On the Transverse Vibration of a Square Plate with Four Clamped Edges.

(read at the annual meeting of the physical society of Japan, 1948)

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I. Introduction

1. The problem of the transverse vibration of a square plate with four clamped edges is one of the most important and interesting characteristic value problem in elastokinetics. Prof. S. Tomotika has discussed this by two different method i. e., the method of treatment similar to that used by G. I. Taylor and the method of solving a minimal problem, and obtained the results that $\rho a^4 hp^2/D\pi^2=(3.6462)^2=13.2948$, and $(3.6461)^2=13.2940$. In this equation, p is the frequency in 2π seconds of the fundamental mode, ρ the density of the material of the plate, and a the length of the side of the square. Also D is the flexural rigidity and is given by the formula $D=\frac{Eh^3}{12}(1-\sigma^2)^{-1}$, E and σ being Young's modulus and Poisson's ratio of the material of the plate respectively.

K. Sezawa and S. Iguchi have discussed the same problem by their own methods. However, they did not considered the rotatory inertia treating this problem. Now we shall discuss the same problem by considering the rotatory inertia.

We shall apply in this problem the method of solving a minimal problem.

II. Transverse Vibration of a Square Plate Clamped at four Edges.

2. Let us take the coordinate axes (x, y) in the middle surface of a square plate of uniform small thickness such that the origin coinsides with the center of the plate and the axes are parallel to the sides, we denote the length of the square and the thickness of the plate by a and h respectively. Let the density, Young's modulus and poisson's ratio of the material of the plate, which is assumed to be uniform and isotropic, be denoted by ρ, E and σ respectively.

Then, if w be the transverse displacement of a point on the middle surface the differential equation for the transverse vibration of the plate is

$$D\left(\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}\right)+\rho h\frac{\partial^2}{\partial t^2}\left\{w-\frac{h^2}{12}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)\right\}=0, \qquad (1)$$

where D denotes the flexural rigidity, given by the formula :

$$D=\frac{1}{12}\,\frac{Eh^3}{1-\sigma^2}.$$

When the four edges of the square plate are clamped, the boundary conditions at the edges are

$$\left.\begin{aligned} &w=0,\ \frac{\partial w}{\partial x}=0 \quad \text{at} \quad x=\pm\frac{a}{2},\\ &w=0,\ \frac{\partial w}{\partial y}=0 \quad \text{at} \quad y=\pm\frac{a}{2}.\end{aligned}\right\} \qquad (2)$$

Writing $x=a\xi/\pi$, $y=a\eta/\pi$, the square whose sides are $x=\pm a/2$, $y=\pm a/2$ is transformed into a square whose sides are $\xi=\pm\pi/2$, $\eta=\pm\pi/2$, and equation (1) is transformed as follows

$$\frac{\partial^4 w}{\partial\xi^4}+2\frac{\partial^4 w}{\partial\xi^2\partial\eta^2}+\frac{\partial^4 w}{\partial\eta^4}+\frac{\rho h a^4}{D\pi^4}\frac{\partial^2}{\partial t^2}\left\{w-\frac{h^2\pi^2}{12a^2}\left(\frac{\partial^2 w}{\partial\xi^2}+\frac{\partial^2 w}{\partial\eta^2}\right)\right\}=0, \tag{3}$$

while the boundary conditions become as follows;

$$\left.\begin{aligned} w=0,\ \frac{\partial w}{\partial\xi}=0, \quad &\text{at} \quad \xi=\pm\frac{\pi}{2},\\ w=0,\ \frac{\partial w}{\partial\eta}=0, \quad &\text{at} \quad \eta=\pm\frac{\pi}{2}. \end{aligned}\right\} \tag{4}$$

