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On Homogeneous Systems III

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The investigation of various properties of homogeneous systems in [2], [3], [4] is continued to this paper. The notions of normal subsystems and quotient homogeneous systems are introduced in § 1. In § 2, analytic homomorphisms of analytic homogeneous systems are treated. It is shown that the tangent Lie triple algebra of a closed normal subsystem of a geodesic homogeneous system G is an ideal of the tangent Lie triple algebra of G (Theorem 3).

§ 1. Normal subsystems of homogeneous systems

In this paper, we use the same terminologies and notations as used in the preceding papers [2], [3] and [4]. Let $G=(G, \eta)$ be an abstract homogeneous system and H a subsystem of G. For an element x of G we denote by xH the subset $\eta(H, x, H) =\{\eta(u, x, v) \mid u, v \in H\}$ of G. The element x is contained in xH since $\eta(u, x, u)=x$ for any $u \in H$. The subsystem H is said to be *invariant* ([3]) if it satisfies

$$\eta(x, y)(xH)=yH \quad \text{for} \quad x, y \in G. \tag{1.1}$$

It is said to be *normal* if

$$\eta(xH, yH, zH)=\eta(x, y, z)H \quad \text{for} \quad x, y, z \in G. \tag{1.2}$$

REMARK. In the case of a homogeneous system of a group (c.f. Example in § 1 of [2]), a subsystem containing the identity element is normal if and only if it is the homogeneous system of a normal subgroup.

LEMMA 1. *A normal subsystem is invariant.*

PROOF. Suppose that H is a normal subsystem of G and $x, y \in G$. Then $\eta(xH, yH, xH)=yH$ implies $\eta(x, y)(xH) \subset yH$, and $xH \subset \eta(y, x)(yH)$. Since x and y can be chosen arbitrarily, we have $\eta(x, y)(xH)=yH$. *q.e.d.*

Let $\tilde{G}=(\tilde{G}, \tilde{\eta})$ be a homogeneous system. A *homomorphism* of G into $\tilde{G}$ is a map $f: G \rightarrow \tilde{G}$ satisfying $f\eta(x, y, z)=\tilde{\eta}(fx, fy, fz)$ for $x, y, z \in G$. If $f: G \rightarrow \tilde{G}$ is a homomorphism, then it is clear that the image (resp. inverse image) of any subsystem of G (resp. $\tilde{G}$) under f is a subsystem of $\tilde{G}$ (resp. G).

LEMMA 2. *Let $f: G \rightarrow \tilde{G}$ be a homomorphism of homogeneous systems. For some* *fixed* $e \in G$, *set* $H=f^{-1}(\tilde{e})$, $\tilde{e}=f(e)$. *Then* H *is a subsystem of* G *and* $xH=f^{-1}(\tilde{x})$, $\tilde{x}=f(x)$ *for any* $x \in G$.

PROOF. Since the subset $\{\tilde{e}\}$ of a single element forms a subsystem of $\tilde{G}$, $H=f^{-1}(\tilde{e})$ is a subsystem of G. Now, let $u \in H$ and $w \in f^{-1}(\tilde{x})$ for $\tilde{x}=f(x)$, $x \in G$, and set $v=\eta(x, u, w)$. The element v belongs to H since $fv=\tilde{\eta}(fx, fu, fw)=\tilde{\eta}(\tilde{x}, \tilde{e}, \tilde{x})=\tilde{e}$. Hence $w=\eta(u, x, \eta(x, u, w))=\eta(u, x, v) \in \eta(H, x, H)=xH$ and $f^{-1}(\tilde{x}) \subset xH$. On the other hand $xH \subset f^{-1}(\tilde{x})$ holds since $f(xH)=f\eta(H, x, H)=\tilde{\eta}(\tilde{e}, \tilde{x}, \tilde{e})=\tilde{x}$. *q.e.d.*

PROPOSITION 1. *A non-empty subset* H *of a homogeneous system* G *is a normal subsystem of* G *if and only if there exists a homomorphism* $f: G \to \tilde{G}$ *of* G *into a homogeneous system* $\tilde{G}$ *such that* $H=f^{-1}(\tilde{e})$ *for some* $\tilde{e}=f(e)$, $e \in H$.

