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On Fundamental Regular * Semigroups

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A semigroup S is called to be *fundamental* if its only one congruence contained in the Green's relation $\mathscr{H}$ on S is the trivial one. Let S be a regular $*$ semigroup, and let E and P be the sets of idempotents and projections of S, respectively. In his paper [3], T. E. Hall gives us the construction of a fundamental regular semigroup $T_{\langle E \rangle}$ which is a generalization of [2] and [4]. In this paper, we shall show that the set of projections plays an important role in the theory of regular $*$ semigroups, and construct a fundamental regular $*$ semigroup T_P, say, by using P instead of $\langle E \rangle$.

A semigroup S with a unary operation $*: S \to S$ is called a $*$ *semigroup* if it satisfies

(i) $(x^*)^* = x$,

(ii) $(xy)^* = y^*x^*$.

Let S and T be $*$ semigroups. A mapping $\phi: S \to T$ is called a $*$ *homomorphism* if ϕ is a (semigroup) homomorphism and $x^*\phi = (x\phi)^*$ for all x in S. A relation ν on S is called a $*$ *relation* on S if $(x, y) \in \nu$ implies $(x^*, y^*) \in \nu$. A $*$ semigroup S is called a *regular* $*$ *semigroup* if it satisfies

(iii) $xx^*x = x$.

An idempotent e in S such that $e^* = e$ is called a *projection*.

The following result due to Nordahl and Scheiblich is a very basic property of regular $*$ semigroups.

RESULT 1 ([5]). *Let S be a regular $*$ semigroup. Then each $\mathscr{L}$-class and each $\mathscr{R}$-class in S contain one and only one projection. Let e and f be projections in S. Then ef is an idempotent in S.*

Hereafter, a regular $*$ semigroup $S(P)$ means that S is a regular $*$ semigroup with the set of projections P. The notations and terminologies are those of [1] and [3], unless otherwise stated.

LEMMA 2. *Let $S(P)$ be a regular $*$ semigroup with the set of idempotents E. Then we have the followings:*

(i) $E=P^2$. *More precisely, for any idempotent e, there exist projections f and g such that* $e\mathscr{R}f$, $e\mathscr{L}g$ *and* $e=fg$.

(ii) *For* e, f *in* P, $ef\in P$ *if and only if* $ef=fe$.

PROOF. (i) It follows immediately from Result 1 that $P^2\subset E$. Let e be any idempotent in S. Then we have that $e\mathscr{R}ee^*=f$, say, $e\mathscr{L}e^*e=g$, say, and $e=ee^*e=ee^*e^*e=fg$.

(ii) Clear.

LEMMA 3. *Let* $S(P)$ *be a regular * semigroup. For any a in S and any e in P,* a^*ea *is a projection.*

PROOF. Since aa^* is a projection, eaa^* is an idempotent. Then $(a^*ea)^2=a^*eaa^*eaa^*a=a^*eaa^*a=a^*ea$, and $(a^*ea)^*=a^*e^*(a^*)^*=a^*ea$. Thus we have $a^*ea\in P$.

A [*] congruence ν on a regular [*] semigroup S is called an *idempotent-separating* [*] *congruence* if $\nu\subset\mathscr{H}_S$. Compare the following with Cor. 4.6[5].

THEOREM 4. *Let* μ $[\mu']$ *be the maximum idempotent-separating* [*] *congruence on a regular * semigroup* $S(P)$. *Then we have* $\mu=\mu'=\{(a, b)\in S\times S\colon a^*ea=b^*eb$ *and* $aea^*=beb^*$ *for all* $e\in P\}$.

PROOF. Let us denote the given relation by ν. It is clear that ν is a * equivalence. Let $(a, b)\in\nu$. For any $e\in P$ and $c\in S$,

$$(ac)^*eac=c^*a^*eac=c^*b^*ebc=(bc)^*ebc,$$

$$(ca)^*eca=a^*c^*eca=b^*c^*ecb=(cb)^*ecb,$$

since c^*ec is a projection. Similarly we have $ace(ac)^*=bce(bc)^*$ and $cae(ca)^*=cbe(cb)^*$, and hence ν is a * congruence.