When the plate vibrates in a normal mode, the displacement w takes of the form

$$w=W\cos(pt+\varepsilon), \tag{5}$$

where W is a function of ξ, η, and p is the frequency of vibration in 2π seconds. $\cos(pt+\varepsilon)$ is the normal coordinate and W is the normal function. If we put (5) in (3) we get the following partial differential equation for W;

$$\frac{\partial^4 W}{\partial\xi^4}+2\frac{\partial^4 W}{\partial\xi^2\partial\eta^2}+\frac{\partial^4 W}{\partial\eta^4}-k^2\left\{W-2g^2\left(\frac{\partial^2 W}{\partial\xi^2}+\frac{\partial^2 W}{\partial\eta^2}\right)\right\}=0 \tag{6}$$

or

$$\Delta\Delta W-k^2\{W-2g^2\Delta W\}=0 \tag{6'}$$

where

$$\Delta\equiv\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2},$$

and

$$k^2\equiv\frac{\rho h a^4 p^2}{D\pi^4}, \tag{7}$$

$$g^2=\frac{h^2\pi^2}{24a^2}. \tag{8}$$

The boundary conditions to be satisfied by the normal function W at the edges of the plate now become

$$\left.\begin{aligned} W=0,\ \frac{\partial W}{\partial\xi}=0 \quad &\text{at} \quad \xi=\pm\frac{\pi}{2},\\ W=0,\ \frac{\partial W}{\partial\eta}=0 \quad &\text{at} \quad \eta=\pm\frac{\pi}{2}. \end{aligned}\right\} \tag{9}$$

Our problem is therefore to find the characteristic values k^2 of the differential equation (6)′ under the clamped edge conditions (9). The least value of k for a certain value of g corresponds evidently with the smallest value of p i. e., the frequency of the fundamental mode of vibration.

Now if we confine ourselves to the most important case of the fundamental mode of vibration of a square plate with four clamped edges, the problem of finding its frequency is equivalent to solving the following minimal problem ; Problem ; It is required to find the least value of the expression;

$$\frac{V(W)}{T(W)}=\frac{\iint_S(\Delta W)^2\,d\xi\,d\eta}{\iint_S\left[W^2+2g^2\left\{\left(\frac{\partial W}{\partial\xi}\right)^2+\left(\frac{\partial W}{\partial\eta}\right)^2\right\}\right]d\xi\,d\eta}, \tag{10}$$

for all functions $W(\xi,\eta)$ which have continous derivatives up to the fourth order in the square C : $|\xi| \leqq \frac{\pi}{2}$, $|\eta| \leqq \frac{\pi}{2}$, and which also satisfy the conditions:

$$W=0, \quad \frac{\partial W}{\partial n}=0, \tag{11}$$

on the boundary C: $|\xi| = \frac{\pi}{2}$, $|\eta| = \frac{\pi}{2}$ of the square. The double integrals are taken over the square S, n being the normal to the boundary and g a constant.

Ⅱ. A Modified Minimum Problem and its Solution

3. But as it is very difficult to find the least value of the above problem. We shall now consider the following modified problem:

It is required find the least value μ^2_m of the expression:

$$\frac{V(v)}{T(v)} \equiv \frac{\iint_S (\Delta v)^2 d\xi\, d\eta}{\iint_S \left\{ v^2 + 2g^2 \left\{ \left(\frac{\partial v}{\partial \xi}\right)^2 + \left(\frac{\partial v}{\partial \eta}\right)^2 \right\} \right\} d\xi\, d\eta}, \tag{12}$$

for all functions v (ξ, η) which vanish on the boundary of the square plate and satisfy following m conditions on C:

$$G_{2j-1} = \int_C \frac{\partial v}{\partial n} g_{2j-1} ds = 0, \qquad (j=1, 2, \cdots, m), \tag{13}$$

instead of $\frac{\partial v}{\partial n}=0$, ds being a line element along C and g a constant. The double integrals are taken, as before, over S, while the single integrals (13) are taken along the boundary C and the functions g_{2j-1} are taken as follows:

$$\begin{aligned} g_{2j-1} &= C_j \cos(2j-1)\xi \quad \text{on} \quad \eta = \pm\frac{\pi}{2}, \\ &= C_j \cos(2j-1)\eta \quad \text{on} \quad \xi = \pm\frac{\pi}{2}, \end{aligned} \tag{14}$$

where the C_j's are certain constants.

We shall apply to this modified problem the general principle in the calculus of variations, which may be expressed as follows:

If in a minimal problem some of the conditions are made less stringent, the minimum value in the modified problem cannot be greater than that in the original problem.

