_{02}[-0.062143\,b_{00}-0.005297\,b_{02}-0.016332\,b_{20}-0.000585\,b_{22}]$$
$$+a_{22}[-0.004554\,b_{00}-0.001226\,b_{02}-0.001552\,b_{20}-0.000473\,b_{22}]$$
$$+\mu a_{00}[-0.009657\,b_{00}+0.017215\,b_{02}+0.015536\,b_{20}+0.006234\,b_{22}]$$
$$+\mu a_{02}[+0.045348\,b_{00}+0.009173\,b_{02}+0.013702\,b_{20}+0.003792\,b_{22}]$$
$$+\mu a_{22}[+0.001518\,b_{00}+0.000484\,b_{02}+0.001222\,b_{20}+0.000614\,b_{22}]$$
$$+0.026623\,a^3_{00}+0.150197\,a^2_{00}a_{02}+0.007011\,a^2_{00}a_{22}+0.032985\,a_{00}a^2_{02}+0.003880\,a_{00}a_{02}a_{22}$$
$$+0.009870\,a^3_{02}+0.000291\,a_{00}a^2_{22}+0.001945\,a^2_{02}a_{22}+0.000269\,a_{02}a^2_{22}+0.000014\,a^3_{22}$$
$$-\frac{1}{6}\left(\delta-\frac{2}{3}\delta^3+\frac{1}{5}\delta^5\right)\left(\frac{1}{3}\delta^3-\frac{2}{5}\delta^5+\frac{1}{7}\delta^7\right)q_{00}-\frac{1}{6}\left\{\left(\delta-\frac{2}{3}\delta^3+\frac{1}{5}\delta^4\right)\left(\frac{1}{5}\delta^5-\frac{2}{7}\delta^7+\frac{1}{9}\delta^9\right)\right.$$
$$\left.+\left(\frac{1}{3}\delta^3-\frac{2}{5}\delta^5+\frac{1}{7}\delta^7\right)^2\right\}q_{02}=0. \quad (23.\ 2)$$

$$0.011258\,a_{00}+0.036372\,a_{02}+0.009841\,a_{22}$$
$$+a_{00}[-0.008818\,b_{00}-0.004942\,b_{02}-0.003133\,b_{20}+0.001324\,b_{22}]$$
$$+a_{02}[-0.004554\,b_{00}-0.001226\,b_{02}-0.001552\,b_{20}-0.000473\,b_{22}]$$
$$+a_{22}[-0.000388\,b_{00}-0.000114\,b_{02}-0.000152\,b_{20}-0.000061\,b_{22}]$$
$$+\mu a_{00}[-0.005458\,b_{00}+0.002810\,b_{02}-0.002099\,b_{20}+0.001001\,b_{22}]$$
$$+\mu a_{02}[0.001518\,b_{00}+0.000484\,b_{02}+0.001222\,b_{20}+0.000614\,b_{22}]$$
$$+\mu a_{22}[0.000357\,b_{20}+0.000114\,b_{22}]$$
$$-0.005493\,a^3_{00}+0.007011\,a^2_{00}a_{02}+0.001510\,a^2_{00}a_{22}+0.001940\,a_{00}a^2_{02}+0.000582\,a_{00}a_{02}a_{22}$$
$$+0.000020\,a_{00}a^2_{22}+0.000648\,a^3_{02}+0.000269\,a^2_{02}a_{22}+0.000043\,a_{02}a^2_{22}+0.000003\,a^3_{22}$$
$$-\frac{1}{12}\left(\frac{1}{3}\delta^3-\frac{2}{5}\delta^5+\frac{1}{7}\delta^7\right)^2 q_{00}-\frac{1}{6}\left(\frac{\delta^3}{3}-\frac{2}{5}\delta^5+\frac{1}{7}\delta^7\right)\left(\frac{1}{5}\delta^5-\frac{2}{7}\delta^7+\frac{1}{9}\delta^9\right)q_{02}=0. \quad (23.3)$$

The values of $b_{00}$, $b_{02}$, $b_{20}$, and $b_{22}$ can be obtained from equations (22. 1) to (22. 4); for $\mu=0.3$, the values are:

$$\left.\begin{aligned}
b_{00}&=0.405052\,a^2_{00}+0.307930\,a_{00}a_{02}+0.003310\,a_{00}a_{22}+0.034605\,a^2_{02}+0.003999\,a_{02}a_{22}\\
&\quad+0.000292\,a^2_{22}\\
b_{02}&=-1.231092\,a^2_{00}-0.134661\,a_{00}a_{02}+0.077909\,a_{00}a_{22}-0.009763\,a^2_{02}+0.014283\,a_{02}a_{22}\\
&\quad+0.000674\,a^2_{22}\\
b_{20}&=-0.910273\,a^2_{00}-0.243842\,a_{00}a_{02}+0.001338\,a_{00}a_{22}-0.015872\,a^2_{02}-0.003371\,a_{02}a_{22}\\
&\quad-0.000469\,a^2_{22}\\
b_{22}&=1.946906\,a^2_{00}-0.190484\,a_{00}a_{02}-0.059594\,a_{00}a_{22}-0.056702\,a^2_{02}-0.005404\,a_{02}a_{22}\\
&\quad+0.000294\,a^2_{22}
\end{aligned}\right\}\quad(24)$$

When the expressions (24) for b's are introduced in (23.1) to (23.3), we have the three equations containing $a_{00}$, $a_{02}$, $a_{22}$, $q_{00}$ and $q_{02}$. If we put $a_{00}$ and $q_{02}$ equal to certain values, the remaining constant values $a_{02}$, $a_{22}$ and $q_{00}$ will be determined. Then we may consider $a_{02}$, $a_{22}$ and $q_{00}$ as functions of $a_{00}$ and $q_{02}$. Next, we shall discuss several important cases of the problems.

I. Uniform load $q_{02}=0$.

We shall consider the case in which the uniform load $q_{00}$ is on the square part $-\delta\leqq x\leqq\delta$, $-\delta\leqq y\leqq\delta$ of the plate.

We may consider the cases $\delta=1$, 0.8, 0.6, 0.4, 0.2 and 0.

Case I. $\delta=1$.

When the uniform load is on the whole plate, the calculated values of $a_{02}$, $a_{22}$, $q_{00}$, $b_{00}$, $b_{02}$, $b_{20}$ and $b_{22}$ for various assumed values of $a_{00}$ are given in Table I.I.

Table I. I, $\delta=1.0$ for $\mu=0.3$

Values of $q_{00}$ and displacement constants for $\mu=0.3$ and various $a_{00}$ values

| $a_{00}$ | $a_{02}$ | $a_{22}$ | $q_{00}$ | $b_{00}$ | $b_{02}$ | $b_{20}$ | $b_{22}$ |
|---|---|---|---|---|---|---|---|
| 0.4 | 0.125125 | 0.142953 | 21.5061 | 0.081028 | −0.199143 | −0.158090 | 0.297585 |
| 0.8 | 0.349623 | 0.412522 | 53.5206 | 0.351310 | −0.798871 | −0.652841 | 1.16542 |
| 1.0 | 0.513210 | 0.642803 | 76.7848 | 0.575767 | −1.24770 | −1.04004 | 1.79425 |
| 1.2 | 0.713210 | 0.956489 | 106.932 | 0.869939 | −1.79271 | −1.52778 | 2.54084 |
| 1.6 | 1.18402 | 1.86931 | 194.343 | 1.68857 | −3.15340 | −2.81959 | 4.35456 |
| 2.0 | 1.72965 | 3.14002 | 329.585 | 2.83360 | −4.84565 | −4.54191 | 6.55893 |

The ratio $\frac{w_{max}}{h}$, that is the ratio of the maximum deflection to the thickness is identically equal to $a_{00}$. The values of $q_{00}$ the dimensionless load parameter, corresponding to various values of $a_{00}$ are give in Table I. I. and Fig. 2 shows a curve for $\frac{w_{max}}{h}$ as a function of $q_{00}$. Fig. 2 shows that the straight line $w_{max}/h=0.0202305\ q_{00}$ obtained by the linear theory becomes tangent to this curve at $q_{00}=0$.

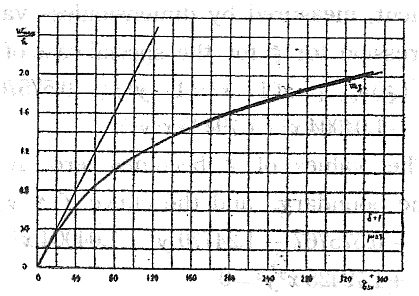

Fig. 2

Table II. I

Ratio of deflection to maximum deflection along $x$-axis

| $a_{00}\diagdown x$ | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|---|---|---|---|---|---|---|
| 0.4 | 1 | 0.933132 | 0.740915 | 0.455726 | 0.155046 | 0 |
| 0.8 | 1 | 0.937711 | 0.754939 | 0.474043 | 0.165849 | 0 |
| 1.0 | 1 | 0.940519 | 0.763539 | 0.485276 | 0.172168 | 0 |
| 1.2 | 1 | 0.943417 | 0.772415 | 0.496868 | 0.178688 | 0 |
| 1.6 | 1 | 0.948880 | 0.789144 | 0.518719 | 0.190980 | 0 |
| 2.0 | 1 | 0.953463 | 0.803179 | 0.537050 | 0.201291 | 0 |

S. Way has also calculated this case. He used for the expression $\zeta$ of the following one:

$$\zeta=(1-x^2)^2(1-y^2)^2(a_{00}+a_{02}x^2+a_{02}y^2) \quad . \tag{25}$$

This expression will be identically equal to (17), if we put $a_{22}=0$.

For a moderate deflection it was seen the above expression (25) gives a good approximation. When the deflction becomes more and more large, this expression will become less exact and to obtain a better approximation, we must use (17) for the expression of $\zeta$. We shall also see later that, in the linear theory, the expression (17) gives a better approximation than Way's one. In order to obtain a more exact solution than Way's one, we must consider the constant $a_{22}$ and adopt (17) for the expression of $\zeta$. When the deflection becomes large, the influence of $a_{22}$ will becomes important. We see for instance this effect, in our values and Way's in $q_{00}$:

| $a_{00}$ | $q_{00}$ (Way) | $q_{00}$ (our case) |
|---|---|---|
| 0.4 | 21.5477 | 21.5061 |
| 1.0 | 77.1729 | 76.7848 |
| 2.0 | 335.0131 | 329.5850 |

Fig. 2. shows also the comparison between our calculated values and Way's one for $q_{00}$. It was seen from Way's paper that the square plate is more rigid than long rectangular plate and more flexible than the circular plate, and these results also hold in our case. The general character of the $x$-component of the displacement, measured by dimensionless variable $\xi$, can be observed by writting the expression for $\xi$ for the special case of $a_{00}=1$:

$$\left\{\begin{array}{l}(\xi)_{a_{00}=1}=x(1-x^2)(1-y^2)(-0.575767-1.24770y^2\\ -1.04004x^2+1.79425x^2y^2).\end{array}\right.$$

The values of $\xi$ becomes zero on $y$-axis, the boundary, and the curve $f(x, y)=0$:

$$\left\{\begin{array}{l}-0.575767-1.24770y^2-1.04004x^2\\ +1.79425x^2y^2=0\end{array}\right.$$

The curve $f(x, y)$ is shown in Fig. 3. The values of $\xi$ for $y=o$, $a_{00}=1$, are given in Table III.

