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Injective Hulls of Certain Right Reductive Semigroups as Right S-Systems

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Let S be a right reductive semigroup. Then the semigroup S is embedded in the semigroup $\Lambda(S)$ of all left translations of S as its left ideal. Thus we regard S as a left ideal of $\Lambda(S)$. Then $\Lambda(S)$ is an essential extension of S as a right S-system. By Berthiaume [2] there exists the injective hull $I(S)$ of S containing $\Lambda(S)$ as a right S-subsystem. In § 1, we give necessary and sufficient conditions that $\Lambda(S)$ equals $I(S)$. It turns out that both left zero semigroups and right reductive primitive regular semigroups satisfy any one of these conditions. Consequently we show that full transformation semigroups (written on the left) and the direct product of column-monomial matrix semigroups over groups are right self-injective. We also study right non-singular semigroups, semilattices of groups S which satisfy the condition that $\Lambda(S)=I(S)$. In § 2, we state some results on right self-injective semigroups. In particular it is shown that any direct product of right self-injective semigroups with 0 and 1 is right self-injective. Consequently we show that any direct product of self-injective semigroups is self-injective.

§ 1. Injective hulls

Let S be a semigroup (neither necessarily with 0 nor 1), and let M_S be a non-empty set with an operation of S on the right side. Then M_S is called a *right S-system* if $(ms)t=m(st)$ for all $m \in M_S$ and for all $s, t \in S$. If N_S is a non-empty subset of M_S such that N_S is closed under the operation of S on M_S, then N_S is called a *right S-subsystem* of M_S. A mapping ϕ of A_S to B_S, where both A_S and B_S are right S-systems, is called an *S-homomorphism* if $\phi(as)=\phi(a)s$ for all $a \in A_S$ and for all $s \in S$. A right S-system I_S is called *injective* if for any injective S-homomorphism ϕ: $A_S \to B_S$ and for any S-homomorphism ξ: $A_S \to I_S$ there exists an S-homomorphism δ: $B_S \to I_S$ such that $\delta\phi=\xi$. A right S-system I_S is called *weakly injective**) if for any right ideal R of S and for any S-homomorphism α: $R \to I_S$, α is extended to an S-homomorphism β: $S \to I_S$. In particular, S is called *right* [*weakly*] *self-injective* if S is [weakly] injective as a right S-system. Further if S is both left and right [weakly] self-injective, then S is simply called [*weakly*] *self-injective*. Let M_S be a right S-system and N_S a right S-subsystem of M_S. N_S is said to be *essential in* M_S if every S-homomorphism ϕ: $M_S \to K_S$ such that $\phi | N_S$ is injective is injective itself. In this case, M_S is called

*) The concept of "weakly injective" introduced here is defined in a weaker sense than those of [2] and [4].

an essential extension of N_S.

The following result is due to P. Berthiaume [2].

THEOREM 1. *Let* S *be a semigroup, and* M_S *an arbitrary right* S*-system. Then there exists an injective right* S*-system* K_S *such that* K_S *is an essential extension of* M_S *and there exists no injective right* S*-system between* M_S *and* K_S. *In this case,* K_S *is called the injective hull of* M_S *and it is denoted by* $I(M_S)$.

Let S be a right reductive semigroup. Then S is naturally embedded in the semigroup $\Lambda(S)$ of all left translations of S as a left ideal, where the product ab of elements a, b of $\Lambda(S)$ is written as $ab(s)=a(b(s))$ $(s \in S)$. Hereafter we regard S as a left ideal of $\Lambda(S)$, where for $a \in \Lambda(S)$ and for $t \in S$, the product $at=a(t)$ and $ta=b \in \Lambda(S)$, $b(s)=t(a(s))$ $(s \in S)$. Thus the semigroup $\Lambda(S)$ satisfies the following condition: (1.1) For any a, $b \in \Lambda(S)$ $as=bs$ for all $s \in S$ implies $a=b$.

LEMMA 1. *Let* S *be a right reductive semigroup. Then* $\Lambda(S)$ *is an essential extension of* S *as a right* S*-system.*

PROOF. Let K_S be any right S-system and ϕ any S-homomorphism of $\Lambda(S)$ to K_S such that the restriction $\phi \mid S$ of ϕ to S is injective. Let a, b be elements of $\Lambda(S)$ such that $\phi(a)=\phi(b)$. Since $\phi(ax)=\phi(bx)$ for all $x \in S$, it follows that $ax=bx$ for all $x \in S$. We have $a=b$, by (1.1). Therefore $\Lambda(S)$ is an essential extension of S.