Then, it follows from this general principles that

$$\mu_1^2 \leqq \mu_2^2 \leqq \cdots \leqq \mu^2_m,$$

and that μ^2_m is not greater than the true minimum value of $k^2 = \rho h a^4 p^2 / D\pi^4$ for a certain value of g. The values $\mu_1^2, \mu_2^2, \cdots, \mu^2_m$ are therefore a non-decreasing sequence of lower limits for the true value of $\rho h a^4 p^2/D\pi^4$.

4. We shall now obtain the Euler equation and the boundary conditions for the the modified problem. (This can be done easily by applying the usal analysis in the calculus of variations.) Thus, if we denote by μ^2_m, $a_1, a_2, \cdots, a_m$ Lagrangian indeterminate multipliers, we obtain the Euler equation and the boundary conditions, for this problem by putting

$$\delta\Big[\iint_S (\varDelta v)^2 d\xi\, d\eta - \mu^2_m \iint_S v^2 d\xi\, d\eta - 2g^2\mu^2_m \iint_S \Big\{\Big(\frac{\partial v}{\partial \xi}\Big)^2 + \Big(\frac{\partial v}{\partial \eta}\Big)^2\Big\} d\xi\, d\eta$$

$$-2\sum_{j=1}^{m} a_j \int_C \frac{\partial v}{\partial n} g_{2j-1}\, ds\Big] = 0, \qquad (15)$$

where δ is the symbol of variation.

Performing the variation we get

$$\iint_S \delta v\{\varDelta\varDelta v - \mu_m^2 v + 2\mu^2_m g^2 \varDelta v\} d\xi\, d\eta$$

$$+\int_C \Big[\Big\{\varDelta v - \sum a_j g_{2j-1}\Big\}\frac{\partial \delta v}{\partial n} - \Big\{\frac{\partial \varDelta v}{\partial n} + 2g^2\mu^2_m \frac{\partial v}{\partial n}\Big\}\delta v\Big] ds = 0. \qquad (16)$$

Since, however δv is arbitrary in the inside of the square and $\partial\delta v/\partial n$ is also arbitrary on the boundary of the square, while δv on the boundary is zero, we obtain from (16) in the inside of the square S,

$$\varDelta\varDelta v - \mu_m^2 v + 2\mu_m^2 g^2 \varDelta v = 0 \qquad (17)$$

and on the boundary C,

$$v = 0, \qquad (18)$$

$$\varDelta v = \sum_{j=1}^{m} a_j g_{2j-1} \qquad (19)$$

The equation (17) is the Euler equation for the modified minimal problem under consideration, and (18) and (19) are the corresponding boundary conditions.

Now we can show that μ_m^2 which has been hitherto used as one of the Lagrangian indeterminate multiplers is really equal to the minimum value of the expression $V(v)/T(v)$ defined by (12) subject to the boundary conditions $v=0$ and $G_{2j-1} = \int_C \Big(\frac{\partial v}{\partial n}\Big) g_{2j-1}\, ds = 0$. For this purpose we multiply both sides of the equation (17) by v and integrate over S. Then we get, using Green's theorem

$$\iint_S [(\varDelta v)^2 - \mu_m^2 v^2 - 2\mu_m g^2 \varDelta v]\, d\xi d\eta + \int_C \Big[\Big(v\frac{\partial \varDelta v}{\partial n} - \varDelta v \frac{\partial v}{\partial n}\Big) + \frac{\partial v}{\partial n}\Big] ds = 0$$

But, by the boundary conditions (18), (19), we get

$$\mu_m^2 \Big[\iint_S \{v^2 + 2g^2 \varDelta v\} d\xi\, d\eta\Big] = \iint_S (\varDelta v)^2 d\xi\, d\eta - \sum_{j=1}^{m} a_j \int_C \frac{\partial v}{\partial n} g_{2j-1} ds. \qquad (20)$$

Thus, we see from this equation that the Lagrangian multipier μ_m^2 is indeed equal to the minimum value of $V(v)/T(v)$ with the boundary conditions $v=0$ and $G_{2j-1} = \int_C \Big(\frac{\partial v}{\partial n}\Big) g_{2j-1}\, ds = 0$ $(j=1, 2, \cdots, m)$ in the edges of the square.