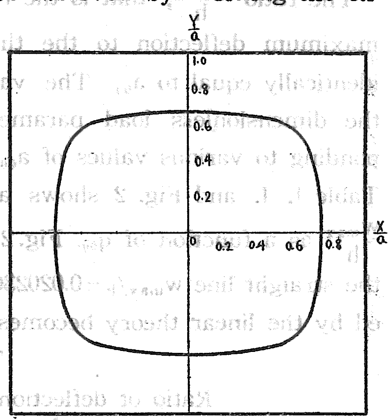


Fig. 3

Table III the values of $\xi$ for $y=0$, $a_{00}=1$

| $x$ | 0 | 0.2 | 0.37379 | 0.4 | 0.6 | 0.8 | 0.89019 | 1.0 |
|---|---|---|---|---|---|---|---|---|
| $\xi$ | 0 | 0.102572 | 0.138418 | 0.137545 | 0.077320 | −0.025879 | −0.045889 | 0 |

We will now consider the calculation of the dimensionless stresses. The membrane stresses at any point are given by

$$\left.\begin{aligned} s'_x &= \left(\xi_x + \frac{\zeta^2_x}{2}\right) + \mu\left(\eta_y + \frac{\zeta^2_y}{2}\right) \\ s'_y &= \left(\eta_y + \frac{\zeta^2_y}{2}\right) + \mu\left(\xi_x + \frac{\zeta^2_x}{2}\right) \\ s'_{xy} &= \eta_x + \xi_y + \zeta_x \cdot \zeta_y . \end{aligned}\right\} \quad (26)$$

The membrane stresses along the $y$-axis and the side $x=1$ are of particular interest. Along the $y$-axis:

$$\left\{\begin{aligned} (s'_x)_{x=0} &= (1-y^2)(b_{00}+b_{02}y^2)+\mu[2b_{20}y^2(1-y^2)+(1-3y^2)(b_{00}+b_{20}y^2) \\ &\quad +2y^2(1-y^2)^2\{a_{02}(1-y^2)-2(a_{00}+a_{02}y^2)\}^2] \\ (s'_{xy})_{x=0} &= 0 \\ (s'_y)_{x=0} &= [2b_{20}y^2(1-y^2)+(1-3y^2)(b_{00}+b_{20}y^2)+2y^2(1-y^2)^2\{a_{02}(1-y^2) \\ &\quad -2a_{00}+a_{02}y^2\}^2]+\mu(1-y^2)(b_{00}+b_{02}y^2) . \end{aligned}\right. \quad (27)$$

These stresses are given in Table IV and plotted in Fig. 4 for $a_{00}=1$

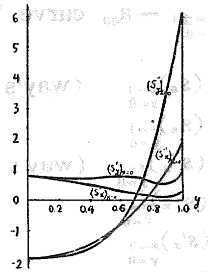

Fig. 4(a)

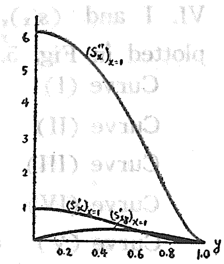

Fig. 4(b)

Table IV. I

(a) The values of $(s'_x)_{x=0}$

| $a_{00} \diagdown y$ | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.74850 | 0.74568 | 0.67491 | 0.59438 | 0.50091 | 0.40014 | 0.30097 | 0.20068 | 0.10498 | 0.06716 | 0.27856 |
| 2 | 3.6837 | 3.5634 | 3.2656 | 2.8248 | 2.3427 | 1.9071 | 1.4493 | 1.2385 | 0.8877 | 0.4154 | 1.0250 |

(b) The values of $(s'_y)_{x=0}$

| $a_{00} \diagdown y$ | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.74850 | 0.74427 | 0.71840 | 0.70193 | 0.71131 | 0.73358 | 0.75746 | 0.72401 | 0.59319 | 0.47451 | 0.92854 |
| 2 | 3.6837 | 3.5476 | 3.1981 | 2.7982 | 2.5644 | 2.6665 | 3.0410 | 3.2773 | 3.2512 | 3.0137 | 3.4166 |

Along the side $x=1$, the membrane stress are expressed as:

$$\left\{\begin{aligned} (s'_x)_{x=1} &= -2(1-y^2)\{b_{00}+b_{20}+(b_{02}+b_{22})y^2\} \\ (s'_y)_{x=1} &= \mu(s'_x)_{x=1} \\ (s'_{xy})_{x=1} &= -2y(1-y^2)\{b_{00}+b_{02}+(b_{20}+b_{22})y^2\} \end{aligned}\right. \quad (28)$$

And these values for $a_{00}=1$, and 2 are given in Table V. I.

Table V. I

The values of $(s'_x)_{x=1}$

| $a_{00} \diagdown y$ | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|---|---|---|---|---|---|---|
| 1 | 0.92854 | 0.84943 | 0.63307 | 0.34241 | 0.08243 | 0 |
| 2 | 3.1417 | 3.1484 | 2.4094 | 1.3972 | 0.44408 | 0 |

The values of $(s'_{xy})_{x=1}$

| $a_{00} \backslash y$ | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|---|---|---|---|---|---|---|
| 1 | 0 | 0.24617 | 0.36997 | 0.30698 | 0.15639 | 0 |
| 2 | 0 | 0.7416 | 1.1352 | 0.9876 | 0.4154 | 0 |

The membrane stresses at the center of the plate and at the middle point of the sides are

$$(s'_x)_{\substack{x=0\\y=0}} = (s'_y)_{\substack{x=0\\y=0}} = b_{00}(1+\mu) \qquad (29)$$

$$(s'_x)_{\substack{x=\pm1\\y=0}} = (s'_y)_{\substack{x=0\\y=\pm1}} = -2(b_{00}+b_{20}) \quad (30)$$

The values of these stresses corresponding to various values of $a_{00}$ are given in Table VI. I and $(s'_x)_{\substack{x=\pm1\\y=0}} - a_{00}$ curve has been plotted in Fig. 5.

Curve (I) $(s_x)_{\substack{x=1\\y=0}}$ (way's curve)
Curve (II) $(s_x)_{\substack{x=1\\y=0}}$
Curve (III) $(s'_x)_{\substack{x=1\\y=0}}$ (way's cuvre)
Curve (IV) $(s''_x)_{\substack{x=1\\y=0}}$
Curve (V) $(s'_x)_{\substack{x=0\\y=0}}$
Curve (VI) $(s'_x)_{\substack{x=0\\y=0}}$

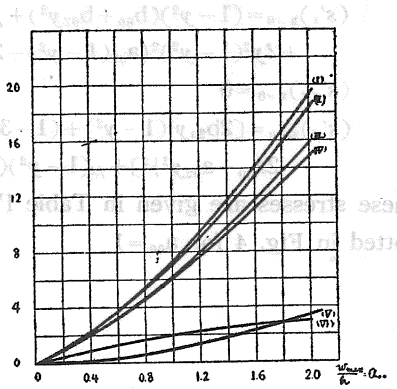


Fig. 5

Table VI. I

| $a_{00}$ | $q_{00}$ | $(s'_x)_{\substack{x=0\\y=0}}$ | | $(s'_x)_{\substack{x=1\\y=0}}$ | |
|---|---|---|---|---|---|
| | | Present Case | Way | Present Case | Way |
| 0.4 | 21.5061 | 0.10534 | 0.1070 | 0.15412 | 0.1540 |
| 0.8 | 53.5206 | 0.45451 | 0.4655 | 0.60306 | 0.6023 |
| 1.0 | 76.7848 | 0.74850 | 0.7655 | 0.92854 | 0.9270 |
| 1.2 | 106.932 | 1.1309 | 1.1592 | 1.3157 | 1.3130 |
| 1.6 | 194.343 | 2.1951 | 2.2647 | 2.2620 | 2.2542 |
| 2.0 | 329.585 | 3.6837 | 3.8247 | 3.4166 | 3.4062 |

In this Table Way's values are also given for comparision.

The dimensionless bending stresses on the upper surface $Z=-\frac{h}{2}$ are given by:

$$s''_x = \frac{1}{2}(\zeta_{xx}+\mu\zeta_{yy}); \quad s''_y = \frac{1}{2}(\zeta_{yy}+\mu\zeta_{xx}); \quad s''_{xy} = \zeta_{xy} \qquad (31)$$

The values of these stresses along the $y$-axis and the side $x=\pm1$ are as follows:

$$\left.\begin{aligned} &(s''_y)_{x=0} = (1-y^2)^2\{(a_{02}-2a_{00})+(a_{22}-2a_{02})y^2\}+\mu\{(6y^2-2)a_{00}+(15y^4-12y^2+1)a_{02}\} \\ &(s''_y)_{x=0} = \{(6y^2-2)a_{00}+(15y^4-12y^2+1)a_{02}\}+\mu(1-y^2)^2\{(a_{02}-2a_{00})+(a_{22}-2a_{02})y^2\} \\ &(s''_{xy}) = 0 \end{aligned}\right\} (32)$$

$$\left.\begin{aligned} &(s''_y)_{x=\pm1} = 4(1-y^2)^2\{(a_{00}+a_{02})+(a_{02}+a_{22})y^2\} \\ &(s''_y)_{x=\pm1} = \mu(s''_x)_{x=\pm1} \\ &(s''_{xy})_{x=\pm1} = 0 \end{aligned}\right\} (33)$$

The values of these stresses corresponding $y=0,\ 0.2,\ \cdots,1.0$ for $a_{00}=1$, and 2 are given in Tables VII. 1 and VIII. 1.

Table VII. 1

The values of $(s'_x)_{x=0}$

| $a_{00}\diagdown y$ | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | −1.9328 | −1.9072 | −1.8286 | −1.6913 | −1.4869 | −1.2039 | −0.8294 | −0.3750 | 0.2450 | 0.9658 | 1.8158 |
| 2 | −2.9528 | −2.9361 | −2.8789 | −2.7590 | −2.5404 | −2.1733 | −1.5947 | −0.7302 | 0.5053 | 2.2063 | 4.4744 |

The values of $(s''_y)_{x=0}$

| $a_{00}\diagdown y$ | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | −1.9328 | −1.9259 | −1.8961 | −1.8166 | −1.6428 | −1.3124 | −0.7458 | −0.1532 | 1.4976 | 3.4161 | 6.0528 |
| 2 | −2.9528 | −3.0250 | −3.2111 | −3.4196 | −3.4981 | −3.2334 | −2.3518 | −0.5194 | 2.6572 | 7.6313 | 14.915 |

Table VIII. I

The values of $(s''_x)_{x=\pm 1}$

| $a_{00}\diagdown y$ | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 6.0528 | 5.9771 | 5.7488 | 5.3570 | 4.7939 | 4.0550 | 3.1612 | 2.1637 | 1.1680 | 0.3537 | 0 |
| 2 | 14.915 | 14.809 | 14.472 | 13.802 | 12.722 | 11.128 | 8.980 | 6.361 | 3.548 | 1.108 | 0 |

The bending stresses at the center of the plate and at the middle of a side are:

$$(s''_x)_{\substack{x=0\\y=0}}=(s''_y)_{\substack{x=0\\y=0}}=(1+\mu)(a_{02}-2a_{00}) \tag{34}$$

$$(s''_x)_{\substack{x=\pm1\\y=0}}=(s''_y)_{\substack{x=0\\y=\pm1}}=4(a_{00}+a_{02}) \tag{35}$$

The values of these stresses corresponding to various values of $a_{00}$ are given in Table IX. I and have been plotted in Fig. 5. for the case of $\mu=0.3$.

Table IX. I

Bending stresses at center and at middle of side

| $a_{00}$ | $q_{00}$ | $(s''_x)_{\substack{x=0\\y=0}}$ Present Case | Way's value | $(s''_x)_{\substack{x=\pm1\\y=0}}$ Present Case | Way's value |
|---|---|---|---|---|---|
| 0.4 | 21.5061 | −0.87864 | −0.8623 | 2.1005 | 2.1469 |
| 0.8 | 53.5206 | −1.6255 | −1.5848 | 4.5985 | 4.7237 |
| 1.0 | 76.7848 | −1.9328 | −1.8716 | 6.0528 | 6.2411 |
| 1.2 | 106.932 | −2.1968 | −2.1088 | 7.6408 | 7.9113 |
| 1.6 | 194.343 | −2.6208 | −2.4564 | 11.1361 | 11.642 |
| 2.0 | 329.585 | −2.9528 | −2.6838 | 14.915 | 15.742 |

In this Table, Way's values are given for comparison.