Let $(x, y)\in\nu\cap E\times E$, where E denotes the set of idempotents of S. By Lemma 2, there exist projections e, f, g and h such that $x=ef$ and $y=gh$. Then

$$x=x^2=efef=ef^*e^*eef=e(ef)^*eef=ex^*ex=ey^*ey\in Sy.$$

Similarly $y\in Sx$, and so $x\mathscr{L}y$. Similarly $x\mathscr{R}y$, and hence $x\mathscr{H}y$. Since x and y are both idempotents, we have $x=y$. Thus ν is an idempotent-separating * congruence on S.

Finally, we shall show that ν is the maximum idempotent-separating [*] congruence. Let ρ be any idempotent-separating [*] congruence on S. If $(a, b)\in\rho$, then $(a, b)\in\mathscr{H}$. Since $aa^*\mathscr{R}a$ and $bb^*\mathscr{R}b$, projections aa^* and bb^* are contained in a same $\mathscr{R}$-class. Since each $\mathscr{R}$-class contains one and only one projection, we have $aa^*=bb^*$. Similarly $a^*a=b^*b$. Then $a^*=a^*aa^*=a^*bb^*$ and $b^*=b^*bb^*=a^*ab^*$. Since ρ is a congruence, $(a^*bb^*, a^*ab^*)\in\rho$ and we have $(a^*, b^*)\in\rho$. Using again the fact that ρ is a congruence, $(a^*ea, b^*eb)\in\rho$ and $(aea^*, beb^*)\in\rho$ for all $e\in P$. But ρ is an idempotent-separating congruence, we have that $a^*ea=b^*eb$ and $aea^*=beb^*$ for all $e\in P$. Then $(a, b)\in\nu$, and hence $\rho\subset\nu$.

Let P be the set of projections of a regular $*$ semigroup S. For any $a\in S$, let ρ_a and λ_a be mappings of P into P defined by

$$e\rho_a=a^*ea,$$

$$e\lambda_a=aea^*.$$

It is clear that $\rho_{ab}=\rho_a\rho_b$ and $\lambda_{ab}=\lambda_b\lambda_a$. Let $\mathscr{T}_P^*$ be the dual semigroup of $\mathscr{T}_P$ and denote its product by $\cdot$, that is, $\alpha\cdot\beta=\beta\alpha$ where the right side is the usual product of transformations β and α.

THEOREM 5. *Let $S(P)$ be a regular $*$ semigroup, and let ξ be a mapping of S into $\mathscr{T}_P\times\mathscr{T}_P^*$ defined by $a\xi=(\rho_a, \lambda_a)$. Then ξ is a homomorphism whose kernel is the maximum idempotent-separating congruence on S.*

PROOF. It is obvious ξ is a homomorphism. Let E be the set of idempotents of S. Firstly, suppose that $(e, f)\in\ker\xi\cap(E\times E)$. Then $e=eee=e\rho_e=e\rho_f=fef$. Similarly $f=efe$, and so $e\mathscr{H}f$. Then $e=f$. Thus $\ker\xi$ is an idempotent-separating congruence on S, and hence $\ker\xi\subset\mu$.

Conversely, if $(a, b)\in\mu$, it follows from Theorem 4 that $a^*ea=b^*eb$ and $aea^*=beb^*$ for all $e\in P$. Then $(\rho_a, \lambda_a)=(\rho_b, \lambda_b)$, and hence $(a, b)\in\ker\xi$.