LEMMA 2. *Let* S *be a right reductive semigroup and* $\Lambda(S)$ *the semigroup of all left translations of* S. *Then* $\Lambda(S)$ *is the injective hull of* S *as a right* S*-system if and only if* $\Lambda(S)$ *is a right self-injective semigroup. Further in this case,* S *is right weakly self-injective.*

PROOF. The "only if" part: Let $M_{\Lambda(S)}$ be a right $\Lambda(S)$-system which is an essential extension of $\Lambda(S)$ as a right $\Lambda(S)$-system. Consider $M_{\Lambda(S)}$ as a right S-system. Since $\Lambda(S)$ is an injective right S-system, there exists an S-homomorphism $\psi\colon M_{\Lambda(S)} \to \Lambda(S)$ such that $\psi \mid \Lambda(S)=1_{\Lambda(S)}$, where $1_{\Lambda(S)}$ denotes the identity mapping on $\Lambda(S)$. Then it will be seen that ψ is an $\Lambda(S)$-homomorphism: For each $a \in M_{\Lambda(S)}$ and for each $x \in \Lambda(S)$, $\psi(a)(xs)=\psi(a(xs))=\psi((ax)s)=\psi(ax)s$ for all $s \in S$. Hence $\psi(a)x=\psi(ax)$ and hence, ψ is a $\Lambda(S)$-homomorphism. Therefore we have $M_{\Lambda(S)}=\Lambda(S)$. Thus it follows from Theorem 1 that $\Lambda(S)$ is injective as a right $\Lambda(S)$-system, that is, $\Lambda(S)$ is a right self-injective semigroup.

The "if" part: Suppose that $\Lambda(S)$ is a right self-injective semigroup. Let $I(S)$ be the injective hull of S containing $\Lambda(S)$ as a right S-subsystem. Note first that for any m, $n \in I(S)$, $mt=nt$ for all $t \in S$ implies $m=n$. To prove this, define a relation ρ by $a \rho b$ $(a, b \in I(S))$ if and only if $at=bt$ for all $t \in S$. It is clear that ρ is an S-congruence on $I(S)$, that is, $a \rho b$ $(a, b \in I(S))$ implies at ρ bt for all $t \in S$. Then there exists an S-homomorphism ψ: $I(S)\to I(S)/\rho$ defined by $\psi(m)=\rho m$ $(m\in I(S))$, where $I(S)/\rho$ is the factor right S-system of $I(S)$ by ρ and, ρm denotes the ρ-class containing m (see [II, 1, p. 251]). Since $\psi\,|\,S$ is injective and S is essential in $I(S)$, it follows that ψ is injective. This means that ρ is the identity relation on $I(S)$. Hence $mt=nt$ $(m, n\in I(S))$ for all $t\in S$ implies $m=n$. Next, for each $m\in I(S)$ and for each $\lambda\in \Lambda(S)$ define a mapping $\psi_{m,\lambda}$: $S\to I(S)$ by $\psi_{m,\lambda}(s)=m(\lambda s)$ for all $s\in S$. Since $I(S)$ is an injective right S-system and $\psi_{m,\lambda}$ is an S-homomorphism, there exists an S-homomorphism $\tilde{\psi}_{m,\lambda}$: $\Lambda(S)\to I(S)$ such that $\tilde{\psi}_{m,\lambda}\,|\,S=\psi_{m,\lambda}$. It also follows from the above that the existence of $\tilde{\psi}_{m,\lambda}$ is unique. This gives the operation $(\circ)$ of $\Lambda(S)$ on $I(S)$ defined by $m\circ\lambda=\tilde{\psi}_{m,\lambda}(1_S)$ for each $m\in I(S)$ and for each $\lambda\in\Lambda(S)$, where 1_S denotes the identity of $\Lambda(S)$. Then it will be seen that $(m\circ\lambda)\circ\xi=m\circ(\lambda\xi)$ for any $m\in I(S)$ and for any λ, $\xi\in\Lambda(S)$: $((m\circ\lambda)\circ\xi)s=\tilde{\psi}_{m\circ\lambda,\xi}(1_S)s=\psi_{m\circ\lambda,\xi}(s)=(m\circ\lambda)(\xi s)=\tilde{\psi}_{m,\lambda}(1_S)(\xi s)=\psi_{m,\lambda}(\xi s)=m(\lambda(\xi s))$ $=m((\lambda\xi)s)=\psi_{m,\lambda\xi}(s)=\tilde{\psi}_{m,\lambda\xi}(1_S)s=(m\circ(\lambda\xi))s$ for all $s\in S$, equivalently, $(m\circ\lambda)\circ\xi=m\circ(\lambda\xi)$. Thus $I(S)$ becomes a right $\Lambda(S)$-system. It is clear that $m\circ s=ms$ for all $s\in S$ and for all $m\in I(S)$. Since $\Lambda(S)$ is essential in $I(S)$ as right S-systems, we have that $\Lambda(S)$ is essential in $I(S)$ as right $\Lambda(S)$-systems. Since $\Lambda(S)$ is right self-injective, it follows that $\Lambda(S)=I(S)$. In this case, let R be a right ideal of S and ϕ an S-homomorphism of R to S. Since $\Lambda(S)$ is an injective right S-system, there exists an S-homomorphism $\bar{\phi}$: $\Lambda(S)\to\Lambda(S)$ such that $\bar{\phi}(r)=\phi(r)$ for all $r\in R$.