From Table II. I, VI. I and IX. I, we see that the discrepancy between our solution and Way's one becomes more and more large when the deflection increases.

The Total stresses at middle of edge are given by

$$(s_x)_{\substack{x=1\\y=0}} = (s'_x)_{\substack{x=1\\y=0}} + (s''_x)_{\substack{x=1\\y=0}}$$

The values of theses stresses for various values of $a_{00}$ are given in Table X. I.

Table X. I

| $a_{00}$ | 0.4 | 0.8 | 1.0 | 1.2 | 1.6 | 2.0 |
|---|---|---|---|---|---|---|
| $(s_x)_{\substack{x=1\\y=0}}$ | 2.2546 | 5.2016 | 6.9813 | 8.9565 | 13.398 | 18.332 |
| Way's value | 2.3009 | 5.3260 | 7.1681 | 9.2242 | 13.896 | 16.148 |

The values of $(s_x)_{\substack{x=1\\y=0}}$, $(s''_x)_{\substack{x=1\\y=0}}$ and $(s''_x)_{\substack{x=0\\y=0}}$ against $a_{00}$ are plotted in Fig. 5. Way's values for $(s_x)_{\substack{x=1\\y=0}}$ and $(s''_x)_{\substack{x=1\\y=0}}$ have been also plotted in this figure for comparison.

We now consider the linear theory. In this case we may take $\xi=\eta=0$. The constants $a_{00}$; $a_{02}$ and $a_{22}$ are then found from the equations

$$\frac{\partial A}{\partial a_{00}}=0\ ;\quad \frac{\partial A}{\partial a_{02}}=0\ ;\quad \frac{\partial A}{\partial a_{22}}=0.$$

These equations give the values

$$\left.\begin{aligned} a_{00}&=0.0202305\,q_{00}\\ a_{02}&=0.0053485\,q_{00}\\ a_{22}&=0.000002532\,q_{00}\end{aligned}\right\}\qquad(36)$$

The approximate expression for the deflection in dimensionless form is therefore

$$\zeta=(1-x^2)^2(1-y^2)^2\{0.0202305+0.0053485(x^2+y^2)+0.000002532\,x^2y^2\}q_{00}.$$

From this expression, $\frac{W_{max}}{h}$, $m_x$ at center, $m_x$ at the middle of side $x=1$, and $Q_x$ at the middle of the side $x=1$ can easily be calculated by (1) and (3).

These values are given in Table XI, and the values found by Hencky and Way are also given for comparison.

Table XI

| | Present case | Hencky | Way |
|---|---|---|---|
| $\frac{W_{max}}{h}$ | $0.0202305\ \frac{pa^4}{Dh}$ | $0.020243\ \frac{pa^4}{Dh}$ | $0.020202\ \frac{pa^4}{Dh}$ |
| $(m_x)_{\substack{x=a\\y=0}}$ | $-0.204632\ pa^2$ | $-0.2052\ pa^2$ | $-0.208482\ pa^2$ |
| $(m_x)_{\substack{x=0\\y=0}}$ | $0.0912925\ pa^2$ | $0.092\ pa^2$ | $0.092524\ pa^2$ |
| $(Q_x)_{\substack{x=a\\y=0}}$ | $-0.870724\ pa$ | $-0.88\ pa$ | $-0.9066\ pa$ |

The reason of the better agreement found here between the present and exact solutions than Way's solution are due to use the expression (17) for the deflection in place of (26). Thus the better approximate solution was found.

Case II. $\delta=0.8$

This is the case in which the lateral load is on the square $-0.8\leqq x\leqq 0.8$, $-0.8\leqq y\leqq 0.8$.

We can treat this case as the same method as Case I. The following results are obtained:

### Table I. II.

Values of $q_{00}$ and displacement constants for $\mu=0.3$ and various $a_{00}$ values

| $a_{00}$ | $a_{02}$ | $a_{22}$ | $q_{00}$ | $b_{00}$ | $b_{02}$ | $b_{20}$ | $b_{22}$ |
|---|---|---|---|---|---|---|---|
| 0.4 | 0.110974 | 0.059023 | 22.0727 | 0.079009 | −0.201138 | −0.156656 | 0.300910 |
| 0.8 | 0.310756 | 0.210742 | 54.8445 | 0.339961 | −0.808219 | −0.644744 | 1.18280 |
| 1.0 | 0.462284 | 0.361033 | 78.6988 | 0.556694 | −1.26491 | −1.02653 | 1.82441 |
| 1.2 | 0.645927 | 0.573143 | 110.667 | 0.840246 | −1.82213 | −1.50690 | 2.58935 |
| 1.6 | 1.09109 | 1.25353 | 198.334 | 1.62826 | −3.22145 | −2.77754 | 4.45759 |
| 2.0 | 1.60474 | 2.23886 | 334.920 | 2.72827 | −4.97816 | −4.47301 | 6.74546 |

### Table II. II

The values of w/h for various values of $a_{00}$ along $x$-axis

| $a_{00}$ \ $x$ | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|---|---|---|---|---|---|---|
| 0.4 | 1 | 0.931827 | 0.736921 | 0.45051 | 0.152612 | 0 |
| 0.8 | 1 | 0.935953 | 0.749454 | 0.466879 | 0.161819 | 0 |
| 1.0 | 1 | 0.938642 | 0.757790 | 0.477767 | 0.167947 | 0 |
| 1.2 | 1 | 0.941443 | 0.766369 | 0.489372 | 0.174246 | 0 |
| 1.6 | 1 | 0.946739 | 0.782587 | 0.510155 | 0.186162 | 0 |
| 2.0 | 1 | 0.951179 | 0.796184 | 0.527914 | 0.196152 | 0 |

### Table IV. II.

The values of $(s'_x)_{x=0}$

| $a_{00}$ \ $y$ | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.72370 | 0.70567 | 0.72663 | 0.57637 | 0.48438 | 0.38699 | 0.28556 | 0.28136 | 0.15963 | 0.15744 | 0.28190 |
| 2 | 3.5468 | 3.4412 | 3.1461 | 2.7223 | 2.2544 | 1.8231 | 1.4620 | 1.1295 | 1.8378 | 0.4183 | 1.0468 |

The values of $(s'_y)_{x=0}$

| $a_{00}$ \ $y$ | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.72370 | 0.71847 | 0.72732 | 0.69846 | 0.71184 | 0.74292 | 0.75514 | 0.73493 | 0.50401 | 0.49484 | 0.93968 |
| 2 | 3.5468 | 3.4270 | 3.1221 | 2.7802 | 2.5926 | 2.4016 | 3.05596 | 3.3179 | 3.3156 | 2.1459 | 3.4895 |

### Table V. II.

The values of $(s'_x)_{x=1}$

| $a_{00}/y$ | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|---|---|---|---|---|---|---|
| 1 | 0.93968 | 0.85639 | 0.62940 | 0.32722 | 0.06411 | 0 |
| 2 | 3.4895 | 3.2142 | 2.4561 | 1.4478 | 0.45809 | 0 |

The values of $(s'_{xy})_{x=1}$

| $a_{00}/y$ | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|---|---|---|---|---|---|---|
| 1 | 0 | 0.25970 | 0.39014 | 0.32332 | 0.11380 | 0 |
| 2 | 0 | 0.82901 | 1.2676 | 1.0996 | 0.4582 | 0 |

## Table VI. II.

The values of $(s'_x)_{\substack{x=0\\y=0}}$ and $(s'_x)_{\substack{x=1\\y=0}}$

| $a_{00}$ | $(S'_x)_{\substack{x=0\\y=0}}$ | $(S'_x)_{\substack{x=1\\y=0}}$ |
|---|---|---|
| 0.4 | 0.10270 | 0.15529 |
| 0.8 | 0.44195 | 0.60957 |
| 1.0 | 0.72370 | 0.93968 |
| 1.2 | 1.0923 | 1.3333 |
| 1.6 | 2.1167 | 2.2986 |
| 2.0 | 3.5468 | 3.4695 |

## Table VII. II.

The values of $(s''_x)_{x=0}$

| $a_{00} \diagdown y$ | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | −1.9990 | −1.9724 | −1.8905 | −1.7394 | −1.5351 | −1.2417 | −0.8559 | −0.3646 | 0.2316 | 0.9385 | 1.7547 |
| 2 | −3.1138 | −3.0968 | −3.0374 | −2.9119 | −2.6818 | −2.2961 | −1.6907 | −0.7982 | 0.4548 | 2.1411 | 4.3257 |

The values of $(s''_y)_{x=0}$

| $a_{00} \diagdown y$ | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | −1.9990 | −1.9863 | −1.9399 | −1.8330 | −1.6324 | −1.2745 | −0.6900 | 0.2082 | 1.5184 | 3.3469 | 5.8491 |
| 2 | −3.1138 | −3.1726 | −3.3200 | −3.4702 | −3.5801 | −3.1514 | −2.2254 | −0.3957 | 2.7026 | 7.4853 | 14.419 |

## Table VIII. II.

The values of $(s''_x)_{x=\pm1}$

| $a_0 \diagdown y$ | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 5.8491 | 5.7408 | 5.4209 | 4.9050 | 4.2200 | 3.4058 | 2.5171 | 1.6262 | 0.8263 | 0.2352 | 0 |
| 2 | 14.419 | 14.283 | 13.855 | 13.086 | 11.910 | 10.273 | 8.173 | 5.710 | 3.144 | 0.970 | 0 |

## Table X. II.

Bending stresses at center and at middle of side.

| $a_{00}$ | $(s''_x)_{\substack{x=0\\y=0}}$ | $(s''_x)_{\substack{x=\pm1\\y=0}}$ |
|---|---|---|
| 0.4 | −0.8957 | 2.0439 |
| 0.8 | −1.6760 | 4.4430 |
| 1.0 | −1.9990 | 5.8491 |
| 1.2 | −2.2803 | 7.3837 |
| 1.6 | −2.7416 | 11.764 |
| 2.0 | −3.1138 | 14.419 |

## Table X. II.

The total stresses $(s_x)_{\substack{x=1\\y=0}}$ at middle of edge.

| $a_{00}$ | 0.4 | 0.8 | 1.0 | 1.2 | 1.6 | 2.0 |
|---|---|---|---|---|---|---|
| $(s_x)_{\substack{x=1\\y=0}}$ | 2.1992 | 5.0526 | 6.7888 | 8.7170 | 14.063 | 17.789 |

Case III. $\delta=0.6$

Table I. III

Values of $q_{00}$ and displacements constants for $\mu=0.3$ and various $a_{00}$ values

| $a_{00}$ | $a_{02}$ | $a_{22}$ | $q_{00}$ | $b_{00}$ | $b_{02}$ | $b_{20}$ | $b_{22}$ |
|---|---|---|---|---|---|---|---|
| 0.4 | 0.031770 | −0.094596 | 26.3303 | 0.008621 | −0.201682 | −0.145803 | 0.311301 |
| 0.8 | 0.151623 | −0.235278 | 65.5399 | 0.296630 | −0.819594 | −0.612675 | 1.23304 |
| 1.0 | 0.251363 | −0.256369 | 93.8898 | 0.483554 | −1.28641 | −0.972726 | 1.91109 |
| 1.2 | 0,380016 | −0.264451 | 130.032 | 0.727242 | −1.86170 | −1.42440 | 2.72797 |
| 1.6 | 0.704603 | −0.158971 | 233.749 | 1.39971 | −3.33059 | −2.61283 | 4.75731 |
| 2.0 | 1.08946 | +0.125487 | 380.459 | 2.33342 | −5.20786 | −4.19137 | 7.28958 |

