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SYSTEM OF LOCAL LOOPS ON A MANIFOLD AND AFFINE CONNECTION

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1. Introduction. The concepts of topological loops (Hofmann [2]) and analytic loops (Malcév [5]) lead us to the concept of differentiable local loops (§ 2, Definition 1) and local loops on manifolds have been studied by the author ([3]). Namely, in a differentiable manifold with an affine connection, each point has a neighbourhood which is a differentiable local loop with a binary operation defined by means of the parallel displacement of geodesics ([3] Theorem 1).

In the present paper, differentiable manifold with a system which assigns to each point a neighbourhood with a structure of local loop will be introduced (§ 2, Definition 2) and it will be shown that an affine connection of a manifold is determined by such a system (§ 3, Theorem 1). In particular, it will be proved that if a differentiable manifold M with an affine connection Γ is given then Γ coincides with the affine connection Γ_Σ of M which is determined by the system Σ of local loops associated with Γ (Theorem 2).

2. We shall begin with some definitions about local loops.

DEFINITION 1. A *local loop* $\mathscr{L} = \mathscr{L}(U, f)$ is a pair of a Hausdorff space U and a binary operation $f: (x, y) \longrightarrow f(x, y) \in U$, defined for certain pairs of points x and y in U, satisfying the following conditions :

(1) There exists a point e in U such that $f(x, e) = f(e, x) = x$ hold for all $x \in U$.

(2) If x, y and $f(x, y)$ belong to U, there exists an open set V which contains the point x and such that the point $f(x', y)$ are defined in U for all $x' \in V$, and the mapping : $x' \longrightarrow f(x', y)$, $x' \in V$, is a homeomorphism of V onto an open subset V' of U containing $f(x, y)$.

(3) In the above, there exists also an open set W containing y such that the mapping : $y' \longrightarrow f(x, y')$, $y' \in W$, is defined and it is a homeomorphism of W onto an open subset of U.

The point e in (1) is called the unit of the local loop $\mathscr{L}$ and the point $f(x, y)$, if defined, is called the product of x and y in $\mathscr{L}$ and is denoted by $x \cdot y$.

Let M be a differentiable manifold. [1)]

A *differentiable local loop* in M is a local loop $\mathscr{L}(U, f)$ whose underlying set U is an open set of M and in the equation $z = f(x, y)$ any argument of x, y and z depends

1) The differentiability is always supposed to be of class C^∞ in the rest of the paper.

differentiably on the other two wherever it is defined. A differentiable local loop in M having a point $p \in M$ as its unit may be denoted by $\mathscr{L}_p = \mathscr{L}_p(U_p, f_p)$, and in this case the product $f_p(x, y)$ may be denoted by $x \overset{p}{\cdot} y$. Then the mappings (called a right translation and a left translation of $\mathscr{L}_p$)

$$R_x(p) : y \longrightarrow R_x(p)y = y \overset{p}{\cdot} x, \quad x, y \in U_p,$$

$$L_x(p) : y \longrightarrow L_x(p)y = x \overset{p}{\cdot} y, \quad x, y \in U_p,$$

if defined in $\mathscr{L}_p$, are local diffeomorphisms in M and they bring p to x.

DEFINITION 2. A *system of differentiable local loops* on M is an assignment of a differentiable local loop $\mathscr{L}_p$ to each point $p \in M$, where the underlying set of $\mathscr{L}_p$ is a neighbourhood of p and the unit of $\mathscr{L}_p$ is the point p. It may be denoted by $\Sigma = \{\mathscr{L}_p ; p \in M\}$. It is said to be *differentiable* if the mapping : $(x, y, p) \longrightarrow x \overset{p}{\cdot} y$ is differentiable in the arguments x, y and p as far as such products are defined in M.

3. Let M be a differentiable manifold with an affine connection Γ. Then at each point p of M it is possible to form a local loop $\mathscr{L}_p$ whose underlying set is a neighbourhood of p and whose unit is the point p, defining its binary operation by means of the parallel displacement of geodesic curves in M with respect to Γ ([3] Theorem 1), that is, a system of local loops $\Sigma_\Gamma = \{\mathscr{L}_p ; p \in M\}$ on M is *associated* with Γ. From the process of construction of Σ_Γ, it is seen that the system Σ_Γ is differentiable.