$$\begin{array}{ccc} \Lambda(S) & \xrightarrow{\ \bar{\phi}\ } & \Lambda(S) \\ {\scriptstyle\iota_R}\uparrow & & \uparrow{\scriptstyle\iota_S} \\ R & \xrightarrow[\ \phi\ ]{} & S \end{array}$$

: ι_R, ι_S denote the inclusion mappings of R to $\Lambda(S)$, of S to $\Lambda(S)$, respectively.

Since $\bar{\phi}(S)=\bar{\phi}(1_S)S\subseteq S$, it follows that $\bar{\phi}$ induces an S-homomorphism ϕ^*: $S\to S$ such that $\phi^*\,|\,R=\phi$. Therefore S is right weakly injective.

THEOREM 2. *Let S be a right reductive semigroup, and $\Lambda(S)$ the semigroup of all left translations of S. Let $I(S)$ be the injective hull of S containing $\Lambda(S)$ as a right S-subsystem. Then the following conditions are equivalent:*

(I) *$\Lambda(S)$ is the injective hull of S as a right S-system, that is, $\Lambda(S)=I(S)$.*

(II) *$\Lambda(S)$ is a right self-injective semigroup.*

(III) *For each $m\in I(S)$ $\{x\in S\,|\,mx\in S\}=S$.*

(IV) i) *S is a right weakly self-injective semigroup,*

ii) *for any $m\in I(S)$ and for any pair of distinct elements a, $b\in S$ there exists $x\in S$ such that $mx\in S$ and $ax\neq bx$.*

PROOF. (I)⇔(II): This follows from Lemma 2.

(I)⇒(III): Obvious.

(III)⇒(I): Consider the mapping ϕ: $I(S)\to\Lambda(S)$ defined by $\phi(m)s=ms$ for each $m\in I(S)$ and for each $s\in S$. Then ϕ is an S-homomorphism and $\phi\,|\,\Lambda(S)=1_{\Lambda(S)}$. Since S is essential in $I(S)$ by Theorem 1, we have $\Lambda(S)=I(S)$.

(I)⇒(IV): Condition i) follows from Lemma 2. Condition ii) follows from the assumption that S is right reductive.

(IV)⇒(I): Let $I(S)$ be the injective hull of S containing $\Lambda(S)$ as a right S-subsystem. Define a relation ρ on $I(S)$ as follows: For $a, b \in I(S)$ $a \rho b$ if and only if there exists $m \in I(S)$ such that $ax = bx$ for all $x \in \{x \in S \mid mx \in S\}$. Then by i) $\rho \mid S$ is the identity relation on S. It is easy to check that the relation ρ is an S-congruence on $I(S)$, that is, $a \rho b$ $(a, b \in I(S))$ implies $ax \rho bx$ for all $x \in S$. Note that ρ is the identity relation on $I(S)$. Because that S is essential in $I(S)$. Let m be any element of $I(S)$. Put $\{x \in S \mid mx \in S\} = (m: S)$. If $(m: S)$ is an empty set, then by ii) S is trivial, that is, S is a single-element semigroup. In this case, it is clear that $I(S) = S$. Hence $I(S) = \Lambda(S)$. So we can assume that $(m: S)$ is a right ideal of S. Define a mapping $\psi_m: (m: S) \to S$ by $\psi_m(x) = mx$ for each $x \in (m: S)$. Then ψ_m is an S-homomorphism. Since S is weakly right self-injective, it follows that ψ_m is extended to an S-homomorphism $\bar{\psi}_m: S \to S$. Then $\bar{\psi}_m \in \Lambda(S)$ and $\bar{\psi}_m x = mx$ for all $x \in (m: S)$. This implies that $\psi_m \rho m$. Hence $\bar{\psi}_m = m$. Therefore we have $\Lambda(S) = I(S)$. This completes the proof of the theorem.

