On the Vibration under Shearing Forces of an Elastic Plate.

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I. Introduction

Let us now consider the problem of an elastic thin square plate under shearing forces. We assume the plate is clamped at edges. It is very interest and important in theoretically and practically to solve these problems. But owing the boundary conditions that edges are clamped, it is almost impossible to obtain exact solutions and the solutions hitherto obtained are in most approximations.

The problem of the stability of the clamped square plate under shearing forces at the four edges discussed by Prof. S. Tomotika, Prof. K. Hidaka and Prof. S. Iguchi. S. Tomotika and I. Imai used the Rayleigh principle to find the solution, K. Hidaka collatz's method and S. Iguchi his own method. The transverse vibration of the square plate with four clamped edges are discussed also S. Tomitika and K. Sezawa. The former used at first Taylor's method and then the variations method and found the more exact solutions.

We will now treat the problems of transverse vibration of a square plate under shearing forces at the four clamped edges. We will now apply Reyleigh's principle to the problem. K. Munekata was also discussed this problem by the method which Lamb used in the problem of hydrodynamics. The allied problem has been discussed by R. V. Southwell and S. W. Skan . They treated the case of a transverse vibration of a flat elastic strip under shearing forces at edges. The boundary conditions they used are a simple support and clamping.

II. Fundamental equations

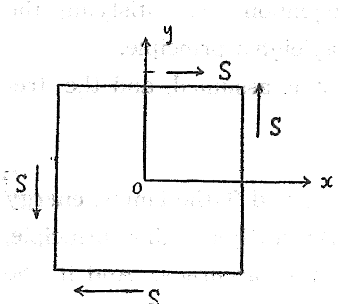

Fig. 1. (a)

We shall employ the usual approximate theory of thin plates, rotatory inertia being neglected. Let us take axes Ox, Oy in the middle plane of the undisturved plate as in Fig. 1. (a), O being its center and and axis Oz perpendicular to the plate. The edges $x=\pm a$, $y=\pm a$ are assumed to be subjected to shearing forces, of uniform intersity S per unit length of edges, acting in the directions shown. The stress-resultants N_1, N_2 and S and stress-couples G_1, G_2, H_1 acting on a distorted element of the plate are shown in Fig. 1 (b) and the equations governing the transverse displacement of the middle surface are

$$\left.\begin{aligned} &\frac{\partial N_1}{\partial x}+\frac{\partial N_2}{\partial y}+2S\frac{\partial^2 w}{\partial x\partial y}=\rho h\frac{\partial^2 w}{\partial t^2} \\ &\frac{\partial G_1}{\partial x}-\frac{\partial H_1}{\partial y}-N_1=0; \\ &\frac{\partial H_1}{\partial x}-\frac{\partial G_2}{\partial y}+N_2=0; \end{aligned}\right\} \qquad (1)$$

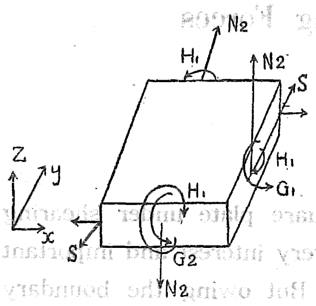

Fig. 1 (b)

in which

2a = length of side of the plate,

h = thickness of the plate,

w = displacement of points in the middle plane in direction of z,

t = time,

ρ = density of the plate,

$D=\frac{Eh^3}{1-\sigma^2}$ the flexural rigidity,

σ = Poisson's ratio.

E = Young's modulus.

$$\left.\begin{aligned} G_1&=-D\left(\frac{\partial^2 w}{\partial x^2}+\sigma\frac{\partial^2 w}{\partial y^2}\right)\\ G_2&=-D\left(\frac{\partial^2 w}{\partial y^2}+\sigma\frac{\partial^2 w}{\partial x^2}\right)\\ H_1&=D(1-\sigma)\frac{\partial^2 w}{\partial x\partial y}\end{aligned}\right\}\quad(2)$$

Eliminating N_1, and N_2 from (1) and substituting for G_1, G_2, and H_1 from(2), we obtain as the differential equation to be satisfied by w,

$$D\Delta\Delta w-2S\frac{\partial^2 w}{\partial x\partial y}+\rho h\frac{\partial^2 w}{\partial t^2}=0 \quad (3)$$

in which $\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$.

The boundary conditions are

$$\left.\begin{aligned} &w=0 \text{ and } \frac{\partial w}{\partial x}=0 \text{ when } x=\pm a,\\ &w=0 \text{ and } \frac{\partial w}{\partial y}=0 \text{ when } y=\pm a.\end{aligned}\right\}\quad(4)$$

Since it is very difficult to find the solution of the equation (3) satisfying the boundary conditions, as an alternative, we will apply Rayleigh's principle.