5. Now we shall show that μ_1^2 is not equal to the smallest characteristic value of the problem of transverse vibration of a square plate with four supported edges. In the case of a square plate with four supported edges, the differential equation for the normal function is, denoting it simply by W^*,

$$\varDelta\varDelta W^* - k^2\{W^* - 2g^2 \varDelta W^*\} = 0, \qquad (21)$$

and the boundary conditions on the supported edges are

$$W^* = 0, \quad \varDelta W^* = 0, \qquad (22)$$

and it will be proved that the smallest characteristic value is $k^2>2$.

Thus, the already mentioned general principle in the calculus of variations yields immediately the result that

$$\mu_1^2 > 2 \tag{23}$$

6. We shall proceed to find to the solution of the differential equation (17) subject the conditions (18) and (19).

Now, we know that, as shown in the preceeding paragraph,

$$2<\mu_1^2 \leqq \mu_2^2 \leqq \cdots \leqq \mu_m^2.$$

On the other hand, if we apply the Rayleigh's principle to the problem of transverse vibration of a square plate with clamped edges by assuming the normal function W in the form $W = w_0(1-\xi^2)^2(1-\eta^2)^2$, we can get the result that $k^2=13.30\left(1+\frac{2h^2}{a^2}\right)^{-1}$ for the fundamental mode of vibration. This Rayleigh value is evidently greater than the true value of $\rho h a^4 p^2/D\pi^4$. Therefore we have the following inequalities

$$2<\mu_m<13, \tag{24}$$

or

$$1.4<\mu_m^2<3.6, \tag{24'}$$

for all values of m for small values of g. It follows immediately from these inequalities that

$$1<\sqrt{\mu_m}<3. \tag{25}$$

Now the solution of the differential equation (17) under the conditions (18) and (19) can be obtained in the form:

$$v=\sum_{i=1}^{m} A_i\, v_{2i-1} \tag{26}$$

where

(i) i=1:

$$\begin{aligned} v_1 = \Big(& \cosh\frac{\pi}{2}\sqrt{1-g^2\mu_m^2+\sqrt{\mu_m^2+g^4\mu_m^4}}\,\cos\sqrt{g^2\mu_m^2-1+\sqrt{\mu_m^2+g^4\mu_m^4}}\;\eta \\ & -\cos\frac{\pi}{2}\sqrt{g^2\mu_m^2-1+\sqrt{\mu_m^2+g^4\mu_m^4}}\,\cosh\sqrt{1-g^2\mu_m^2+\sqrt{\mu_m^2+g^4\mu_m^4}}\;\eta\Big)\cos\xi \\ & +\Big(\cosh\frac{\pi}{2}\sqrt{1-g^2\mu_m^2+\sqrt{\mu_m^2+g^4\mu_m^4}}\,\cos\sqrt{g^2\mu_m^2-1+\sqrt{\mu_m^2+g^4\mu_m^2}}\;\xi \\ & -\cos\frac{\pi}{2}\sqrt{g^2\mu_m^2-1+\sqrt{\mu_m^2+g^4\mu_m^4}}\,\cosh\sqrt{1-g^2\mu^2\mu_m^2+\sqrt{\mu_m^2+g^4\mu_m^4}}\;\xi\Big)\cos\eta \end{aligned} \tag{27 a}$$