REMARK 1. Let S be a right reductive semigroup such that $\Lambda(S) = I(S)$. By Theorem 2 $\Lambda(S)$ is a right self-injective semigroup. Hence $\Lambda(S)$ must have left zero elements. Thus S has left zero elements.

LEMMA 3. *Let S be a semigroup with left zero elements and $I(S)$ the injective hull of S. Then for each $m \in I(S) \setminus S$, $|mS \cap S| \geqq 2$.*

PROOF. Suppose that there exists $m \in I(S) \setminus S$ with $|mS \cap S| \leqq 1$. Then $mS \cap S$ is empty or $mS \cap S$ consists of a left zero element. Define a mapping $\phi: S \cup mS \cup \{m\} \to I(S)$ by $\phi(s) = s$ for each $s \in S$ and $\phi(n) = z_0$ for all $n \in mS \cup \{m\}$, where $\{z_0\} = mS \cap S$ if $|mS \cap S| = 1$ or z_0 is an arbitrarily fixed left zero element of S if $|mS \cap S| = 0$. Since ϕ is an S-homomorphism, it follows that ϕ is extended to an S-homomorphism $\bar{\phi}: I(S) \to I(S)$. Since S is essential in $I(S)$, we obtain that $\bar{\phi}$ is injective. This is a contradiction.

By Theorem 2 and Lemma 3 we have

THEOREM 3.*) *Let S be either a left zero semigroup or a right reductive primitive regular semigroup. Then $\Lambda(S)$ is the injective hull of S as a right S-system and is a right self-injective semigroup.*

PROOF. We first consider the case where S is a left zero semigroup. For any $m \in I(S)$ and for any $s \in S$, there exists $z \in S$ such that $msz \in S$, by Lemma 3. Since S

*) Theorem 3 of this paper has been independently (and in a general form) proved by H. J. Weinert [S-sets and semigroups of quotients, Semigroup Forum, **19** (1980), 1–78].

is left zero, it follows that $ms = msz \in S$. Thus we have $mS \subseteq S$ for all $m \in I(S)$. Therefore S satisfies the condition (III) of Theorem 2. We next consider the case where S is a right reductive primitive regular semigroup. For any $m \in I(S) \backslash S$ and for any $s \in S$, there exists $z \in S$ such that $msz \in S$ and $msz \neq m0$, by Lemma 3. Since S is the 0-disjoint union of completely 0-simple semigroups, it follows that $ms \in mszS \subseteq S$. Thus we have $mS \subseteq S$ for all $m \in I(S)$. Therefore S satisfies the condition (III) of Theorem 2. The theorem follows from Theorem 2.

Immediately we have

COROLLARY 1. ([3, Theorem 7.16]) *Any direct product of column-monomial matrix semigroups over groups is a right self-injective semigroup.*

Let X be a non-empty set and $\mathcal{T}_X$ the full transformation semigroup on X (written on the left side). Given a binary operation ($\circ$) on X with $x \circ y = x$ (x, $y \in X$), the set X with the binary operation ($\circ$) is a left zero semigroup. Then it easily follows that $\Lambda(X) = \mathcal{T}_X$.

By Theorem 3 we have

COROLLARY 2. *Full transformation semigroups on sets (written on left side) are right self-injective.*

Immediately we have

COROLLARY 3. *Every [finite] semigroup is embedded in a right self-injective [finite] regular semigroup.*

THEOREM 4. *Let S be a right reductive completely simple semigroup but not a left zero semigroup. Then $\Lambda(S_0)$*) is the injective hull of S as a right S-system and is a right self-injective semigroup.*