Let P be the set of projections of a regular $*$ semigroup S. Let A, B be subsets of P. A mapping α: $A\to B$ is called a *partial homomorphism* if for $a_1, a_2,\dots, a_n$ in A, $a_1a_2\cdots a_n\in A$ implies that $(a_1\alpha)(a_2\alpha)\cdots(a_n\alpha)\in B$ and $(a_1a_2\cdots a_n)\alpha=(a_1\alpha)(a_2\alpha)\cdots(a_n\alpha)$. If a partial homomorphism α: $A\to B$ is bijective, we call α a *partial isomorphism*. In this case, we say A is partial isomorphic to B and denote it by $A\overset{p}{\cong}B$. For each $e\in P$, let $\langle e\rangle=\{f\in P: f\leq e\}=ePe$. Let $\mathscr{U}=\{(e, f)\in P\times P: \langle e\rangle\overset{p}{\cong}\langle f\rangle\}$ and for each $(e, f)\in\mathscr{U}$ let $T_{e,f}$ be the set of all partial isomorphisms of $\langle e\rangle$ onto $\langle f\rangle$. Let $T_P=\{(\rho_e\alpha, \lambda_f\alpha^{-1}): \alpha\in T_{e,f}, (e, f)\in\mathscr{U}\}$. Notice that $\rho_e=\lambda_e$ for any $e\in P$. For convenience, we shall sometimes denote $(\rho_e\alpha, \lambda_f\alpha^{-1})$ simply by $\phi(\alpha)$. Let a be any element of S. Denote aa^* by e and a^*a by f. It is clear that mappings θ: $x\mapsto a^*xa$ and θ': $y\mapsto aya^*$ are mutually inverse partial isomorphisms of $\langle e\rangle$ onto $\langle f\rangle$ and of $\langle f\rangle$ onto $\langle e\rangle$, respectively. We have easily $(\rho_a, \lambda_a)=(\rho_e\theta, \lambda_f\theta^{-1})$ and hence $S\xi\subset T_P\subset\mathscr{T}_P\times\mathscr{T}_P^*$, where ξ is the homomorphism in Theorem 5.

THEOREM 6. *Let P be the set of projections of a regular $*$ semigroup. Define a unary operation $*$: $T_P\to T_P$ by $(\rho_e\alpha, \lambda_f\alpha^{-1})^*=(\rho_f\alpha^{-1}, \lambda_e\alpha)$. Then we have the followings:*

(i) *T_P is a regular $*$ subsemigroup of $\mathscr{T}_P\times\mathscr{T}_P^*$,*

(ii) *the set of projections of T_P is $\{(\rho_e, \lambda_e): e\in P\}$ and it is partial isomorphic* *to P*,

(iii) *for* $(e, f) \in \mathscr{U}$, $\alpha \in T_{e,f}$ *and* $g \in P$,

$$\phi(\alpha)^*(\rho_g, \lambda_g)\phi(\alpha) = (\rho_{(ege)\alpha}, \lambda_{(ege)\alpha}),$$

(iv) T_P *is fundamental.*

PROOF. (i) Let $(\rho_e\alpha, \lambda_f\alpha^{-1})$, $(\rho_g\beta, \lambda_h\beta^{-1})$ be any elements of T_P. Then the range of $\rho_e\alpha = fPf$ and the range of $\rho_e\alpha\rho_g\beta = \{(gfxfg)\beta : x \in P\}$. Now,