Table II. III

The values of w/h for variouse values of $a_{00}$ along $x$-axis

| $a_{00}\diagdown x$ | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|---|---|---|---|---|---|---|
| 0.4 | 1 | 0.924528 | 0.714567 | 0.421312 | 0.136188 | 0 |
| 0.8 | 1 | 0.928587 | 0.726997 | 0.437547 | 0.145320 | 0 |
| 1.0 | 1 | 0.930866 | 0.733978 | 0.446665 | 0.150449 | 0 |
| 1.2 | 1 | 0.933274 | 0.741352 | 0.456296 | 0.155867 | 0 |
| 1.6 | 1 | 0.937834 | 0.755317 | 0.474536 | 0.166127 | 0 |
| 2.0 | 1 | 0.941681 | 0.767098 | 0.489924 | 0.174782 | 0 |

Table IV. III

The values of $(s'_x)_{x=0}$

| $a_{00}\diagdown x$ | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.62862 | 0.91374 | 0.57115 | 0.50600 | 0.42606 | 0.33626 | 0.23972 | 0.14126 | 0.05633 | 0.04812 | 0.07360 |
| 2 | 3.0334 | 2.9514 | 2.7199 | 2.3786 | 1.9418 | 1.5709 | 1.1722 | 0.7775 | 0.4023 | 0.2522 | 1.1148 |

The values of $(s'_y)_{x=0}$

| $a_{00}\diagdown x$ | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.62862 | 0.632316 | 0.64556 | 0.67146 | 0.71255 | 0.75242 | 0.75939 | 0.69794 | 0.55876 | 0.44225 | 0.97834 |
| 2 | 3.0334 | 2.98710 | 2.8780 | 2.7814 | 2.7300 | 2.8896 | 3.0171 | 2.9296 | 2.4328 | 1.9171 | 3.7157 |

Table V. III

The values of $(s'_x)_{x=1}$

| $a_{00}\diagdown y$ | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|---|---|---|---|---|---|---|
| 1 | 0.97834 | 0.89124 | 0.65390 | 0.33829 | −0.06435 | 0 |
| 2 | 3.7159 | 3.4074 | 2.5618 | 1.4189 | 3.7846 | 0 |

The values of $(s'_{xy})_{x=1}$

| $a_{00}$ \ $y$ | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|---|---|---|---|---|---|---|
| 1 | 0 | 0.29389 | 0.43863 | 0.35716 | 0.05653 | 0 |
| 2 | 0 | 1.0562 | 1.5985 | 1.3510 | 0.5136 | 0 |

Table VI. III

| $a_{00}$ | $(s'_x)_{x=0, y=0}$ | $(s'_x)_{x=1, y=0}$ |
|---|---|---|
| 0.4 | 0.08821 | 0.15436 |
| 0.8 | 0.38562 | 0.63209 |
| 1.0 | 0.62862 | 0.97834 |
| 1.2 | 0.94541 | 1.3942 |
| 1.6 | 1.8196 | 2.4262 |
| 2.0 | 3.0335 | 3.7157 |

Table VII. III

The values of $(s''_x)_{x=0}$

| $a_{00}$ \ $y$ | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | −2.2632 | −2.2368 | −2.1265 | −1.9395 | −1.6720 | −1.3188 | −0.8839 | −0.3660 | 0.2220 | 0.8572 | 1.5016 |
| 2 | −3.7837 | −3.7486 | −3.6362 | −3.4257 | −3.0847 | −2.5732 | −1.8487 | −0.8726 | 0.08119 | 1.9174 | 3.7073 |

The values of $(s''_y)_{x=0}$

| $a_{00}$ \ $y$ | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | −2.2632 | −2.2348 | −2.1151 | −1.9010 | −1.5706 | −1.0937 | −0.4343 | 0.4532 | 1.6184 | 3.1163 | 5.0054 |
| 2 | −3.7837 | −3.7815 | −3.7548 | −3.6438 | −3.3496 | −2.7353 | −1.6276 | 0.1815 | 2.8318 | 6.8923 | 12.358 |

Table VIII. III

The values of $(s''_x)_{x=\pm 1}$

| $a_{00}$ \ $y$ | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 5.0054 | 4.9056 | 4.6123 | 4.1435 | 3.5296 | 2.8128 | 2.0473 | 1.2994 | 0.6470 | 0.1801 | 0 |
| 2 | 12.358 | 12.160 | 11.568 | 10.596 | 9.268 | 7.635 | 5.778 | 3.860 | 2.005 | 0.588 | 0 |

Table IX. III

Bending stresses at center and at middle of side.

| $a_{00}$ | $(s'')_{x=0, y=0}$ | $(s''_x)_{x=\pm 1, y=0}$ |
|---|---|---|
| 0.4 | −0.99870 | 1.7270 |
| 0.8 | −1.8829 | 3.8065 |
| 1.0 | −2.2632 | 5.0054 |
| 1.2 | −2.6260 | 6.3201 |
| 1.6 | −3.4440 | 9.2184 |
| 2.0 | −3.7837 | 12.358 |

Table X. III

The total stresses $(s_x)_{x=1}$ at middle of edge.
$y=0$

| $a_{00}$ | 0.4 | 0.8 | 1.0 | 1.2 | 1.6 | 2.0 |
|---|---|---|---|---|---|---|
| $(s_x)_{x=1, y=0}$ | 1.8814 | 4.4386 | 5.9837 | 7.7144 | 11.645 | 16.074 |

Case 4. $\delta=0.4$

Table I IV

Values of $q_{00}$ and displacement constants for $\mu=0.3$ and various $a_{00}$ values

| $a_{00}$ | $a_{02}$ | $a_{22}$ | $q_{00}$ | $b_{00}$ | $b_{02}$ | $b_{20}$ | $b_{22}$ |
|---|---|---|---|---|---|---|---|
| 0.4 | −0.0708187 | −0.0551028 | 43.4578 | 0.0562027 | −0.194869 | −0.138860 | 0.317909 |
| 0.8 | −0.0837004 | −0.117244 | 108.454 | 0.238589 | −0.787899 | −0.566524 | 1.26391 |
| 1·0 | −0.0595218 | −0.140950 | 155.532 | 0.386419 | −1.23396 | −0.896042 | 1.96640 |
| 1.2 | −0.0150639 | −0.151853 | 215.720 | 0.577129 | −1.78453 | −1.30665 | 2.81783 |
| 1.6 | −0.123365 | −0.090992 | 385.573 | 1.09771 | −3.18982 | −2.37883 | 4.95436 |
| 2.0 | 0.31079 | 0.109552 | 638.640 | 1.81543 | −4.99128 | −3.79372 | 0.765076 |

Table II. IV.

The values of w/h for various values of $a_{00}$ along $x$-axis

| $a_{00}\diagdown x$ | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|---|---|---|---|---|---|---|
| 0.4 | 1 | 0.915073 | 0.685612 | 0.383493 | 0.114915 | 0 |
| 0.8 | 1 | 0.917743 | 0.693788 | 0.394172 | 0.120922 | 0 |
| 1.0 | 1 | 0.921177 | 0.704304 | 0.407909 | 0.128649 | 0 |
| 1.2 | 1 | 0.921137 | 0.704183 | 0.407749 | 0.128559 | 0 |
| 1.6 | 1 | 0.924442 | 0.714305 | 0.420969 | 0.135995 | 0 |
| 2.0 | 1 | 0.927317 | 0.723109 | 0.432469 | 0.142464 | 0 |

Table IV. IV

The values of $(s'_x)_{x=0}$

| $a_{00}\diagdown y$ | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.50234 | 0.49978 | 0.48876 | 0.46048 | 0.40419 | 0.31290 | 0.19047 | 0.05933 | −0.02882 | −0.05505 | 0.30578 |
| 2 | 2.3601 | 2.3230 | 2.2102 | 2.0164 | 1.7349 | 1.3621 | 0.8994 | 0.27770 | 0.0621 | 0.1123 | 1.1870 |

The values of $(s'_y)_{x=0}$

| $a_{00}\diagdown y$ | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.502345 | 0.54250 | 0.64736 | 0.77412 | 0.86761 | 0.86408 | 0.74514 | 0.53319 | 0.34239 | 0.36891 | 1.0192 |
| 2 | 2.3601 | 2.4416 | 2.6622 | 2.9701 | 3.1921 | 3.2490 | 2.9935 | 2.4090 | 1.7129 | 1.6582 | 3.9565 |

Table V. IV

The values of $(s'_x)_{x=1}$

| $a_{00}\diagdown y$ | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|---|---|---|---|---|---|---|
| 1 | 1.0192 | 0.92222 | 0.65929 | 0.31482 | 0.02942 | 0 |
| 2 | 3.9566 | 3.5945 | 2.6100 | 1.3090 | 0.2012 | 0 |

The values of $(s'_{xy})_{x=1}$

| $a_{00}$ \ $y$ | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|---|---|---|---|---|---|---|
| 1 | 0 | 0.30901 | 0.45446 | 0.35498 | 0.09361 | 0 |
| 2 | 0 | 1.1603 | 1.7195 | 1.3728 | 0.4074 | 0 |

Table VI. IV

| $a_{00}$ | $(s'_x)_{x=0, y=0}$ | $(s'_x)_{x=1, y=0}$ |
|---|---|---|
| 0.4 | 0.07306 | 0.16531 |
| 0.8 | 0.31017 | 0.65587 |
| 1.0 | 0.50234 | 1.0192 |
| 1.2 | 0.75027 | 1.4590 |
| 1.6 | 1.4270 | 2.5622 |
| 2.0 | 2.3601 | 3.9566 |

Table VII. IV

The values of $(s''_x)_{x=0}$

| $a_{00}$ \ $y$ | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | −2.6774 | −2.6165 | −2.4366 | −2.1459 | −1.7581 | −1.2926 | −0.7742 | −0.2336 | 0.2928 | 0.7630 | 1.1286 |
| 2 | −4.6968 | −4.6951 | −4.4178 | −3.9658 | −3.3351 | −2.5462 | −1.6187 | −0.5798 | 0.5336 | 1.6722 | 2.7722 |

The values of $(s''_y)_{x=0}$

| $a_0$ \ $y$ | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | −2.6774 | −2.5961 | −2.3622 | −1.9746 | −1.4448 | −0.7852 | −0.01214 | 0.8546 | 1.7913 | 2.7708 | 3.7619 |
| 2 | −4.6968 | −4.6905 | −4.3750 | −3.8352 | −3.0446 | −1.9738 | −0.5828 | 1.1760 | 3.3576 | 6.0234 | 9.2407 |

Table VIII. IV

The values of $(s''_x)_{x=\pm 1}$

| $a_{00}$ \ $y$ | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2.9600 | 3.6792 | 3.4274 | 3.0555 | 2.5639 | 2.0033 | 1.4226 | 0.8763 | 0.4210 | 0.1124 | 0 |
| 2 | 10.920 | 9.7033 | 8.8569 | 8.3409 | 7.5635 | 6.4974 | 5.1476 | 0.35812 | 1.9643 | 0.6038 | 0 |

Tacle IX. IV

Bending stresses at center and at middle of side

| $a_{00}$ | $(s''_x)_{x=0, y=0}$ | $(s''_x)_{x=\pm 1, y=0}$ |
|---|---|---|
| 0.4 | −1.1321 | 1.3168 |
| 0.8 | −2.1888 | 2.8652 |
| 1.0 | −2.6774 | 3.7619 |
| 1.2 | −3.1396 | 4.7398 |
| 1.6 | −3.6996 | 6.8935 |
| 2.0 | −4.7968 | 9.2407 |

Table X. IV

The total stresses $(s_x)_{\substack{x=1\\y=0}}$ at middle of edge.

| $a_{00}$ | 0.4 | 0.8 | 1.0 | 1.2 | 1.6 | 2.0 |
|---|---|---|---|---|---|---|
| $(s_{\cdot x})_{\substack{x=1\\y=0}}$ | 1.4821 | 3.5211 | 4.7811 | 6.1988 | 9.4557 | 13.1973 |