We shall now consider the converse of the above.

THEOREM 1. *Let M be a differentiable manifold. If a differentiable system $\Sigma = \{\mathscr{L}_p ; p \in M\}$ of differentiable local loops is given on M, then it determines an affine connection of M.*

PROOF. At each point $p \in M$, Σ gives a local loop $\mathscr{L}_p = \mathscr{L}_p(U_p, f_p)$ where the underlying set U_p is a neighbourhood of p and the point p is the unit of $\mathscr{L}_p$. Let X_p be a tangent vector to M at p and let $c : t \longrightarrow c(t)$ be a differentiable curve in U_p passing through $p = c(0)$ with the tangent vector X_p at p. The mapping $R_{c(t)}(p)$ (a right translation of $\mathscr{L}_p$, see § 2) maps diffeomorphically a neighbourhood $V(p)$ of p onto a neighbourhood $V'(c(t))$ of $c(t)$, and it induces the tangent linear isomorphism $dR_{c(t)}(p)$ of the tangent space $T_p(M)$ at p onto $T_{c(t)}(M)$ at $c(t)$.

Now, let n denote the dimension of M and let $P(M, G)$ be the bundle of frames over M with structural group $G = GL(n, R)$ and with the projection π. Then the above linear isomorphism $dR_{c(t)}(p)$ defines the mapping $\Phi_{c(t)}$ of the fibre $\pi^{-1}(p)$ over p onto the fibre $\pi^{-1}(c(t))$ over $c(t)$, and

$$(*) \qquad \Phi_{c(t)}(ua) = (\Phi_{c(t)}u)a$$

holds for every $u \in \pi^{-1}(p)$ and $a \in G$. The curve $t \longrightarrow u(t) = \Phi_{c(t)}u$ in P is a differentiable curve through $u = u(0)$, and the tangent vector $Y_u = \frac{d}{dt} u(t) / _{t=0} \in T_u(P)$ to the curve at u depends only on the tangent vector X_p. The mapping $X_p \longrightarrow Y_u$ of the tangent space $T_p(M)$ at p into the tangent space $T_u(P)$ at $u \in \pi^{-1}(p)$ is a linear isomorphism and hence its image Q_u in $T_u(P)$ is a linear subspace of $T_u(P)$.

By the assumption of our theorem the right translations $R_x(p)$ of $\mathscr{L}_p$ depend differentiably on p. Therefore, the distribution $u \longrightarrow Q_u$ is a differentiable distribution on P if Q_u's are assigned to each $u \in P$ with respect to $\mathscr{L}_{\pi(u)}$. The relation (*) shows that this distribution is right invariant. Since $dim.\ Q_u = dim.\ T_{\pi(u)}(M) = n$, we have

$$T_u(P) = G_u + Q_u \qquad \text{(direct sum)}$$

where G_u is the subspace of $T_u(P)$ consisting of the vectors tangent to the fibre through $u \in P$. Thus we have a connection in $P(M, G)$ with Q_u's as its horizontal subspaces, that is, an affine connection of M is obtained and the proof of the theorem is completed.

We shall apply the above considerations to the special case when an affine connection is previously given on the manifold M.

THEOREM 2. *Let M be a differentiable manifold with an affine connection Γ and $\Sigma = \{\mathscr{L}_p; p \in M\}$ the system of local loops associated with Γ. Let Γ_Σ be the affine connection of M given by the system Σ as in* Theorem 1. *Then these affine connections Γ and Γ_Σ coincide with each other.*

PROOF. Let $u \longrightarrow H_u$ be the distribution of the horizontal subspaces with respect to Γ on the frame bundle $P(M, G)$ over M. If the geodesic curve with respect to Γ is chosen as the curve $c(t)$ in the process of construction of Γ_Σ from Σ (in the proof of the Theorem 1), then the mapping $dR_{c(t)}(p)$ means the parallel displacement along $c(t)$ with respect to Γ (see [3], p. 201). Hence $Q_u \subset H_u$ at each point $u \in P$. Since $dim.\ Q_u = dim. H_u = dim. M$, we have $Q_u = H_u$. Therefore Γ_Σ coincides with Γ.

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