(ii) i=2, 3, ..., m

$$\begin{aligned} v_{2i-1} = \Big(& \cosh\frac{\pi}{2}\sqrt{(2i-1)^2-g^2\mu_m^2+\sqrt{\mu_m^2+g^4\mu_m^4}}\,\cosh\sqrt{(2i-1)^2-g^2\mu_m^2-\sqrt{\mu_m^2+g^4\mu_m^4}}\;\eta \\ & -\cosh\frac{\pi}{2}\sqrt{(2i-1)^2-g^2\mu_m^2-\sqrt{\mu_m^2+g^4\mu_m^4}}\,\cosh\sqrt{(2i-1)^2-g^2\mu_m^2+\sqrt{\mu_m^2+g^4\mu_m^4}}\;\eta\Big) \\ & \times\cos(2i-1)\,\xi \\ & +\Big(\cosh\frac{\pi}{2}\sqrt{(2i-1)^2-g^2\mu_m^2+\sqrt{\mu_m^2+g^4\mu_m^4}}\,\cosh\sqrt{(2i-1)^2-g^2\mu_m^2-\sqrt{\mu_m^2+g^4\mu_m^4}}\;\xi \\ & \cosh\sqrt{(2i-1)^2-g^2\mu_m^2-\sqrt{\mu_m^2+g^4\mu_m^4}}\,\cosh\sqrt{(2i-1)^2-g^2\mu_m^2+\sqrt{\mu_m^2+g^4\mu_m^4}}\;\xi\Big) \\ & \times\cos(2i-1)\,\eta. \end{aligned} \tag{27 b}$$

The A_i's are constants, and the i-th constat A_i is proportional to the corresponding i-th constant a_i in (19), so that A_i's are not simultaneously zero.

It found that the boundary condition for the modified minimum problem that $v=0$ on the boundary is satisfied by the above expression for v. The A_i's must then satisfied the second boundary condition (13), which may be written in the form:

$$\sum_{i=1}^{m} A_i \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{\partial v_{2i-1}}{\partial \xi}\right)_{\xi=\frac{\pi}{2}} \cos(2j-1)\eta \, d\eta = 0, \quad (j=1,2,\cdots,m) \tag{28}$$

since the expression for v given by (26) is symmetrical with respect to ξ and η.

We denot for brevity

$$C_{ij} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{\partial v_{2i-1}}{\partial \xi}\right)_{\xi=\frac{\pi}{2}} \cos(2j-1)\eta \, d\eta. \quad (i, j, =1, 2, \cdots, m) \tag{29}$$

Then the equations (28) become

$$\sum_{i=1}^{m} A_i C_{ij} = 0 \quad . \qquad (j=1,2,\cdots,m) \tag{30}$$

Since, as mentioned already, all the A_i's do not vanish simultaneously, we must have

$$\Delta_m = \begin{vmatrix} C_{11} & C_{12} & \cdots & C_{1m} \\ C_{21} & C_{22} & \cdots & C_{2m} \\ \vdots & \vdots & & \vdots \\ C_{m1} & C & \cdots\cdots & C_{mm} \end{vmatrix} = 0 \quad . \tag{31}$$

This is a transcendental equation for determing μ_m and its least positive root gives the required lower limit for the true value of $\rho h a^4 p^2/D\pi^4$.

7. The expression for the C_{ij}'s are easily obtained in the form:

for $i=1,\ j=1,2,\cdots,m$

$$C_{1j} = \cos\frac{\pi}{2}\sqrt{g^2\mu_m^2-1+\sqrt{\mu_m^2+g^4\mu_m^4}}\cosh\frac{\pi}{2}\sqrt{1-g^2\mu_m^2+\sqrt{\mu_m^2+g^4\mu_m^4}}$$
$$\times\Bigg[(-1)^{j-1}\frac{4(2j-1)\sqrt{\mu_m^2+g^4\mu_m^4}}{\mu_m^2+g^4\mu_m^4-\{1-g^2\mu_m^2+(2j-1)^2\}^2}-\delta_{1j}\frac{\pi}{2}\Bigg\{\sqrt{g^2\mu_m^2-1+\sqrt{\mu_m^2+g^4\mu_m^4}}$$
$$\times\tan\frac{\pi}{2}\sqrt{g^2\mu_m^2-1+\sqrt{\mu_m^2+g^4\mu_m^4}}+\sqrt{1-g^2\mu_m^2+\sqrt{\mu_m^2+g^4\mu_m^4}}$$
$$\times\tanh\frac{\pi}{2}\sqrt{1-g^2\mu_m^2+\sqrt{\mu_m^2+g^4\mu_m^4}}\Bigg\}\Bigg] \tag{32a}$$