PROOF. Let S be a right reductive completely simple semigroup but not a left zero semigroup. First we shall show that $\Lambda(S_0)$ is an essential extension of S as a right S-system. Let K_S be any right S-system and ϕ any S-homomorphism of $\Lambda(S)$ to K_S such that $\phi \mid S$ is injective. Let x, y be elements of $\Lambda(S_0)$ with $\phi(x) = \phi(y)$. Then it will be seen that for each $s \in S$, $xs \in S$ if and only if $ys \in S$: If $xt \in S$ and $yt = 0$ [$xt = 0$ and $yt \in S$] for some $t \in S$, then $\phi(xts) = \phi(yts) = \phi(yt) = \phi(xt)$ [$\phi(yts) = \phi(xts) = \phi(xt) = \phi(yt)$] implies $xts = xt$ [$yts = yt$] for all $s \in S$. Thus xt [yt] is a left zero element of S. Then it easily follows that S is a left zero semigroup. This is a contradiction. Therefore we have $xs \in S$ if and only if $ys \in S$ ($s \in S$). Thus $\phi(xs) = \phi(ys)$ implies $xs = ys$ for each $s \in S$. Also, $x0 = y0 = 0$, since S has no left zero element. Thus we have $xs = ys$ for all $s \in S_0$. Hence $x = y$ and hence ϕ is injective itself. Thus $\Lambda(S_0)$ is an essential

*) S_0 is the semigroup obtained from S by adjoining with zero.

extension of S. Next we shall show that $\Lambda(S_0)$ is the injective hull of S. Let $I(S)$ be the injective hull of S containing $\Lambda(S_0)$ as a right S-subsystem. We can make $I(S)$ into the right S_0-system with $x0=0$ for all $x \in I(S)$. Then it is clear that $\Lambda(S_0)$ is essential in $I(S)$ as right S_0-systems. By Theorem 3 we have $\Lambda(S_0)=I(S)$, that is, $\Lambda(S_0)$ is the injective hull of S. Also by Theorem 3 $\Lambda(S_0)$ is a right self-injective semigroup. Thus the proof of the theorem is complete.

Let S be a semigroup with 0 and I a right ideal of S. If for any non-zero $x \in S$, there exists $s \in S^1$ *) such that $0 \neq xs \in I$, then the right ideal I is called $\cap$*-large*. While if for each triple of a, b and $c \in S$ with $a \neq b$ there exists $z \in S$ such that $cz \in I$ and $az \neq bz$, then I is called *dense*. A semigroup S with 0 is called *right non-singular* if every $\cap$-large right ideal of S is dense. Of course, a right non-singular semigroup is right reductive.

The latter half of the following theorem was proved by C. Hinkle [4].

THEOREM 5. *Let S be a right non-singular semigroup. Then $\Lambda(S)=I(S)$ if and only if S is a right weakly self-injective semigroup. Further, if S is a regular semigroup, then so is $\Lambda(S)$.*

PROOF. The "only if" part: This follows from Lemma 2. The "if" part: Let $I(S)$ be the injective hull of S containing $\Lambda(S)$ as a right S-subsystem. For each $m \in I(S)$, put $\{x \in S \mid mx \in S\}=(m\colon S)$. Then it easily follows from Lemma 3 that $(m\colon S)$ is an $\cap$-large right ideal of S. Since S is right non-singular, $(m\colon S)$ is dense. Therefore S satisfies the condition (IV) of Theorem 2. By Theorem 2 we have $\Lambda(S)=I(S)$. Further if S is a regular semigroup, then it follows from Theorem 4.2 of [4] that $\Lambda(S)$ is a regular semigroup. The theorem holds.

COROLLARY 4. *Let S be a right non-singular semigroup. Then S is right weakly self-injective and has identity if and only if S is right self-injective.*

We next investigate semilattices of groups which satisfy the condition that $\Lambda(S)=I(S)$.

Let S be a semilattice of groups. It is well known that a) every idempotent of S is in the center of S, b) S is reductive and c) $\Lambda(S)$ is a semilattice of groups. (See [1, I, Lemma 4.8] [5, V. 6.5 Corollary])

THEOREM 6. *Let S be a semilattice of groups and E_S the set of all idempotents of S. Then the following conditions are equivalent:*

(I) $\Lambda(S)=I(S)$.

(II) *$\Lambda(S)$ is a self-injective semigroup which is a semilattice of groups.*

(III) *For each $e \in E_S$ eS is an injective right S-system.*

(IV) *For each $e \in E_S$ eS is a self-injective semigroup.*

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*) S^1 is the semigroup obtained from S by adjoining with identity.

PROOF. (I)$\Leftrightarrow$(II): This follows from Theorem 2 and the above fact c).