Case 5. $\delta=0.2$

Table I. V

Values of $q_{00}$ and displacement constants for $\mu=0.3$ and various $a_{00}$ values

| $a_{00}$ | $a_{02}$ | $a_{22}$ | $q_{00}$ | $b_{00}$ | $b_{02}$ | $b_{20}$ | $b_{22}$ |
|---|---|---|---|---|---|---|---|
| 0.4 | −0.160216 | 0.173846 | 143.210 | 0.0460901 | −0.183556 | −0.130252 | 0.318272 |
| 0.8 | −0.285974 | 0.400639 | 358.886 | 0.192265 | −0.734447 | −0.637803 | 1.26653 |
| 1.0 | −0.330415 | 0.553055 | 516.620 | 0.308274 | −1.14698 | −0.820224 | 1.97177 |
| 1.2 | −0.362588 | 0.741758 | 716.062 | 0.455874 | −1.64958 | −1.20495 | 2.82754 |
| 1.6 | −0.395543 | 1.24773 | 1280.65 | 0.852554 | −2.91837 | −2.18143 | 4.97992 |
| 2.0 | −0.400287 | 1.93212 | 2117.63 | 1.39002 | −4.52560 | −3.44239 | 7.70603 |

Table II. V

The values of w/h for various values of $a_{00}$ along $x$-axis

| $a_{00} \diagdown x$ | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|---|---|---|---|---|---|---|
| 0.4 | 1 | 0.906834 | 0.660381 | 0.350538 | 0.096278 | 0 |
| 0.8 | 1 | 0.908422 | 0.665243 | 0.356889 | 0.099950 | 0 |
| 1.0 | 1 | 0.909697 | 0.668297 | 0.360878 | 0.102194 | 0 |
| 1.2 | 1 | 0.910461 | 0.670488 | 0.365045 | 0.104538 | 0 |
| 1.6 | 1 | 0.912487 | 0.677690 | 0.373147 | 0.109593 | 0 |
| 2.0 | 1 | 0.914222 | 0.683005 | 0.380088 | 0.113000 | 0 |

Table IV. V

The values of $(s'_x)_{x=0}$

| $a_{00} \diagdown y$ | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|---|---|---|---|---|---|---|
| 1 | 0.40076 | 0.42179 | 0.38758 | 0.16394 | −0.06373 | +0.30717 |
| 2 | 1.8070 | 1.8309 | 1.6123 | 0.76056 | −0.13721 | +1.2314 |

The values of $(s'_y)_{x=0}$

| $a_{00} \diagdown y$ | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|---|---|---|---|---|---|---|
| 1 | 0.40076 | 0.64188 | 0.96804 | 0.74959 | 0.25255 | +1.0239 |
| 2 | 1.8070 | 2.5825 | 3.6776 | 2.9958 | 1.1876 | +4.1047 |

Table V. V

The Values of $(s'_x)_{x=1}$

| $a_{00} \diagdown y$ | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|---|---|---|---|---|---|---|
| 1 | 1.0239 | 0.91987 | 0.63837 | 0.27523 | −0.01146 | 0 |
| 2 | 4.1047 | 3.6974 | 2.5931 | 1.1615 | 0.0122 | 0 |

The values of $(s'_{xy})_{x=1}$

| $a_{00}$ \ $y$ | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|---|---|---|---|---|---|---|
| 1 | 0 | 0.30438 | 0.43980 | 0.32575 | 0.05858 | 0 |
| 2 | 0 | 1.1386 | 1.6487 | 1.2293 | 0.2343 | 0 |

Table VI. V

| $a_{00}$ | $(s'_x)_{x=0, y=0}$ | $(s'_x)_{x=1, y=0}$ |
|---|---|---|
| 0.4 | 0.05992 | 0.16832 |
| 0.8 | 0.24994 | 0.89108 |
| 1.0 | 0.40076 | 1.0239 |
| 1.2 | 0.59264 | 1.4982 |
| 1.6 | 1.1083 | 2.6578 |
| 2.0 | 1.8070 | 4.1048 |

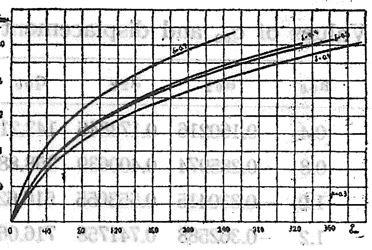

Fig. 6

Table VII. V

The values of $(s''_x)_{x=0}$

| $a_{00}$ \ $y$ | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | −3.0295 | −2.9415 | −2.6517 | −2.2815 | −1.7596 | −1.1848 | −0.5911 | −0.0427 | 0.4038 | 0.6982 | 0.8035 |
| 2 | −5.7204 | −5.5558 | −5.0759 | −4.3212 | −3.3560 | −2.2633 | −1.1382 | −0.0787 | 0.8248 | 1.5024 | 1.9196 |

The valves of $(s''_y)_{x=0}$

| $a_{00}$ \ $y$ | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | −3.0295 | −2.9129 | −2.5607 | −2.0255 | −1.3131 | 0.1831 | 0.3820 | 1.2270 | 1.9567 | 2.4749 | 2.6783 |
| 2 | −5.7204 | −5.5186 | −4.9241 | −3.9687 | −2.7044 | −1.2020 | 0.4510 | 2.1529 | 3.7915 | 5.2474 | 6.3988 |

Table VIII. V

The values of $(s''_x)_{x=\pm 1}$

| $a_{00}$ \ $y$ | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2.6783 | 2.6338 | 2.4919 | 2.2842 | 1.9902 | 1.6316 | 1.2282 | 0.8100 | 0.4209 | 0.1227 | 0 |
| 2 | 6.3988 | 6.3316 | 6.1231 | 5.6703 | 5.2068 | 4.4610 | 3.5245 | 2.4453 | 1.3375 | 0.4102 | 0 |

Table IX. V

Bending stresses at center and at middle of side

| $a_{00}$ | $(s''_x)_{x=0, y=0}$ | $(s''_x)_{x=\pm 1, y=0}$ |
|---|---|---|
| 0.4 | −1.2483 | 0.95914 |
| 0.8 | −2.4518 | 2.0561 |
| 1.0 | −3.0295 | 2.6783 |
| 1.2 | −3.5914 | 3.3497 |
| 1.6 | −4.6742 | 4.8178 |
| 2.0 | −5.7204 | 6.3988 |

Table X' V

The total stresses $(s_x)_{x=1, y=0}$ at middle of edge

| $a_{00}$ | 0.4 | 0.8 | 1.0 | 1.2 | 1.6 | 2.0 |
|---|---|---|---|---|---|---|
| $(s_x)_{x=1, y=0}$ | 1.1274 | 2.9472 | 3.7022 | 4.8479 | 7.4756 | 10.504 |

Case 6 $\delta \to 0$. Concentrated load.

If the value of $q_{00}$ is finite, the terms containing $q_{00}$ in equations (23) vanish and this case becomes a trivial case and we does not consider it. But we suppose that the value of $q_{00}$ becomes more and more large as $\delta$ decreases and the product $q_{0}, \delta^2$ has a limiting finite value when $\delta$ tends to zero. We write for brevity $Q = 4 \lim_{\delta \to 0} q_{00} \delta^2$.

This is the case in which the concentrated load Q acts its center of the square plate.

The constants $a_{02}$, $a_{22}$, Q, $b_{00}$, $b_{02}$, $b_{20}$, and $b_{22}$ for various assumed values of $a_{00}$ are gven in Table I. VI.

Table I. VI

Values of Q and displacement constants for $\mu = 0.3$ and various $a_{00}$ values

| $a_{00}$ | $a_{02}$ | $a_{22}$ | Q | $b_{00}$ | $b_{02}$ | $b_{20}$ | $b_{22}$ |
|---|---|---|---|---|---|---|---|
| 0.4 | −0.194836 | 0.295326 | 21.4598 | 0.0423101 | −0.178411 | −0.126933 | 0.317495 |
| 0.6 | −0.284121 | 0.471804 | 35.6336 | 0.0965846 | −0.400735 | −0.286684 | 0.712701 |
| 0.8 | −0.366339 | 0.686732 | 53.8767 | 0.174582 | −0.710217 | −0.511880 | 1.26299 |
| 1.0 | −0.437056 | 0.936225 | 77.4992 | 0.278797 | −1.10641 | −0.804511 | 1.96600 |
| 1.2 | −0.501473 | 1.24481 | 107.719 | 0.409576 | −1.58568 | −1.16467 | 2.81872 |
| 1.4 | −0.555493 | 1.60397 | 145.790 | 0.569727 | −2.14727 | −1.59460 | 3.81832 |
| 1.6 | −0.604451 | 2.02654 | 192.900 | 0.758800 | −2.78704 | −2.09373 | 4.96218 |
| 1.8 | −0.648174 | 2.50795 | 250.266 | 0.977917 | −3.50300 | −2.66289 | 6.24800 |
| 2.0 | −0.688229 | 3.04699 | 319.094 | 1.22724 | −4.29253 | −3.30210 | 7.67384 |

The values of dimensionless load Q corresponding to various values of $a_{00} = \frac{w_{max}}{h}$ are given in table I. VI and the curve for $\frac{w_{max}}{h}$ as a function of Q are plotted in Fig. 7.

Lorentz found that

$$a_{00} = 0.02055\, Q.$$

Ths straight line $a_{00} = 0.02055\, Q$ is also plotted in Fig. 7 for comparison.

We find from this figure that for small deflection there is a good agreement between Lorentz and the present solution, but the discrepancy arises for large deflection. We shall compare later this present solution and the experimental data.

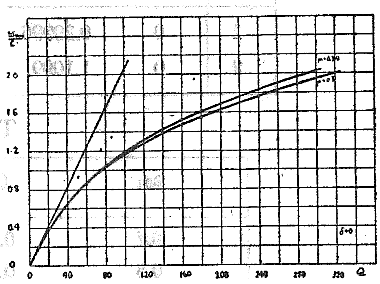


Fig. 7

The ratio of deflection to maximam deflection along $x$-axis for various values of

$a_{00}$ are given in Table II. VI.