and for $i=2,3,\cdots m,\ j=1,2,\cdots,m$

$$C_{ij}=\cosh\frac{\pi}{2}\sqrt{(2i-1)^2-g^2\mu_m^2-\sqrt{\mu_m^2+g^4\mu_m^4}}\cosh\frac{\pi}{2}\sqrt{(2i-1)^2-g^2\mu_m^2+\sqrt{\mu_m^2+g^4\mu_m^4}}$$
$$\times\Bigg[(-1)^{i+j-1}\frac{4(2i-i)(2j-1)\sqrt{\mu_m^2+g^4\mu_m^4}}{\{(2i-1)^2-g^2\mu_m^2+(2j-1)^2\}^2-(\mu_m^2+g^4\mu_m^4)}$$
$$+\delta_{ij}\frac{\pi}{2}\Bigg\{\sqrt{(2i-1)^2-g^2\mu_m^2-\sqrt{\mu_m^2+g^4\mu_m^4}}\tanh\frac{\pi}{2}\sqrt{(2i-1)^2-g^2\mu_m^2-\sqrt{\mu_m^2+g^4\mu_m^4}}$$
$$-\sqrt{(2i-1)^2-g^2\mu_m^2+\sqrt{\mu_m^2+g^4\mu_m^4}}\tanh\frac{\pi}{2}\sqrt{(2i-1)^2-g^2\mu_m^2+\sqrt{\mu_m^2+g^2\mu_m^4}}\Bigg\}\Bigg] \tag{32b}$$

where

$$\delta_{ij} = \begin{cases} 1 & (i=j) \\ 0 & (i \neq j) \end{cases} .$$

Now, we put

$$b_{ij} = (-1)^{i-1} \frac{4(2j-1)\sqrt{\mu_m^2+g^4\mu_m^4}}{\mu_m^2+g^4\mu_m^4-\{1-g^2\mu^2+(2j-1)^2\}^2}$$

$$-\delta_{ij}\frac{\pi}{2}\left\{\sqrt{g^2\mu_m^2-1+\sqrt{\mu_m^2+g^4\mu_m^4}}\tan\frac{\pi}{2}\sqrt{g^2\mu_m^2-1+\sqrt{\mu_m^2+g^4\mu_m^4}}\right.$$

$$\left.+\sqrt{1-g^2\mu_m^2+\sqrt{\mu_m^2+g^4\mu_m^4}}\tanh\frac{\pi}{2}\sqrt{1-g^2\mu_m^2+\sqrt{\mu_m^2+g^4\mu_m^4}}\right\} \quad (j=1,2,\cdots,m) \tag{33a}$$

$$b_{ij} = (-1)^{i+j-1} \frac{4(2j-1)(2i-1)\sqrt{\mu_m^2+g^4\mu_m^4}}{\{(2i-1)^2-g^2\mu_m^2+(2j-1)^2\}^2-(\mu_m^2+g^4\mu_m^4)}$$

$$+\delta_{ij}\frac{\pi}{2}\left\{\sqrt{(2i-1)^2-g^2\mu_m^2-\sqrt{\mu_m^2+g^4\mu_m^4}}\tanh\frac{\pi}{2}\sqrt{(2i-1)^2-g^2\mu_m^2-\sqrt{\mu_m^2+g^4\mu_m^4}}\right.$$

$$\left.-\sqrt{(2i-1)^2-g^2\mu_m^2+\sqrt{\mu_m^2+g^4\mu_m^4}}\tanh\frac{\pi}{2}\sqrt{(2i-1)^2-g^2\mu_m^2+\sqrt{\mu_m^2+g^4\mu_m^4}}\right\}$$

$$\begin{pmatrix} i=2,3,\cdots,m \\ j=1,2,\cdots,m \end{pmatrix} \tag{33b}$$

It is easily found that when $(i, j=1,2,\cdots,m)$

$$b_{ij} = b_{ji}$$

Then we have

$$C_{ij} = \cos\frac{\pi}{2}\sqrt{g^2\mu_m^2-1+\sqrt{\mu_m^2+g^4\mu_m^4}}\cosh\frac{\pi}{2}\sqrt{1-g^2\mu_m^2+\sqrt{\mu_m^2+g^4\mu_m^4}}\, b_{ij},$$