PROOF. Suppose that $f: G \to \tilde{G}$ is a homomorphism of G into $\tilde{G}$ and $H=f^{-1}(\tilde{e})$ for $\tilde{e} \in \tilde{G}$. Then, by Lemma 2 above, we see that H is a subsystem of G and $xH=f^{-1}(\tilde{x})$, $\tilde{x}=fx$ for any $x \in G$. For $x, y, z \in G$ we have $\eta(xH, yH, zH) \subset \eta(x, y, z)H$ since $f\eta(xH, yH, zH)=\tilde{\eta}(fx, fy, fz)=f\eta(x, y, z)$. If $u, v \in H$ and $x, y, z \in G$, we have $f\eta(\eta(y, x, u), z, \eta(y, x, v))=\tilde{\eta}(\tilde{\eta}(fy, fx, \tilde{e}), fz, \tilde{\eta}(fy, fx, \tilde{e}))=fz$ and we get $\eta(\eta(y, x, u), z, \eta(y, x, v)) \in zH$. This fact implies $\eta(H, \eta(x, y, z), H) \subset \eta(xH, yH, zH)$. In fact, $\eta(u, \eta(x, y, z), v)=\eta(x, y)\eta(\eta(y, x, u), z, \eta(y, x, v)) \in \eta(x, y, zH) \subset \eta(xH, yH, zH)$ hold for $u, v \in H$ and $x, y, z \in G$. Thus, (1.2) is shown and hence H is a normal subsystem. Conversely, let H be a normal subsystem of G. By Lemma 1, H is an in invariant subsystem, and $y \in xH$ if and only if $xH=yH$ (Lemma 2 of [3]). For $x, y \in G$ we define an equivalence relation $\sim$ on G as $x \sim y$ if $y \in xH$. The quotient set $\tilde{G}=G/\sim$ is the collection of subsets of G given by $\{xH \mid x \in G\}$. If we set $\tilde{\eta}(xH, yH, zH)=\eta(x, y, z)H$ for $x, y, z \in G$, then, by (1.2), we get a well defined homogeneous system $\tilde{G}=(\tilde{G}, \tilde{\eta})$ so that the natural projection $f: G \to \tilde{G}$ is a homomorphism of G onto $\tilde{G}$, and $H=f^{-1}(\tilde{e})$ for $\tilde{e}=f(H) \in \tilde{G}$. *q.e.d.*

If H is a normal subsystem of a homogeneous system G and $\tilde{G}=\{xH \mid x \in G\}$, the homogeneous system $(\tilde{G}, \tilde{\eta})$ defined in the proof above will be called a *quotient homogeneous system* of G modulo H, and denoted by G/H.

§2. Analytic homomorphisms

Let $G=(G, \eta)$ be an analytic homogeneous system whose underlying space G is a separable analytic manifold of dimension n. In the followings we assume that G is a geodesic homogeneous system (cf. [3]). We denote by $\mathfrak{G}$ the tangent Lie triple algebra at some fixed point $e \in G$. Suppose that H is a closed invariant analytic subsystem of G containing e. Then H is an auto-parallel submanifold with respect to the canonical connection of G, and hence the tangent Lie triple algebra $\mathfrak{H}$ of H at e is an invariant Lie triple subalgebra of $\mathfrak{G}$ (cf. the proof of Theorem 5 in [3]). For each

$x \in G$, xH is also an invariant subsystem of G obtained as the image of H under an analytic automorphism $\eta(e, x)$ of G, and xH is an integral manifold of the distribution $\mathfrak{S}$: $x \mapsto \mathfrak{S}_x = \eta_*(e, x)\mathfrak{H}$ on G. Since G is assumed to be separable and H is closed, there exists a cubical coordinate neighborhood U around e such that $xH \cap U$ is a single slice of U whenever $xH \cap U \neq \emptyset$ (cf. p. 94 in [1]). Let $\tilde{G} = G/\sim$ be the quotient set of G under the equivalence relation; $x \sim y$ if $y \in xH$. Then, from the results of [5] (Theorem X, p. 20) it follows that $\tilde{G}$ has an analytic structure determined by an atlas consisting of local coordinate systems $\{(\tilde{U}_x, \phi_x) \mid x \in G\}$ such that $\tilde{U}_x = f \circ \eta(e, x)U$ and $\phi_x \circ f = p \circ \eta(x, e)$, where f: $G \to \tilde{G}$ is the natural projection and p: $U \to R^h$ is defined by $p(x) = (x^1, \dots, x^h)$ when $xH \cap U$ is expressed in the cubical coordinate system $(U; u^1, \dots, u^n)$ as a slice defined by $u^1 = x^1, \dots, u^h = x^h$ $(h = n - \dim H)$. Moreover, the projection f is analytic. If H is normal, then, in the same way as in the case of the factor group G/H of a Lie group G by a closed normal subgroup H, it is shown that the operation $\tilde{\eta}$: $\tilde{G} \times \tilde{G} \times \tilde{G} \to \tilde{G}$ of the quotient homogeneous system $\tilde{G} = G/H$ is analytic and f is an analytic homomorphism of G onto $\tilde{G}$. Thus we have;