According this principle, the mode of the displacement is assumed, and the frequency is determined from the energy condition

$$V + T = \text{const.}$$

in which V denotes the total increase in potential energy, and T the kinetic energy of the motion, in a vibration of the assumed type. By applying this principle, much labour can usually be saved, without serious loss of accuracy. And if the appropriate boundary conditions are satisfied, and if the mode is suitably chosen in other respects, this procedure will result in a close estimate of the gravest frequency natural to the system, and the estimate will be too high for the true value. Its accuracy may therefore be improved by including one or more arbitary parameters in the assumed mode, and subsequently adjusting these as as to make the resulting estimate of frequecy a minimum, just as in Ritz's method. The value obtained by this principle is a upper limit of the value.

The expressions for V and T are:

$$V=\frac{1}{2}D\int\int\Big[(\Delta w)^2+2(1-\delta)\Big\{\Big(\frac{\partial^2 w}{\partial x\partial y}\Big)^2-\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\Big\}+\frac{2S}{D}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\Big]dx\,dy, \quad (5)$$

$$T=\rho\frac{h}{2}\int\int\Big(\frac{\partial w}{\partial t}\Big)^2 dx\,dy\ , \quad (6)$$

the integration extending over the whole area of the plate.

It will be noticed that the terms in the integral of V which involve $(1-\sigma)$ are transformed to

$$2(1-\sigma)\Big\{\frac{\partial}{\partial x}\Big(\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x\partial y}\Big)-\frac{\partial}{\partial y}\Big(\frac{\partial^2 w}{\partial x^2}\frac{\partial w}{\partial y}\Big)\Big\},$$

and so vanish in integration and V takes the expression:

$$V=\frac{1}{2}D\int_{-a}^{a}\int_{-a}^{a}\Big[(\Delta w)^2+\frac{2S}{D}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\Big]dxdy \quad (7)$$

For brevity, we put

$$x=a\xi,\quad y=a\eta. \quad (8)$$

Then V, and T are transformed as follows:

$$V=\frac{1}{2}\frac{D}{a^2}\int_{-1}^{1}\int_{-1}^{1}\Big[\Big(\frac{\partial^2 w}{\partial \xi^2}+\frac{\partial^2 w}{\partial \eta^2}\Big)^2+\frac{2Sa^2}{D}\frac{\partial w}{\partial \xi}\frac{\partial w}{\partial \eta}\Big]d\xi d\eta \quad (9)$$

$$T=\frac{\rho h a^2}{2}\int_{-1}^{1}\int_{-1}^{1}\Big(\frac{\partial w}{\partial t}\Big)^2 d\xi d\eta. \quad (10)$$

When the plate vibrates in a normal mode, we assume for the expression of displacement w satisfying the boundary conditions (4) that

$$w=w_0(1-\xi)^2(1-\eta^2)^2\{1+C_1(\xi^2+\eta^2)+2C_2\xi\eta\}\cos(pt+\varepsilon) \quad (11)$$

in which w_0 being proportional to the displacement of center of the plate, and C_1 and C_2 arbitray parameters. The expression (11) is introduced into V and T (9), (10), and then by Rayleigh's principle (5), we obtain the following relation:

$$\Big(\frac{1}{1225}+\frac{4}{13475}C_1+\frac{16}{47775}C_1^2+\frac{268}{1091475}C_2^2\Big)+A\Big(\frac{2}{1091475}C_1C_2-\frac{1}{99225}C_2\Big)$$
$$=K\Big(\frac{1}{99225}+\frac{1}{1091475}C_1+\frac{92}{156080925}C_1^2+\frac{4}{12006225}C_2^2\Big), \quad (12)$$

in which

$$\left.\begin{aligned}A&=\frac{2Sa^2}{D},\\ \text{and}\quad K&=\frac{\rho h a^4 p^2}{D}.\end{aligned}\right\} \quad (13)$$

To make the shearing force S minimum for a given frequency, we must determime C_1 and C_2 as to satisfying the following equations.

$$\frac{\partial A}{\partial C_1}=0,\qquad \frac{\partial A}{\partial C_2}=0. \quad (14)$$

After differenciating equation (12) and substituting (14) for $\frac{\partial A}{\partial C_1}$ and $\frac{\partial A}{\partial C_2}$ from (14). we obtain the following equations determining the parameters C_1 and C_2:

$$\frac{4}{13475}+\frac{32}{47775}C_1+\frac{2A}{1091475}C_2=K\Big(\frac{1}{1091475}+\frac{184}{156080925}C_1\Big), \quad (15)$$

$$\frac{536}{1091475}C_2+A\left(\frac{2}{1091475}C_1-\frac{1}{99225}\right)=\frac{8K}{12006225}C_2. \quad (16)$$

From these equations we obtain the values of C_1 and C_2:

$$C_1=\frac{286(K-81)(5896-8K)-17303A^2}{(52272-92K)(5896-8K)-3146A^2}, \quad (17)$$

and

$$C_2=\frac{121(26136-46K)-3146(K-81)}{(26136-46K)(5896-8K)-1573A^2}A. \quad (18)$$

Substituting (17) and (18) for C_1 and C_2 in equation (12), we obtain the relation between A and K, that is, the shearing force and the frequency of vibration. We then find values of shearing force for various values of the frequencies of vibration and these values C_1 and C_2 are given in Table 1 and plotted in Fig. 2 and Fig. 3.