$$gfPfg = gfg(fPf)gfg \subset gfgPgfg = gf(gPg)fg \subset gfPfg,$$

and hence the range of $\rho_e\alpha\rho_g\beta = \{jxj : x \in P\}$, where $j = (gfg)\beta$. We remark that $\beta | \langle gfg \rangle$ is a partial isomorphism of $\langle gfg \rangle$ onto $\langle j \rangle$, with inverse $\beta^{-1} | \langle j \rangle$. Define mappings $\theta_{gfg}: \langle fgf \rangle \rightarrow \langle gfg \rangle$ and $\theta_{fgf}: \langle gfg \rangle \rightarrow \langle fgf \rangle$ by $x\theta_{gfg} = gfgxgfg$ and $y\theta_{fgf} = fgfyfgf$, respectively. It is clear that θ_{gfg} and θ_{fgf} are mutually inverse partial isomorphisms. Let us denote $(fgf)\alpha^{-1}$ by i. It is obvious that $\alpha | \langle i \rangle$ is a partial isomorphism of $\langle i \rangle$ onto $\langle fgf \rangle$, with inverse $\alpha^{-1} | \langle fgf \rangle$. Let $\gamma = (\alpha | \langle i \rangle)(\theta_{gfg} | \langle fgf \rangle) \cdot (\beta | \langle gfg \rangle)$ be a partial isomorphism of $\langle i \rangle$ onto $\langle j \rangle$. We shall show that $\rho_e\alpha\rho_g\beta = \rho_i\gamma$. For any x in P,

$$\begin{aligned} x\rho_i\gamma &= (iexei)\alpha(\theta_{gfg} | \langle fgf \rangle)(\beta | \langle gfg \rangle) \qquad \text{since} \quad i \leqq e, \\ &= (i\alpha)((exe)\alpha)(i\alpha)(\theta_{gfg} | \langle fgf \rangle)(\beta | \langle gfg \rangle) \\ &= (gfgfgf((exe)\alpha)fgfgfg)\beta \\ &= (g((exe)\alpha)g)\beta \qquad \text{since} \quad (exe)\alpha \in fPf, \\ &= x\rho_e\alpha\rho_g\beta. \end{aligned}$$

Thus we have $\rho_e\alpha\rho_g\beta = \rho_i\gamma$. Similarly $(\lambda_f\alpha^{-1}) \cdot (\lambda_h\beta^{-1}) = \lambda_h\beta^{-1}\lambda_f\alpha^{-1} = \lambda_j\gamma^{-1}$, and hence $(\rho_e\alpha, \lambda_f\alpha^{-1})(\rho_g\beta, \lambda_h\beta^{-1}) = (\rho_i\gamma, \lambda_j\gamma^{-1}) \in T_P$. Then we have T_P is a subsemigroup.

Next, we shall show that T_P is a regular $*$ semigroup. It is obvious $(\phi(\alpha)^*)^* = \phi(\alpha)$. Now

$$\begin{aligned} \phi(\beta)^*\phi(\alpha)^* &= (\rho_h\beta^{-1}, \lambda_g\beta)(\rho_f\alpha^{-1}, \lambda_e\alpha) \\ &= (\rho_k\delta, \lambda_m\delta^{-1}), \end{aligned}$$

where $k = (gfg)\beta$, $m = (fgf)\alpha^{-1}$ and $\delta = (\beta^{-1} | \langle k \rangle)(\theta_{fgf} | \langle gfg \rangle)(\alpha^{-1} | \langle fgf \rangle)$. Then we have $k = j$, $m = i$ and $\delta = \alpha^{-1}$, and hence $\phi(\beta)^*\phi(\alpha)^* = (\rho_j\gamma^{-1}, \lambda_i\gamma) = (\rho_i\gamma, \lambda_j\gamma^{-1})^* = (\phi(\alpha)\phi(\beta))^*$. Now

$$\begin{aligned} \phi(\alpha)\phi(\alpha)^*\phi(\alpha) &= (\rho_{f\alpha^{-1}}\alpha\theta_f\alpha^{-1}, \lambda_{f\alpha^{-1}}\alpha\theta_f^{-1}\alpha^{-1})\phi(\alpha) \\ &= (\rho_e\iota_{\langle e \rangle}, \lambda_e\iota_{\langle e \rangle})(\rho_e\alpha, \lambda_f\alpha^{-1}) \end{aligned}$$

$$=(\rho_e\alpha,\ \lambda_f\alpha^{-1})$$
$$=\phi(\alpha).$$

Thus we have that T_P is a regular $*$ subsemigroup of $\mathscr{T}_P\times\mathscr{T}_P^*$.