$$(j=1,2,\cdots,m)$$

$$C_{ij} = \cosh\frac{\pi}{2}\sqrt{(2i-1)^2-g^2\mu_m^2-\sqrt{\mu_m^2+g^4\mu_m^4}}\cosh\frac{\pi}{2}\sqrt{(2i-1)^2-g^2\mu_m^2+\sqrt{\mu_m^2+g^4\mu_m^4}}\, b_{ij}$$

$$\begin{pmatrix} i=2,3,\cdots,m \\ j=1,2,\cdots,m \end{pmatrix} \tag{34}$$

and the determinal equation (31) becomes

$$\Delta_m = \begin{vmatrix} b_{11} & b_{12} \cdots b_{1m} \\ b_{21} & b_{22} \cdots b_{2m} \\ \vdots & \\ b_{m1} & b_{m2} \cdots b_{mm} \end{vmatrix} = 0 . \tag{35}$$

We shall now find the least positive root μ_m of this determinal equation.

IV. Calculation of the Roots μ_1, μ_2, μ_3 and μ_4.

8. We shall now find the least positive roots μ_m of this determinal equations Δ_1, Δ_2, Δ_3 and Δ_4 for h/a=0.1, 0.2 and 0.3. For this purpose the values of Δ_m for appropriate several values of μ_m are calculated and then the least positive root μ_m is obtained by using Lagrange interpolation formula. The results obtained here are given in Table I, II, III.

Table I. h/a=0.11

μ_m	Δ_2	Δ_3	Δ_4	μ_m	Δ_1
3.605	−0.02109	0.038887	−0.0348405	3.5	0.2150
3.610	0.00674	0.002626	−0.0049380	3.6	−0.0819
3.615	0.03950	−0.039277	0.0335940	3.7	−0.3334
Root	3.609	3.6103	3.6107		3.5705

Table II. h/a=0.2

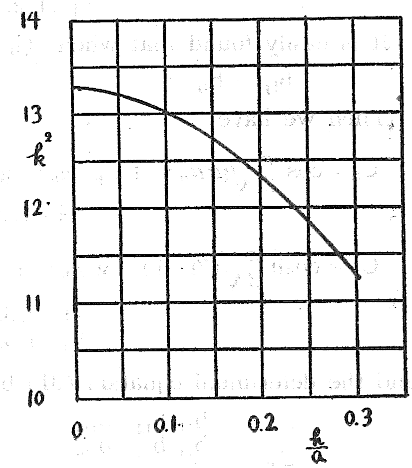
μ_m	Δ_2	Δ_3	Δ_4	μ_m	Δ_1
3.50	−0.03558	0.054348	−0.0478360	3.4	0.1969
3.505	−0.01031	0.023173	−0.0205100	3.5	−0.0937
3.510	0.02097	−0.015520	0.0125117	3.6	−0.3530
Root	3.507	3.5082	3.5082		3.466

Table III. h/a=0.3

μ_m	Δ_2	Δ_3	Δ^4	μ_1	Δ_1
3.350	−0.02618	0.040695	−0.0340722	3.2	0.3586
3.355	0.00509	0.003289	−0.0034244	3.3	0.04697
3.360	0.03476	−0.032247	0.0253866	3.4	−0.2435
Root	3.354	3.3555	3.35558		3.3157

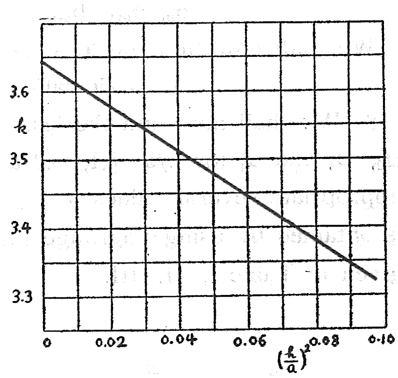
From these Tables we see that the value of μ_m rapidly converges to the limiting value when m increases. And we may take μ_4 as the value of $k=\sqrt{\frac{\rho h a^4 p^2}{D\pi^4}}$.

Fig. 1. shows the relation between k^2 and h/a that is, the frequency of the fundamental mode of vibration and the ratio of the thickness to the lengh of side of a square plate for small value of h/a.