The II. VI

Ratio of deflection to maximum deflection along $x$-axis

| $a_{00}$ \ $y$ | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|---|---|---|---|---|---|---|
| 0.4 | 1 | 0.903644 | 0.650610 | 0.337776 | 0.089199 | 0 |
| 0.8 | 1 | 0.908508 | 0.665505 | 0.356731 | 0.100142 | 0 |
| 1.0 | 1 | 0.905488 | 0.656258 | 0.335154 | 0.093349 | 0 |
| 1.2 | 1 | 0.906195 | 0.658421 | 0.347979 | 0.094938 | 0 |
| 1.6 | 1 | 0.907673 | 0.662950 | 0.353894 | 0.098864 | 0 |
| 2.0 | 1 | 0.908914 | 0.666751 | 0.358857 | 0.101058 | 0 |

Table IV. VI

The values of $(s'_x)_{x=0}$

| $a_{00}$ \ $y$ | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.36244 | 0.37368 | 0.39750 | 0.40926 | 0.38008 | 0.29235 | 0.15419 | 0.01113 | 0.02206 | 0.00153 | 0.31543 |
| 2 | 1.5954 | 1.6285 | 1.6955 | 1.7131 | 1.5831 | 1.2429 | 0.8058 | 0.1468 | −0.1950 | 0.0482 | 1.2549 |

The values of $(s'_y)_{x=0}$

| $a_{00}$ \ $y$ | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.36244 | 0.44159 | 0.64203 | 0.90545 | 1.0076 | 0.96951 | 0.74597 | 0.444501 | 0.25236 | 0.36093 | 1.0514 |
| 2 | 1.5954 | 1.8718 | 2.5780 | 3.3893 | 3.9001 | 3.7923 | 3.0306 | 1.8447 | 1.0099 | 1.4572 | 4.14972 |

Table V. VI

The values of $(s'_x)_{x=1}$

| $a_{00}$ \ $y$ | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|---|---|---|---|---|---|---|
| 1 | 1.05143 | 0.94366 | 0.65214 | 0.27682 | 0.01758 | 0 |
| 2 | 4.1497 | 3.7252 | 2.5769 | 1.0977 | −0.0642 | 0 |

The values of $(s'_{xy})_{x=1}$

| $a_{00}$ \ $y$ | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|---|---|---|---|---|---|---|
| 1 | 0 | 0.29996 | 0.43127 | 0.31448 | 0.04853 | 0 |
| 2 | 0 | 1.1099 | 1.5898 | 1.1454 | 0.1540 | 0 |

Table VI. VI

| $a_{00}$ | $(s'_x)_{x=0, y=0}$ | $(s'_x)_{x=1, y=0}$ |
|---|---|---|
| 0.4 | 0.05500 | 0.16925 |
| 0.8 | 0.22696 | 0.67460 |
| 1.0 | 0.36244 | 1.0514 |
| 1.2 | 0.53242 | 1.5102 |
| 1.6 | 0.98644 | 2.6699 |
| 2.0 | 1.5954 | 4.1497 |

## Table VII. VI

The values of $(s''_x)_{x=0}$

| $a_{00}$ \ $y$ | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | −3.1692 | −3.0684 | −2.7786 | −2.3266 | −1.7569 | −1.1270 | −0.5029 | 0.0464 | 0.4566 | 0.6759 | 0.6755 |
| 2 | −6.0947 | −5.8976 | −5.3259 | −4.4372 | −3.3155 | −2.0660 | −0.8879 | 0.6723 | 0.9740 | 1.4445 | 1 5741 |

The values of $(s''_y)_{x=0}$

| $a_{00}$ \ $y$ | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | −3.1618 | −3.0365 | −2.6515 | −2.0431 | −1.2603 | −0.3705 | 0.5420 | 1.3778 | 2.0246 | 2.3593 | 2.2518 |
| 2 | −6.0947 | −5.8521 | −5.4147 | −4.0144 | −2.5517 | −0.8644 | 0.8866 | 2.5632 | 3.9768 | 4.9362 | 5.2471 |

## Table VIII. VI

The vaules of $(s'')_{x=\pm1}$

| $a_{00}$ \ $y$ | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2.2518 | 2.2265 | 2.1488 | 2.0135 | 1.8142 | 1.5474 | 1.2168 | 0.8402 | 0.4574 | 0.1397 | 0.6755 |
| 2 | 5.2451 | 5.2332 | 5.1817 | 5.0466 | 4.7661 | 4.2772 | 3.5396 | 2.0863 | 1.4590 | 0.4648 | 1.5735 |

## Table IX. VI

Bending stresses at center and at middle of side

| $a_{00}$ | $(s''_x)_{x=0,\ y=0}$ | $(s''_x)_{x=\pm1,\ y=0}$ |
|---|---|---|
| 0.4 | −1.2933 | 0.82066 |
| 0.8 | −2.5563 | 1.7346 |
| 1.0 | −3.1682 | 2.2518 |
| 1.2 | −3.7719 | 2.7941 |
| 1.6 | −4.9458 | 3.9822 |
| 2 0 | −6.0947 | 5.2451 |

## Table X. VI

The total stresses $(s_x)_{x=1,\ y=0}$ at middle of edge

| $a_{00}$ | 0.4 | 0.8 | 1.0 | 1.2 | 1.6 | 2.0 |
|---|---|---|---|---|---|---|
| $(s_x)_{x=1,\ y=0}$ | 0.98991 | 2.4092 | 3.3032 | 4.3043 | 6.6521 | 9.3948 |

We have hitherto discussed the case of Poission's ratio $\mu=0.3$. Steel have this Poission's ratio. We will now treat the case of the Poission's ratio $\mu=0.34$, since alminium has this ratio.

We may consider that these two cases are most important in practice and can now treat the case $\mu=0.34$ by the same method as the case of $\mu=0.30$. The following results are obtained.

## Table XI.

Values of $q_{00}$, Q and displacement constants for $\mu=0.34$ and various $a_{00}$ values.

| δ | $a_{00}$ | $a_{02}$ | $a_{22}$ | $q_{00}$ or Q | $b_{00}$ | $b_{02}$ | $b_{20}$ | $b_{22}$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.5 | 0.16356 | 0.19400 | 27.739 | 0.12371 | −0.31444 | −0.22435 | 0.46322 |
| | 1.0 | 0.47326 | 0.67984 | 73.807 | 0.54487 | −1.2372 | −1.0161 | 1.8013 |
| | 1.5 | 0.94370 | 1.7270 | 157.806 | 0.95290 | −2.7702 | −2.0629 | 4.2591 |
| | 2.0 | 1.5093 | 3.3754 | 301.90 | 2.6166 | −4.8338 | −4.3784 | 6.6534 |
| 0.8 | 0.5 | 0.14094 | 0.087259 | 28.450 | 0.11961 | −0.31705 | −0.24140 | 0.46895 |
| | 1.0 | 0.42529 | 0.38135 | 75.658 | 0.52812 | −1.2734 | −1.0037 | 1.8313 |
| | 1.5 | 0.87398 | 1.0899 | 161.43 | 1.3164 | −2.8537 | −2.3525 | 3.9862 |
| | 2.0 | 1.4269 | 2.3065 | 308.18 | 2.5396 | −5.0023 | −4.3277 | 6.8395 |
| 0.6 | 0.5 | 0.049535 | −0.14128 | 33.995 | 0.10428 | −0.31893 | −0.22978 | 0.48494 |
| | 1.0 | 0.22228 | −0.20973 | 90.341 | 0.45716 | −1.2896 | −0.95040 | 1.9118 |
| | 1.5 | 0.50813 | −0.30554 | 189.86 | 1.1200 | −2.8680 | −2.2059 | 4.1892 |
| | 2.0 | 0.9488 | −0.32718 | 361.03 | 2.1758 | −5.1980 | −4.0650 | 7.3348 |
| 0.4 | 0.5 | −0.083953 | −0.061941 | 56.098 | 0.083762 | −0.30560 | −0.21309 | 0.49385 |
| | 1.0 | −0.077662 | −0.0938752 | 149.51 | 0.36258 | −1.2338 | −0.87457 | 1.9593 |
| | 1.5 | 0.034522 | 0.011284 | 315.87 | 0.88702 | −2.7934 | −2.0241 | 4.3559 |
| | 2.0 | 0.21805 | 0.32708 | 590.61 | 1.6887 | −4.9700 | −3.6848 | 7.6438 |
| 0.2 | 0.5 | −0.19555 | 0.31067 | 185.32 | 0.067836 | −0.15001 | −0.19914 | 0.49104 |
| | 1.0 | −0.32849 | 0.53644 | 495.84 | 0.28933 | −1.1510 | −0.81190 | 1.9606 |
| | 1.5 | −0.38783 | 1.0658 | 1049.9 | 0.69866 | −2.5827 | −1.8646 | 4.3660 |
| | 2.0 | −0.39399 | 1.8572 | 1965.6 | 1.3191 | −4.5599 | −3.3744 | 7.6733 |
| 0 | 0.5 | −0.24109 | 0.38199 | 27.508 | 0.058167 | −0.29110 | −0.19090 | 0.49817 |
| | 1.0 | −0.44105 | 0.94297 | 74.250 | 0.25781 | −1.1068 | −0.78390 | 1.9523 |
| | 1.5 | −0.59226 | 1.8338 | 157.54 | 0.61187 | −2.4584 | −1.7879 | 4.3432 |
| | 2.0 | −0.70732 | 3.0671 | 295.03 | 1.1399 | −4.2994 | −3.2172 | 7.6267 |

In Fig. 7. the curve for $\frac{W_{max}}{h}$ for a concentrated load as a function of Q are plotted. R. G. Sturm and R. L. Moore carried out the experiments of the deflection of rectangular plates under concentrated load. They used several Alminium plates, $55'' \times 55''$, with different thickness $1''$, $\frac{1}{2}''$, $\frac{1}{4}''$, and $\frac{1}{8}''$.

The measured values of $\frac{W_{max}}{h}$ have been also plotted. Form this figure, we find that our present solutions are have better agreement with the measured values than Lorentz's solution. But there are moderate differences measured and the calculated values, if the deflection becomes large. It seems to us that this discrepancy reduces by using more undetermined constants in the assumed expressions for the displacements.

Another method of reducing this discrepancy is to make very careful experiments. Particular care would have to be taken to fulfill the boundary conditions with the experimental plate. It is the boundary conditions in this case that the edges of the plate are all clamped. These conditions are u=v=w=0 at the edges and $\frac{\partial w}{\partial X}$ at $X=\pm a$, and $\frac{\partial w}{\partial Y}$ at $Y=\pm a$. And it is easy to fulfill these conditions for small deflection, but it seen to be very difficult to realize them as the deflection increases, and cause a moderate discrepancy between the present solution and the experimental data.

The ratios of deflection to maximum deflection along $x$-axis for various values of $a_{00}$ and $\delta$ are given in Table XII.

Table XII.

Ratio of deflection to maximum deflection along $x$-axis

| $\delta$ | $a_{00}$ \ $x$ | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 0.93905 | 0.75903 | 0.41939 | 0.16885 | 0 |
| | 2 | 1 | 0.94942 | 0.79080 | 0.52088 | 0.19219 | 0 |
| 0.8 | 1 | 1 | 0.93728 | 0.75361 | 0.41231 | 0.16488 | 0 |
| | 2 | 1 | 0.94790 | 0.78615 | 0.5480 | 0.18878 | 0 |
| 0.6 | 1 | 1 | 0.92979 | 0.73070 | 0.44238 | 0.14804 | 0 |
| | 2 | 1 | 0.93911 | 0.75922 | 0.47963 | 0.16899 | 0 |
| 0.4 | 1 | 1 | 0.91874 | 0.69683 | 0.39815 | 0.12316 | 0 |
| | 2 | 1 | 0.92562 | 0.71791 | 0.42568 | 0.13864 | 0 |
| 0.2 | 1 | 1 | 0.90949 | 0.66952 | 0.36116 | 0.10236 | 0 |
| | 2 | 1 | 0.91434 | 0.68336 | 0.38055 | 0.11326 | 0 |
| 0 | 1 | 1 | 0.90534 | 0.65581 | 0.34456 | 0.09302 | 0 |
| | 2 | 1 | 0.90856 | 0.66567 | 0.35745 | 0.10027 | 0 |

Table XIII.