(ii) It is clear that (ρ_e, λ_e) is a projection for each $e\in P$. Conversely, suppose that $(\rho_e\alpha, \lambda_f\alpha^{-1})$ is a projection of T_P. Since $(\rho_e\alpha, \lambda_f\alpha^{-1})=\phi(\alpha)=\phi(\alpha)^*=(\rho_f\alpha^{-1}, \lambda_e\alpha)$, $fPf=$the range of $\rho_e\alpha=$the range of $\rho_f\alpha^{-1}=ePe$. Then we have $e=f$, and so $\alpha\in T_{e,e}$. Since $\phi(\alpha)$ is an idempotent, $\alpha^2=\alpha$ and so $\alpha=\iota_{\langle e\rangle}$. Therefore $\phi(\alpha)=(\rho_e, \lambda_e)$ and the set of projections of T_P is $\{(\rho_e, \lambda_e): e\in P\}$. It is clear that $\{(\rho_e, \lambda_e): e\in P\}$ is partially isomorphic to P.

(iii) Let $(e, f)\in\mathscr{U}$, $\alpha\in T_{e,f}$ and $g\in P$. Setting $\gamma=(\alpha^{-1}\mid\langle(ege)\alpha\rangle)(\theta_{geg}\mid\langle ege\rangle)$, we have easily that $(geg)\gamma^{-1}=(ege)\alpha$. Then

$$\phi(\alpha)^*(\rho_g, \lambda_g)\phi(\alpha)=(\rho_{(ege)\alpha}\gamma,\ \lambda_{geg}\gamma^{-1})(\rho_e\alpha,\ \lambda_f\alpha^{-1})$$
$$=(\rho_{(ege)\alpha}\psi,\ \lambda_{(ege)\alpha}\psi^{-1}),$$

where $\psi=(\gamma\mid\langle(ege)\alpha\rangle)(\theta_{ege}\mid\langle geg\rangle)(\alpha\mid\langle ege\rangle)=\iota_{\langle(ege)\alpha\rangle}$

$$=(\rho_{(ege)\alpha},\ \lambda_{(ege)\alpha}).$$

(iv) Let μ be the maximum idempotent-separating congruence on T_P, and suppose that $(\rho_e\alpha, \lambda_f\alpha^{-1})\mu(\rho_g\beta, \lambda_h\beta^{-1})$. By Theorem 4, $\phi(\alpha)^*(\rho_i, \lambda_i)\phi(\alpha)=\phi(\beta)^*(\rho_i, \lambda_i)\phi(\beta)$ and $\phi(\alpha)(\rho_i, \lambda_i)\phi(\alpha)^*=\phi(\beta)(\rho_i, \lambda_i)\phi(\beta)^*$ for all $i\in P$. It follows from (iii) above that $(\rho_{(eie)\alpha}, \lambda_{(eie)\alpha})=(\rho_{(gig)\beta}, \rho_{(gig)\beta})$ and $(\rho_{(fif)\alpha^{-1}}, \lambda_{(fif)\alpha^{-1}})=(\rho_{(hih)\beta^{-1}}, \lambda_{(hih)\beta^{-1}})$. Since $\xi: S\to T_P$ is a homomorphism whose kernel is the maximum idempotent-separating congruence on S, we have that $i\rho_e\alpha=(eie)\alpha=(gig)\beta=i\rho_g\beta$ and $i\lambda_f\alpha^{-1}=(fif)\alpha^{-1}=(hih)\beta^{-1}=i\lambda\beta^{-1}$. Then $\rho_e\alpha=\rho_g\beta$ and $\lambda_f\alpha^{-1}=\lambda_h\beta^{-1}$, and hence $\phi(\alpha)=\phi(\beta)$. Thus T_P is fundamental.

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