From this figure we see that when the ratio increases the frequency of the fundamental mode of vibration of the plate decreases, and therefore the frequency of the fundamental mode of vibration of the square plate decreases as the plate thickens.

Fig. 2. shows the relation between k and $\left(\frac{h}{a}\right)^2$. The curve in Fig. 2 represents almost a straight line and we may expect that the following relation hold in a small values of h/a,



$$k=\sqrt{\frac{\rho h a^4 p^2}{D\pi}}=k_0\left\{1+\alpha\left(\frac{h}{a}\right)^2\right\} \quad (36)$$

approximately.

The value of k_0 is the value of k when h=o. and therefore we may use Tomotika's value: $k_0=3.6461$.

By the method of least squares, the value of coefficient α is determined as follows:

$$\alpha=\frac{-3.348}{3.646}=-0.9183 .$$

By using the equation (36), we can find approximately the frequency of the fundamental mode of vibration of a square plate with four clamped edge for any small values of h/a.

V. Summary

3. The problem of calcuating the frequency of the fundamental mode of transverse vibration of a square plate with clamped edges is equivalent to a minimum problem of computing the minimum value of the expression:

$$\frac{V(W)}{T(W)}=\frac{\iint_S (\Delta W)^2\, d\xi\, d\eta}{\iint_S \left[W^2+2g^2\left\{\left(\frac{\partial W}{\partial \xi}\right)^2+\left(\frac{\partial W}{\partial \eta}\right)^2\right\}\right] d\xi d\eta} \tag{10}$$

for all function $W(\xi, \eta)$ which have continuous derivatives up to the fourth order in the square S : $|\xi| \leqq \frac{\pi}{2}$, $|\eta| \leqq \frac{\pi}{2}$ and which also satisfy the clamped edges conditions:

$$W=0, \quad \frac{\partial W}{\partial n}=0 \tag{11}$$

at the bonndary C: $|\xi| =\frac{\pi}{2}$, $|\eta| =\frac{\pi}{2}$ of the square, the double integrals being over the square S and n denoting the normal to the boundary.

In the present paper, we consider the modified minimum problem which may be expressed as follows:

It is required to find the minimum value of the expression:

$$\frac{V(v)}{T(v)}=\frac{\iint_S (\Delta v)^2\, d\xi\, d\eta}{\iint_S \left[v^2+2g^2\left\{\left(\frac{\partial v}{\partial \xi}\right)^2+\left(\frac{\partial v}{\partial \eta}\right)^2\right]d\xi\, d\eta}$$

for all functions $v(\xi, \eta)$ which vanish on the boundary C of the square and satisfy the following boundary m conditions on C:

$$G_{2j-1}=\int_C \frac{\partial v}{\partial n} g_{2j-1} ds=0 \quad , \quad (j=1,2,\cdots,m)$$

where ds is a line element along C so that $ds=d\xi$ on $\eta=\pm\frac{\pi}{2}$, and $ds=d\eta$ on $\xi=\pm\frac{\pi}{2}$. The double integrals are taken over the square S, while the single integrals are taken along the boundary C and functions g_{2j-1} are taken follows:

$$g_{2j-1}=C_j \cos(2j-1)\xi \qquad \text{on} \quad \eta=\pm\frac{\pi}{2},$$

$$=C_j \cos(2j-1)\eta \qquad \text{on} \quad \xi=\pm\frac{\pi}{2},$$

where the C_i's are certain constants.

Applying to the modified problem the general principle in the calculus of variation, a non-decreasing sequence of lower limits for the true minimum value of $k^2=\frac{\rho h a^4 p^2}{D\pi^4}$ is calculated for $\frac{h}{a}=0.1$, 0.2 and 0.3. From these sequences we find the Table IV, giving the frequency p of the fundamental mode of transverse vibration of a square plate with clamped edges.

Table IV.

h/a	0.1	0.2	0.3
$\frac{\rho h a^4 p^2}{D\pi^4}$	13.037	12.308	11.260

In conclusion, I wish to thank Prof. Tomotika for his encouragement during this work.

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