The values of $(s'_x)_{x=0}$ for $a_{00}=1$

| $\delta$ \ $y$ | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|---|---|---|---|---|---|---|
| 1.0 | 0.73013 | 0.66276 | 0.61576 | 0.31609 | 0.21410 | 0.30342 |
| 0.8 | 0.70768 | 0.64075 | 0.50420 | 0.29767 | 0.19516 | 0.32341 |
| 0.6 | 0.61259 | 0.57156 | 0.45038 | 0.25099 | 0.13862 | 0.33541 |
| 0.4 | 0.48586 | 0.52855 | 0.41778 | 0.20805 | 0.07608 | 0.34815 |
| 0.2 | 0.38770 | 0.42060 | 0.40462 | 0.18487 | 0.03566 | 0.35533 |
| 0 | 0.34547 | 0.39524 | 0.40112 | 0.17489 | 0.00622 | 0.35775 |

The values of $(s'_y)_{x=0}$ for $a_{00}=1$

| $\delta$ \ $y$ | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|---|---|---|---|---|---|---|
| 1.0 | 0.73013 | 0.71225 | 0.68984 | 0.76407 | 0.61474 | 0.8924 |
| 0.8 | 0.70768 | 0.69295 | 0.77419 | 0.75947 | 0.58924 | 0.9512 |
| 0.6 | 0.61259 | 0.66828 | 0.77660 | 0.75003 | 0.49587 | 0.9865 |
| 0.4 | 0.48586 | 0.77238 | 0.86807 | 0.74774 | 0.37519 | 1.0240 |
| 0.2 | 0.38770 | 0.62954 | 0.95940 | 0.75195 | 0.29472 | 1.0451 |
| 0 | 0.34547 | 0.62925 | 1.0034 | 0.74853 | 0.26078 | 1.0522 |

## Table XIV.

The values af $(s'_x)_{x=1}$ for $a_{00}=1$

| $\delta \diagdown y$ | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|---|---|---|---|---|---|---|
| 1.0 | 0.9424 | 0.8614 | 0.6400 | 0.3432 | 0.07933 | 0 |
| 0.8 | 0.9512 | 0.8703 | 0.6490 | 0.3517 | 0.08535 | 0 |
| 0.6 | 0.9865 | 0.8992 | 0.6614 | 0.3446 | 0.06842 | 0 |
| 0.4 | 1.024 | 0.9272 | 0.6650 | 0.3210 | 0.03429 | 0 |
| 0.2 | 1.045 | 0.9412 | 0.6598 | 0.2702 | −0.01121 | 0 |
| 0 | 1.052 | 0.9452 | 0.6566 | 0.2838 | −0.01082 | 0 |

The values of $(s'_{xy})_{x=1}$

| $\delta \diagdown y$ | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|---|---|---|---|---|---|---|
| 1.0 | 0 | 0.2538 | 0.3808 | 0.3146 | 0.1093 | 0 |
| 0.8 | 0 | 0.2735 | 0.4122 | 0.3436 | 0.1242 | 0 |
| 0.6 | 0 | 0.3049 | 0.4560 | 0.3735 | 0.1250 | 0 |
| 0.4 | 0 | 0.3179 | 0.4688 | 0.3692 | 0.1020 | 0 |
| 0.2 | 0 | 0.3132 | 0.4556 | 0.3442 | 0.0740 | 0 |
| 0 | 0 | 0.3081 | 0.4450 | 0.3290 | 0.0583 | 0 |

The values of $(s'_x)_{\substack{x=0\\y=0}} = (s'_y)_{\substack{x=0\\y=0}}$

| $\delta \diagdown y$ | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|---|---|---|---|---|---|---|
| 0.5 | 0.1658 | 0.1603 | 0.1397 | 0.1122 | 0.0909 | 0.0780 |
| 1.0 | 0.7301 | 0.7077 | 0.6126 | 0.4859 | 0.3877 | 0.3455 |
| 1.5 | 1.277 | 1.794 | 1.501 | 1.187 | 0.9362 | 0.8199 |
| 2.0 | 3.506 | 3.403 | 2.916 | 2.263 | 1.768 | 1.528 |

The values of $(s'_x)_{\substack{x=1\\y=0}} = (s'_y)_{\substack{x=0\\y=1}}$

| $a_{00} \diagdown \delta$ | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|---|---|---|---|---|---|---|
| 0.5 | 0.2013 | 0.2436 | 0.2510 | 0.2587 | 0.2626 | 0.2655 |
| 1.0 | 0.8924 | 0.9512 | 0.9865 | 1.024 | 1.045 | 1.052 |
| 1.5 | 2.220 | 2.072 | 2.172 | 2.631 | 2.332 | 2.352 |
| 2.0 | 3.524 | 3.576 | 3.778 | 3.992 | 4.111 | 4.155 |

## Table XVI.

The values of $(s''_x)_{x=0}$ for $a_{00}=1$

| $\delta \diagdown y$ | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|---|---|---|---|---|---|---|
| 1.0 | −2.046 | −1.928 | −1.547 | −0.8317 | 0.1842 | 2.004 |
| 0.8 | −2.110 | −1.988 | −1.595 | −0.8586 | 0.3052 | 1.938 |
| 0.6 | −2.381 | −2.220 | −1.722 | −0.8742 | 0.3004 | 1.662 |
| 0.4 | −2.783 | −2.525 | −1.799 | −0.7512 | 0.3703 | 1.254 |
| 0.2 | −3.020 | −2.754 | −1.802 | −0.5697 | 0.4827 | 0.9133 |
| 0 | −3.271 | −2.862 | −1.790 | −0.4700 | 0.5410 | 0.7602 |

The values of $(s''_y)_{x=0}$ for $a_{00}=1$

| $\delta\diagdown y$ | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|---|---|---|---|---|---|---|
| 1.0 | −2.046 | −1.974 | −1.670 | −0.7173 | 1.114 | 5.893 |
| 0.8 | −2.110 | −2.028 | −1.664 | −0.6676 | 1.530 | 5.701 |
| 0.6 | −2.381 | −2.204 | −1.611 | −0.4263 | 1.624 | 4.889 |
| 0.4 | −2.783 | −2.452 | −1.495 | −0.194 | 1.792 | 3.689 |
| 0.2 | −3.024 | −2.653 | −1.377 | 0.3475 | 1.947 | 2.686 |
| 0 | −3.271 | −2.742 | −1.319 | 0.5184 | 2.020 | 2.356 |

Table XVIII

The values of $(s''_x)_{x=\pm1}$ for $a_{00}=1$

| $\delta\diagdown y$ | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|---|---|---|---|---|---|---|
| 1.0 | 5.893 | 5.601 | 4.679 | 3.094 | 1.146 | 0 |
| 0.8 | 5.701 | 5.372 | 4.387 | 2.811 | 1.006 | 0 |
| 0.6 | 4.889 | 4.507 | 3.455 | 2.010 | 0.6378 | 0 |
| 0.4 | 3.689 | 3.374 | 2.526 | 1.410 | 0.4212 | 0 |
| 0.2 | 2.686 | 2.506 | 1.989 | 1.223 | 0.4171 | 0 |
| 0 | 2.236 | 2.134 | 1.804 | 1.212 | 0.4563 | 0 |

Table XVII.

The values of $(s''_x)_{\substack{x=0\\y=0}}=(s''_y)_{\substack{x=0\\y=0}}$

| $a_{00}\diagdown\delta$ | 1.0 | 0.8 | 0.6 | 0.4 | 0.2 | 0 |
|---|---|---|---|---|---|---|
| 0.5 | −1.121 | −1.151 | −1.274 | −1.452 | −1.602 | −1.663 |
| 1.0 | −2.046 | −2.110 | −2.381 | −2.783 | −3.020 | −3.271 |
| 1.5 | −2.755 | −2.849 | −3.339 | −3.974 | −4.540 | −4.814 |
| 2.0 | −3.338 | −3.448 | −4.087 | −5.068 | −5.888 | −6.308 |

The values of $(s''_x)_{\substack{x=\pm1\\y=0}}=(s''_y)_{\substack{x=0\\y=\pm1}}$

| $a_{00}\diagdown\delta$ | 1.0 | 0.8 | 0.6 | 0.4 | 0.2 | 0 |
|---|---|---|---|---|---|---|
| 0.5 | 2.654 | 2.564 | 2.198 | 1.664 | 1.218 | 1.036 |
| 1.0 | 5.893 | 5.701 | 4.889 | 3.689 | 2.686 | 2.236 |
| 1.5 | 9.772 | 9.496 | 8.032 | 6.138 | 4.449 | 3.631 |
| 2.0 | 14.04 | 13.71 | 11.80 | 8.872 | 6.424 | 5.171 |

## II. The case $q=q_{00}+q_{02}(x^2+y^2)$ and $\delta=1.0$

We now consider the case the load q does not constant and is expressed as $q=q_{00}+q_{02}(x^2+y^2)$. It is useful in practice to consider the case. We put Poission's ratio equal to 0.3. Then by the same method as the previous problem, we obtain the following results. We shall treat four cases, $q_{02}=-\frac{1}{4}q_{00}$, $-\frac{1}{2}q_{00}$, $-\frac{3}{4}q_{00}$, and $-q_{00}$.

## Table XIX

The values of $a_{02}$, $a_{22}$, $q_{00}$, $b_{00}$, $b_{02}$, $b_{20}$, and $b_{22}$ for various $a_{00}$, and $q_{02}$.

Case I. $q_{02}=-\frac{1}{4}q_{00}$

| $a_{00}$ | $a_{02}$ | $a_{22}$ | $q_{00}$ | $b_{00}$ | $b_{02}$ | $b_{20}$ | $b_{22}$ |
|---|---|---|---|---|---|---|---|
| 0.5 | 0.15277 | 0.11114 | 33.242 | 0.13385 | −0.31546 | −0.25028 | 0.45666 |
| 1.0 | 0.47675 | 0.43044 | 82.093 | 0.56202 | −1.4806 | −1.0303 | 1.8165 |
| 1.5 | 1.0045 | 1.0593 | 181.94 | 1.4201 | −2.8430 | −2.4275 | 3.9359 |
| 2.0 | 1.6731 | 2.4505 | 351.55 | 2.7781 | −4.9579 | −4.5114 | 6.6773 |

Case II. $q_{02}=-\frac{1}{2}q_{00}$

| $a_{00}$ | $a_{02}$ | $a_{22}$ | $q_{00}$ | $b_{00}$ | $b_{02}$ | $b_{20}$ | $b_{22}$ |
|---|---|---|---|---|---|---|---|
| 0.5 | 0.09104 | 0.13552 | 20.781 | 0.11585 | −0.30859 | −0.23876 | 0.49150 |
| 1.0 | 0.17330 | 0.29175 | 84.867 | 0.46065 | −1.2312 | −0.95283 | 1.8946 |
| 1.5 | 0.203777 | 0.95293 | 181.36 | 1.0127 | −2.6960 | −2.1225 | 4.2122 |
| 2.0 | 0.18834 | 2.1141 | 334.76 | 1.7506 | −4.6393 | −3.7313 | 7.4610 |

Case Ⅲ. $q_{02}=-\frac{3}{4}q_{00}$

| $a_{00}$ | $a_{02}$ | $a_{22}$ | $q_{00}$ | $b_{00}$ | $b_{02}$ | $b_{20}$ | $b_{22}$ |
|---|---|---|---|---|---|---|---|
| 0.5 | 0.1141 | −0.10975 | 34.233 | 0.11910 | −0.32022 | −0.24176 | 0.49048 |
| 1.0 | 0.38930 | −0.14944 | 103.20 | 0.52945 | −1.2974 | −0.10076 | 1.8734 |
| 1.5 | 0.85294 | 0.12045 | 189.73 | 1.3315 | −2.9338 | −2.3718 | 4.0843 |
| 2.0 | 1.4032 | 1.0699 | 401.34 | 2.5622 | −5.1326 | −4.3594 | 7.0061 |

Case Ⅳ. $q_{02}=-q_{00}$

| $a_{00}$ | $a_{02}$ | $a_{22}$ | $q_{00}$ | $b_{00}$ | $b_{02}$ | $b_{20}$ | $b_{22}$ |
|---|---|---|---|---|---|---|---|
| 0.5 | 0.074088 | −0.18533 | 37.929 | 0.11251 | −0.32045 | −0.23678 | 0.48709 |
| 1.0 | 0.33086 | −0.38883 | 104.00 | 0.50896 | −1.3087 | −0.99284 | 1.9016 |
| 1.5 | 0.69985 | −0.36278 | 225.02 | 1.2488 | −2.9620 | −2.3118 | 4.1876 |
| 2.0 | 1.1790 | −0.10408 | 431.40 | 2.3897 | −5.2701 | −4.2380 | 7.2727 |

## Table XX

The ratio of deflection to maximum deflection along $x$-axis.