THEOREM 1. *Let (G, η) be an analytic homogeneous system defined on a separable analytic manifold G. Suppose that G is geodesic and H is a closed normal analytic subsystem of G. Then the quotient homogeneous system $\tilde{G} = G/H$ of G modulo H is an analytic homogeneous system and the natural projection f: $G \to G/H$ is an analytic homomorphism.*

PROPOSITION 2. *Let G and $\tilde{G}$ be analytic homogeneous systems and f an analytic homomorphism of G onto $\tilde{G}$. Suppose that the rank of f is maximal at each point of G. If G is geodesic, then so is $\tilde{G}$.*

PROOF. For some fixed point $e \in G$, denote by Λ_e the *left inner mapping group* (or *holonomy group*) of G at e, i.e., Λ_e is the subgroup of $\mathrm{Aut}(G)$ generated by all diffeomorphisms of the form;

$$\lambda_{x,y} = \eta(x \cdot y, e) \circ \eta(x, x \cdot y) \circ \eta(e, x), \quad x, y \in G,$$

where $x \cdot y = \eta(e, x, y)$. The group Λ_e is contained in the isotropy subgroup of $\mathrm{Aut}(G)$ at e (cf. §3 in [2]). Let K_e be the closure of the left inner mapping group Λ_e in the affine transformation group of the canonical connection of G, and $A = G \times K_e$ be the Lie group identified with the subgroup $\{\eta(e, x) \circ \alpha \mid x \in G, \alpha \in K_e\}$ of $\mathrm{Aut}(G)$ under the map $(x, \alpha) \mapsto \eta(e, x) \circ \alpha$ (cf. [2] and §1 in [3]). By Proposition 5 in [2], the homogeneous system G is geodesic if and only if the 1-parameter subgroup $\exp tX$, $t \in \boldsymbol{R}$, of A is contained in $G \times \{1\}$ for each $X \in \mathfrak{G}$ in the decomposition $\mathfrak{A} = \mathfrak{G} + \mathfrak{K}$ of the Lie algebra $\mathfrak{A}$ of A, i.e., $\eta(e, x(t))$, $t \in \boldsymbol{R}$, is a 1-parameter subgroup of $\mathrm{Aut}(G)$ for each geodesic $x(t) = (\exp tX)e$ of G tangent to X at e, since $G = A/K_e$ is a reductive homogeneous space and the canonical connection of G is the canonical connection of the second kind on A/K_e (cf. Theorem 1 in [3]). Let $\tilde{\Lambda}_{\tilde{e}}$, $\tilde{K}_{\tilde{e}}$ and $\tilde{A} = \tilde{G} \times \tilde{K}_{\tilde{e}}$ denote the transformation groups corresponding to the homogeneous system $\tilde{G}$. Suppose that f is an analytic homomorphism of G onto $\tilde{G}$ with maximal rank, $f(e)=\tilde{e}$ and let $\mathfrak{G}$ (resp. $\tilde{\mathfrak{G}}$) be the tangent Lie triple algebra at e (resp. $\tilde{e}$). For any $\tilde{X}\in\tilde{\mathfrak{G}}$ choose $X\in\mathfrak{G}$ such that $(df)_e(X)=\tilde{X}$. If G is geodesic, then $t\mapsto\eta(e, x(t))$ for $x(t)=(\exp tX)e$, $t\in\boldsymbol{R}$, is a 1-parameter group of transformations on G. Since f is a homomorphism, $f\circ\eta(e, x(t))=\tilde{\eta}(\tilde{e}, \tilde{x}(t))\circ f$ for $\tilde{x}(t)=f\circ x(t)$ and $\tilde{\eta}(\tilde{e}, \tilde{x}(t))$ is a 1-parameter subgroup of $\mathrm{Aut}(\tilde{G})$, that is, $t\mapsto(\tilde{x}(t), 1)$ is a 1-parameter subgroup of $\tilde{A}$. The tangent vector $\frac{d}{dt}\Big|_0(\tilde{x}(t), 1)$ at the identity $(\tilde{e}, 1)$ is identified with $(df)_eX=\tilde{X}\in\tilde{\mathfrak{G}}$ in the decomposition $\tilde{\mathfrak{A}}=\tilde{\mathfrak{G}}+\tilde{\mathfrak{K}}$ of the Lie algebra $\tilde{\mathfrak{A}}$ of $\tilde{A}$ since $\tilde{G}=\tilde{A}/\tilde{K}_{\tilde{e}}$ is a reductive homogeneous space. Thus we have $(\tilde{x}(t), 1)=\exp t\tilde{X}$ and we see that $\tilde{G}$ is a geodesic homogeneous system. *q.e.d.*