Table I.

K	0	10	20	30	40	50	60	70	81
A	76.4516	72.323	67.650	62.512	56.620	49.799	41.494	30.451	0
c_1	−0.82027	−0.739189	−0.652130	−0.560659	−0.463289	−0.360493	−0.251500	−0.135868	0
c_2	1.80296	1.70688	1.59627	1.47366	1.33216	1.16823	0.969420	0.707571	0

From Table 1, it is found the minimum value of A for which the stability can become neutral is 76.4516. This value was obtained by S. Tomotika and I. Imai. K. Hidaka found for $2a=1$ the value $\frac{2S}{D\pi^2}=29.7941$ and Tomotika's value was $\frac{2S}{D\pi^2}=30.9847$. These two values are in good agreement.

In the case $A=0$ no shearing forces exert and S. Tomotika found for this case

$$\frac{16\rho ha^4p^2}{D\pi^4}=13.2948$$

In the present case we find,

$$16\frac{\rho ha^4p^2}{D\pi^4}=\frac{16K}{\pi^4}=13.3047$$

By comparing these two values, there is a good agreement. From above two cases we may conclude that the values in the Table 1 express good approximate values for their true values. On the other hand, Munekata obtained the following results which are shown in Table 2.

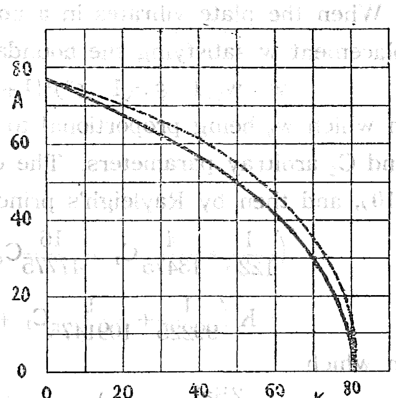

Fig. 2 ···Munekata's curve —Auther's curve

Table 2 (Courtesy of Mr. Munekata, Kyoto Univ.)

K	0	25.94	45.42	63.10	74.72	80.34	81.54
A	76.14	68.00	58.00	44.00	28.00	12.00	0

In Fig. 2, Munekata's curve is shown in a broken line for comparison. From this figure, we see that both curves are in fairly agreement. We can find thus the shearing force necessary to perform the transverse vibration in a certain normal mode of frequency $p/2\pi$.

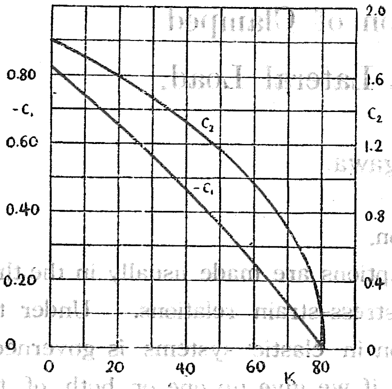

Fig. 3 (a)

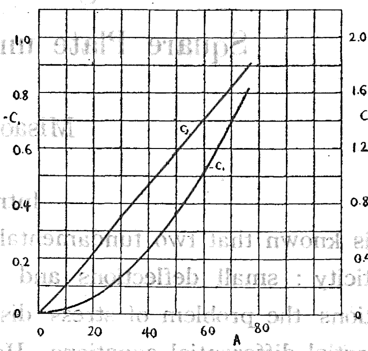

Fig. 3 (b)

If we shall consider to find the values of frequency of transverse vibration corresponding to a certain shearing force, we must determine C_1 and C_2 so to satisfying the following conditions,

$$\frac{\partial K}{\partial C_1}=0 \quad \text{and} \quad \frac{\partial K}{\partial C_2}=0 \qquad (19)$$

And the equations determing C_1 and C_2 are the same with equations (17) and (18) and in this case, we obtain the Table 1. Table II which will serve in both these cases.

Ⅲ. Summary

By applying Rayleigh principle, we have treated the problems of transverse vibration of a square plate under shearing forces at the four clamped edges. The relation between the shearing force and the frequency of the normal mode of vibration was found in Table 1. and showen in Fig. 2.

From this figure, we may evaluate the corresponding values of the frequency of vibration to a given shearing force, or vice versa.

In conclusion, I wish to thank Prof. Tomotika for his encouragement and also to Mr. Munekata for permission to make use of the data contained in Table 2.

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