Case I. $q_{02}=-\frac{1}{4}q_{00}$

| $a_{00}$ \ $x$ | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|---|---|---|---|---|---|---|
| 0.5 | 1.000 | 0.9328 | 0.7401 | 0.4546 | 0.1529 | 0 |
| 1.0 | 1.000 | 0.9392 | 0.7594 | 0.4799 | 0.1691 | 0 |
| 1.5 | 1.000 | 0.9463 | 0.7812 | 0.5083 | 0.1851 | 0 |
| 2.0 | 1.000 | 0.9525 | 0.8021 | 0.5230 | 0.1990 | 0 |

Case II. $q_{02}=-\frac{1}{2}q_{00}$

| $a_{00}\diagdown x$ | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|---|---|---|---|---|---|---|
| 0.5 | 1.000 | 0.9283 | 0.7261 | 0.4366 | 0.1447 | 0 |
| 1.0 | 1.000 | 0.9280 | 0.7252 | 0.4352 | 0.1440 | 0 |
| 1.5 | 1.000 | 0.9266 | 0.7209 | 0.4296 | 0.1409 | 0 |
| 2.0 | 1.000 | 0.9251 | 0.7163 | 0.4235 | 0.1374 | 0 |

Case III. $q_{02}=-\frac{3}{4}q_{00}$

| $a_{00}\diagdown x$ | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|---|---|---|---|---|---|---|
| 0.5 | 1.000 | 0.9300 | 0.7313 | 0.4433 | 0.1486 | 0 |
| 1.0 | 1.000 | 0.9359 | 0.7506 | 0.4670 | 0.1619 | 0 |
| 1.5 | 1.000 | 0.9425 | 0.7656 | 0.4934 | 0.1768 | 0 |
| 2.0 | 1.000 | 0.9475 | 0.7848 | 0.5131 | 0.1878 | 0 |

Case IV. $q_{02}=-q_{00}$

| $a_{00}\diagdown x$ | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|---|---|---|---|---|---|---|
| 0.5 | 1.000 | 0.9266 | 0.7223 | 0.4315 | 0.1419 | 0 |
| 1.0 | 1.000 | 0.9338 | 0.7430 | 0.4584 | 0.1570 | 0 |
| 1.5 | 1.000 | 0.9388 | 0.7583 | 0.4784 | 0.1683 | 0 |
| 2.0 | 1.000 | 0.9434 | 0.7721 | 0.4965 | 0.1785 | 0 |

Table XXI.

The values of $(s'_x)_{x=0}$ for $a_{00}=1$

| $q_{02}\diagdown y$ | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|---|---|---|---|---|---|---|
| $-\frac{1}{4}q_{00}$ | 0.7306 | 0.6517 | 0.4599 | 0.2394 | 0.0416 | 0.2810 |
| $-\frac{1}{2}q_{00}$ | 0.5989 | 0.5601 | 0.4400 | 0.2357 | 0.0242 | 0.2963 |
| $-\frac{3}{4}q_{00}$ | 0.6883 | 0.6151 | 0.4654 | 0.2616 | 0.0607 | 0.2869 |
| $-q_{00}$ | 0.6617 | 0.6040 | 0.4525 | 0.2467 | 0.0436 | 0.2903 |

$(s'_y)_{x=0}$ for $a_{00}=1$

| $q_{02}\diagdown y$ | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|---|---|---|---|---|---|---|
| $-\frac{1}{4}q_{00}$ | 0.7306 | 0.7081 | 0.7047 | 0.7416 | 0.5597 | 0.9366 |
| $-\frac{1}{2}q_{00}$ | 0.5989 | 0.6689 | 0.7947 | 0.7517 | 0.4380 | 0.9874 |
| $-\frac{3}{4}q_{00}$ | 0.6883 | 0.6931 | 0.7314 | 0.7511 | 0.5310 | 0.9564 |
| $-q_{00}$ | 0.6617 | 0.6835 | 0.8300 | 0.8141 | 0.5042 | 0.9678 |

## Table XXII

The values of $(s'_x)_{x=1}$ for $a_{00}=1$

| $q_{00}\diagdown y$ | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|---|---|---|---|---|---|---|
| $-\frac{1}{4}q_{00}$ | 0.9366 | 0.8733 | 0.6964 | 0.4446 | 0.1824 | 0 |
| $-\frac{1}{2}q_{00}$ | 0.9844 | 0.8940 | 0.6485 | 0.3243 | 0.0487 | 0 |
| $-\frac{3}{4}q_{00}$ | 0.9562 | 0.8737 | 0.6484 | 0.3466 | 0.0788 | 0 |
| $-q_{00}$ | 0.9678 | 0.8835 | 0.6536 | 0.3462 | 0.0752 | 0 |

The values of $(s'_{xy})_{x=1}$ for $a_{00}=1$

| $q_{02}\diagdown y$ | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|---|---|---|---|---|---|---|
| $-\frac{1}{4}q_{00}$ | 0 | 0.2931 | 0.4397 | 0.3629 | 0.1256 | 0 |
| $-\frac{1}{2}q_{00}$ | 0 | 0.2683 | 0.4229 | 0.3504 | 0.1231 | 0 |
| $-\frac{3}{4}q_{00}$ | 0 | 0.2814 | 0.4166 | 0.3314 | 0.0967 | 0 |
| $-q_{00}$ | 0 | 0.3407 | 0.5328 | 0.4881 | 0.2393 | 0 |

## Table XX. [illegible]

The values of $(s'_x)_{\substack{x=0\\y=0}}$

| $q_{02}\diagdown a_{00}$ | 0.5 | 1.0 | 1.5 | 2.0 |
|---|---|---|---|---|
| $-\frac{1}{4}q_{00}$ | 0.1740 | 0.7306 | 1.846 | 3.612 |
| $-\frac{1}{2}q_{00}$ | 0.1506 | 0.5989 | 1.317 | 2.276 |
| $-\frac{3}{4}q_{00}$ | 0.1548 | 0.6883 | 1.764 | 3.331 |
| $q_{00}$ | 0.1463 | 0.6617 | 1.623 | 3.107 |

The values of $(s')_{\substack{x=1\\y=0}}$

| $q_{02}\diagdown a_{00}$ | 0.5 | 1.0 | 1.5 | 2.0 |
|---|---|---|---|---|
| $-\frac{1}{4}q_{00}$ | 0.2329 | 0.9366 | 2.015 | 3.467 |
| $-\frac{1}{2}q_{00}$ | 0.2458 | 0.9844 | 2.220 | 3.962 |
| $-\frac{3}{4}q_{00}$ | 0.2454 | 0.9562 | 2.080 | 3.594 |
| $-q_{00}$ | 0.2486 | 0.9676 | 2.1260 | 3.696 |

## Table XXIV.

The values of $(s''_x)_{x=0}$ for $a_{00}=1$

| $q_{00} \diagdown y$ | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|---|---|---|---|---|---|---|
| $-\frac{1}{4}q_{00}$ | −1.980 | −1.873 | −1.522 | −1.5440 | 0.2345 | 1.7721 |
| $-\frac{1}{2}q_{00}$ | −2.375 | −2.1852 | −1.635 | −0.7798 | 0.2828 | 1.8773 |
| $-\frac{3}{4}q_{00}$ | −2.094 | −1.983 | −1.616 | −0.9092 | 0.2037 | 1.6671 |
| $-q_{00}$ | −2.170 | −2.051 | −1.662 | −0.9272 | 0.1953 | 1.5970 |

The values of $(s''_y)_{x=0}$ for $a_{00}=1$

| $q_{00} \diagdown y$ | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|---|---|---|---|---|---|---|
| $-\frac{1}{4}q_{00}$ | −1.980 | −1.928 | −1.657 | −0.915 | 1.512 | 5.907 |
| $-\frac{1}{2}q_{00}$ | −2.375 | −2.172 | −1.521 | −0.308 | 1.875 | 4.693 |
| $-\frac{3}{4}q_{00}$ | −2.094 | −2.004 | −1.621 | −0.602 | 1.456 | 5.557 |
| $-p_{00}$ | −2.170 | −2.053 | −1.606 | −0.547 | 1.572 | 5.323 |

## Table XXV.

The values of $(s''_x)_{x=\pm 1}$ for $a_{00}=1$

| $q_{00} \diagdown y$ | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|---|---|---|---|---|---|---|
| $-\frac{1}{4}q_{00}$ | 5.907 | 5.507 | 4.383 | 1.918 | 0.923 | 0 |
| $-\frac{1}{2}q_{00}$ | 4.693 | 4.394 | 3.522 | 1.660 | 0.762 | 0 |
| $-\frac{3}{4}q_{00}$ | 5.557 | 5.157 | 3.997 | 1.827 | 0.800 | 0 |
| $-q_{00}$ | 5.323 | 4.898 | 3.730 | 1.622 | 0.670 | 0 |

The values of $(s''_y)_{x=\pm 1}$ for $a_{00}=1$

| $q_{00} \diagdown y$ | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|---|---|---|---|---|---|---|
| $-\frac{1}{4}q_{00}$ | 1.772 | 1.652 | 1.315 | 0.5753 | 0.2768 | 0 |
| $-\frac{1}{2}q_{00}$ | 1.408 | 1.318 | 1.057 | 0.4982 | 0.2288 | 0 |
| $-\frac{3}{4}q_{00}$ | 1.667 | 1.547 | 1.199 | 0.5482 | 0.2399 | 0 |
| $-q_{00}$ | 1.597 | 1.469 | 1.199 | 0.4867 | 0.2011 | 0 |

## Table XXVI.

The values of $(s''_x)_{\substack{x=0\\y=0}} = (s''_y)_{\substack{x=0\\y=0}}$

| $q_{00} \diagdown a_{00}$ | 0.5 | 1.0 | 1.5 | 2.0 |
|---|---|---|---|---|
| $-\frac{1}{4}q_{00}$ | −1.101 | −1.980 | −2.594 | −3.025 |
| $-\frac{1}{2}q_{00}$ | −1.182 | −2.378 | −3.635 | −4.955 |
| $-\frac{3}{4}q_{00}$ | −1.151 | −2.094 | −2.791 | −3.372 |
| $-q_{00}$ | −1.204 | −2.170 | −2.990 | −3.667 |

The values of $(s''_x)_{\substack{x=\pm1\\y=0}}$ $(s''_y)_{\substack{x=0\\y=\pm1}}$

| $q_{02} \diagdown a_{02}$ | 0.5 | 1.0 | 1.5 | 2.0 |
|---|---|---|---|---|
| $-\frac{1}{4}q_{00}$ | 2.611 | 5.907 | 10.02 | 14.69 |
| $-\frac{1}{2}q_{00}$ | 2.284 | 4.693 | 6.815 | 8.753 |
| $-\frac{3}{4}q_{00}$ | 2.458 | 5.557 | 9.412 | 13.61 |
| $q_{00}$ | 2.296 | 5.323 | 8.800 | 12.72 |

Table XXVIII.

The values of $(s_x)_{\substack{x=1\\y=0}}$

| $q_{02} \diagdown a_{00}$ | 0.5 | 1.0 | 1.5 | 2.0 |
|---|---|---|---|---|
| $-\frac{1}{4}q_{00}$ | 2.844 | 6.844 | 12.03 | 18.16 |
| $-\frac{1}{2}q_{00}$ | 2.530 | 5.677 | 9.035 | 12.71 |
| $-\frac{3}{4}q_{00}$ | 2.703 | 6.513 | 11.49 | 17.20 |
| $-q_{00}$ | 2.545 | 6.291 | 10.93 | 16.42 |

## Summary

The problems treated here are those the square plate with four clamped edges is under a lateral load. The load has firstly constant value on a part of the plate and secondly symmetrical to its center. The deflection of the plate is assumed of the same order as its thickness. The relation between its maximum deflection i. e., the deflection of its center and the load are obtained. The membrane and bending stresses of the plate are also calculated. These results are compared with the experiment in the case of the concentrated load. In conclusion, I wish to thank Prof. Tomotika for his encouragement during this work. This research has been maintained by the Grant in Aid for Fundamental Scientific Research of the Department of Education, and, certain parts of this paper was read at the meeting of the phyisical society of Japan.

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