THEOREM 2. *Let (G, η) and $(\tilde{G}, \tilde{\eta})$ be two analytic homogeneous systems. Assume that both of G and $\tilde{G}$ are geodesic. If f is an analytic homomorphism of G into $\tilde{G}$ sending $e\in G$ to $\tilde{e}\in\tilde{G}$. Then $F=(df)_e\colon\mathfrak{G}\to\tilde{\mathfrak{G}}$ is a Lie triple algebra homomorphism, where $\mathfrak{G}$ (resp. $\tilde{\mathfrak{G}}$) is the tangent Lie triple algebra of G (resp. $\tilde{G}$) at e (resp. $\tilde{e}$).*

PROOF. For each $X\in\mathfrak{G}=T_e(G)$, denote by X^* the analytic vector field on G defined as;

$$X^*(x)=\eta_*(e, x)X, \quad x\in G,$$

which will be called the *vector field associated with the tangent vector X at e*. In the same manner we define an analytic vector field $\tilde{X}^*$ on $\tilde{G}$ associated with $\tilde{X}=F(X)\in\tilde{\mathfrak{G}}=T_{\tilde{e}}(\tilde{G})$. We first show that X^* and $\tilde{X}^*$ are f-related. In fact,

$$\begin{aligned}\tilde{X}^*(f(x))&=\tilde{\eta}_*(\tilde{e}, fx)\tilde{X}=\tilde{\eta}_*(fe, fx)(df)_eX\\&=d(\tilde{\eta}(fe, fx)\circ f)_eX=d(f\circ\eta(e, x))_eX=F\circ\eta_*(e, x)X\\&=F(X^*(x)), \quad x\in G.\end{aligned}$$

For any analytic curve $c(t)$, $t\in I$, on G defined on an open interval I of $\boldsymbol{R}$, put $\tilde{c}(t)=f\circ c(t)$, $t\in I$. The original definition that G is geodesic is the following (cf. [3]); If $c(t)$ is a geodesic curve with respect to the canonical connection ∇ of G, the parallel displacement $\tau(t_1, t_2)$ of tangent vectors along c from $x_1=c(t_1)$ to $x_2=c(t_2)$ is given by $\tau(t_1, t_2)=\eta_*(x_1, x_2)\colon T_{x_1}(G)\to T_{x_2}(G)$. In particular, if G is geodesic, the tangent vectors $\frac{dc}{dt}$ to the geodesic curve c satisfy

$$\frac{dc}{dt}(t_2)=\tau(t_1, t_2)\frac{dc}{dt}(t_1)=\eta_*(c(t_1), c(t_2))\frac{dc}{dt}(t_1).$$

In this case, the corresponding curve $\tilde{c}=f\circ c$ on $\tilde{G}$ satisfies

$$\frac{d\tilde{c}}{dt}(t_2)=(df)_{x_2}\frac{dc}{dt}(t_2)=(df)_{x_2}\eta_*(x_1, x_2)\frac{dc}{dt}(t_1)$$
$$=\tilde{\eta}_*(\tilde{x}_1, \tilde{x}_2)\ (df)_{x_1}\frac{dc}{dt}(t_1)$$
$$=\tilde{\eta}_*(\tilde{x}_1, \tilde{x}_2)\frac{d\tilde{c}}{dt}(t_1),$$

that is, $\tilde{c}$ is an integral curve of the vector field $\tilde{C}_1^*(\tilde{x})=\tilde{\eta}_*(\tilde{x}_1, \tilde{x})\tilde{C}_1$, $\tilde{x}\in\tilde{G}$, for $\tilde{C}_1=\frac{d\tilde{c}}{dt}(t_1)\in T_{\tilde{x}_1}(\tilde{G})$. If $\tilde{G}$ is geodesic, the curve $\tilde{c}$ is a geodesic curve with respect to the canonical connection $\tilde{\nabla}$ of $\tilde{G}$. Thus, we see that the homomorphism f sends geodesic curves in G to geodesic curves in $\tilde{G}$. Now, by using this fact we show that the vector fields $\nabla_{X^*}Y^*$ and $\tilde{\nabla}_{\tilde{X}^*}\tilde{Y}^*$ are f-related if X^* and $\tilde{X}^*$ (resp. Y^* and $\tilde{Y}^*$) are f-related vector fields. For an arbitrarily fixed point $x_0\in G$, we consider a geodesic curve $c(t)$, $|t|<\varepsilon$, such that $c(0)=x_0$ and $\frac{dc}{dt}(0)=X^*(x_0)$. Let $\tau(t, 0)$ denote the parallel displacement of vectors along c from $c(t)$ to x_0. Then;

$$(\nabla_{X^*}Y^*)_{x_0}=\lim_{h\to 0}\frac{1}{h}(\tau(h, 0)Y^*(c(h))-Y^*(x_0)).$$

From the fact just proved above, it follows;

$$(df)_{x_0}\tau(h, 0)Y^*(c(h))=(df)_{x_0}\eta_*(c(h), x_0)Y^*(c(h))$$
$$=d(f\circ\eta(c(h), x_0))Y^*(c(h))$$
$$=\tilde{\eta}^*(\tilde{c}(h), \tilde{x}_0)(df)_{c(h)}Y^*(c(h))$$
$$=\tilde{\eta}^*(\tilde{c}(h), \tilde{x}_0)\tilde{Y}^*(\tilde{c}(h))$$
$$=\tilde{\tau}(h, 0)\tilde{Y}^*(\tilde{c}(h)),$$

where $\tilde{c}=f\circ c$ and $\tilde{x}_0=f(x_0)$. Therefore, we get

$$(df)_{x_0}(\nabla_{X^*}Y^*)_{x_0}=(\tilde{\nabla}_{\tilde{X}^*}\tilde{Y}^*)_{\tilde{x}_0}.$$

Let X^*, Y^* and Z^* be the vector fields associated with X, Y and Z in $\mathfrak{G}$, respectively. The torsion S and the curvature R of ∇ have their respective values for these vector fields as follows:

$$S(X^*, Y^*)=[X^*, Y^*]-\nabla_{X^*}Y^*+\nabla_{Y^*}X^*,$$
$$R(X^*, Y^*)Z^*=\nabla_{[X^*,Y^*]}Z^*-\nabla_{X^*}\nabla_{Y^*}Z^*+\nabla_{Y^*}\nabla_{X^*}Z^*.$$

Hence, if $\tilde{X}^*$, $\tilde{Y}^*$ and $\tilde{Z}^*$ are vector fields on $\tilde{G}$ associated with $\tilde{X}=F(X)$, $\tilde{Y}=F(Y)$ and $\tilde{Z}=F(Z)$, respectively, then each of the pairs $S(X^*, Y^*)$ and $\tilde{S}(\tilde{X}^*, \tilde{Y}^*)$; $R(X^*,$

$Y^*)Z^*$ and $\tilde{R}(\tilde{X}^*, \tilde{Y}^*)\tilde{Z}^*$ is f-related. Hence we have

$$F\, S_e(X, Y) = \tilde{S}_{\tilde{e}}(\tilde{X}, \tilde{Y}),$$

$$F\, R_e(X, Y)Z = \tilde{R}_{\tilde{e}}(\tilde{X}, \tilde{Y})\tilde{Z},$$

i.e., $F\colon \mathfrak{G} \to \tilde{\mathfrak{G}}$ is a homomorphism of the tangent Lie triple algebras (cf. [3]). *q.e.d.*

Combining Theorem 1 with Proposition 2 and Theorem 2, we have the following;

THEOREM 3. *Let $G = (G, \eta)$ be an analytic homogeneous system and H a closed normal subsystem of G. Suppose that G is geodesic. Then, at any point $e \in H$, the tangent Lie triple algebra $\mathfrak{H}$ of H is a Lie triple algebra ideal of the tangent Lie triple algebra $\mathfrak{G}$ of G, and the tangent Lie triple algebra $\tilde{\mathfrak{G}}$ of the quotient homogeneous system G/H at the origin is isomorphic to the quotient Lie triple algebra $\mathfrak{G}/\mathfrak{H}$.*

PROOF. Let $\tilde{G} = G/H$ be the quotient homogeneous system of G modulo H. By Theorem 1, $\tilde{G}$ is an analytic homogeneous system and the natural projection $f\colon G \to \tilde{G}$ is an analytic homomorphism. Then Proposition 2 and Theorem 2 imply that $\tilde{G}$ is a geodesic homogeneous system and $F = (df)_e\colon \mathfrak{G} \to \tilde{\mathfrak{G}}$ is a Lie triple algebra homomorphism of the tangent Lie triple algebras at e and $\tilde{e} = f(e)$. Since the kernel of any homomorphism of Lie triple algebras is an ideal, we see that $\mathfrak{H} = \operatorname{Ker} F$ is an ideal of $\mathfrak{G}$ and $\tilde{\mathfrak{G}}$ is isomorphic to the quotient Lie triple algebra $\mathfrak{G}/\mathfrak{H}$. *q.e.d.*

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