(I)$\Rightarrow$(III): Since $A(S)$ is an injective right S-system, there exists the injective hull $I(eS)$ of eS such that $I(eS)\subseteq A(S)$, by Theorem 1. Define a mapping $\psi: I(eS)\to eS$ by $\psi(m)=me$ for each $m\in I(eS)$. By using the above fact a), it is easy to check that ψ is an S-homomorphism. Since $\psi\,|\,eS$ is injective and eS is essential in $I(eS)$, we have that ψ is injective itself. Hence $I(eS)=eS$ and hence eS is an injective right S-system.

(III)$\Rightarrow$(IV): Obvious.

(IV)$\Rightarrow$(III): Let M_S, N_S be right S-systems and ϕ, ξ an injective S-homomorphism of M_S to N_S, an S-homomorphism of M_S to eS, respectively. Consider M_S, N_S as right eS-systems and ϕ, ξ as eS-homomorphisms. Since eS is an injective right eS-system, there exists an eS-homomorphism $\psi: N_S\to eS$ such that $\psi\phi=\xi$. Since $\psi(nt)=\psi(nt)e=\psi(nte)=\psi(n)te=\psi(n)t$ for all $n\in N_S$ and for all $t\in S$, it follows that ψ is an S-homomorphism. Thus it follows that eS is injective as a right S-system.

(III)$\Rightarrow$(I): By Theorem 2, it is sufficient to show that $xS\subseteq S$ for all $x\in I(S)$. Let x be any element of $I(S)$ and e any idempotent of S. Then $xeS\cap S\subseteq eS$. Since eS is injective as a right S-system, there exists an S-homomorphism $\lambda: xeS\cup eS\to eS$ such that $\lambda\,|\,eS=1_{eS}$. It is clear that λ is extended to an S-homomorphism $\delta: xeS\cup S\to S$ such that $\delta\,|\,S=1_S$. Since S is essential in $xeS\cup S$, it follows that δ is injective. Hence $xe\in S$. This implies that $xS\subseteq S$ for all $x\in I(S)$, since S is a semilattice of groups. This completes the proof of the theorem.

§2. Right self-injective semigroups

In general, right self-injective semigroups have left identities and left zero elements, but neither necessarily identity nor zero element (see Theorem 7 below). Right reductive right self-injective semigroups have identity but not necessarily zero element, for example, full transformation semigroups on sets X ($|X|\geqq 2$). Self-injective semigroups have identity and zero element.

THEOREM 7. *Let S be a right simple semigroup. Then S_0 is a right self-injective semigroup.*

PROOF. Let $I(S_0)$ be the injective hull of S_0. Suppose that there is $m\in I(S_0)\backslash S_0$. By Lemma 3 $|mS_0\cap S_0|\geqq 2$. Hence there exists a non-zero $x\in S_0$ such that $mx\in S$. Since S is right simple, we have $mS_0=S_0$. Define a mapping $\psi: S_0\cup\{m\}\to I(S_0)$ by $\psi(s)=s$ for each $s\in S_0$ and $\psi(m)=mz$, where z is an arbitrarily fixed idempotent element of S. Then ψ is well defined and is an S_0-homomorphism. Hence ψ is extended to an S_0-homomorphism $\phi: I(S_0)\to I(S_0)$. This contradicts that S_0 is essential in $I(S_0)$. Therefore we have $I(S_0)=S_0$. The theorem follows.

The following results show that the class of right self-injective semigroups is not closed under taking direct products but the class of right self-injective semigroups with 0 and 1, and the class of self-injective semigroups are closed under taking direct products.

Theorem 8. *Let L, R be a left zero semigroup, a right zero semigroup, respectively. Then (I) $\Lambda(L)$ and R_0 are right self-injective semigroups. (II) If $|L|\geqq 2$, $|R|\geqq 2$, then the direct product $\Lambda(L)\times R_0$ of $\Lambda(L)$ and R_0 is not a right self-injective semigroup.*

Proof. (I): This follows from Theorem 3 and Theorem 7.

(II): Let a, b be distinct elements of L. Then we have, both (a, R_0) and (b, R_0) are right ideals of $\Lambda(L)\times R_0$, and $(a, R_0)\cap(b, R_0)=\square$. Here suppose that $\Lambda(L)\times R_0$ is a right self-injective semigroup. Then there exists a $\Lambda(L)\times R_0$-homomorphism $\psi: (\Lambda(L)\times R_0)^1\to\Lambda(L)\times R_0$ such that $\psi((a, R_0))=\{(a, 0)\}$ and $\psi((b, R_0))=(b, R_0)$. Put $\psi(1)=(c, d)\in\Lambda(L)\times R_0$, where $c\in\Lambda(L)$ and $d\in R_0$. It follows that $dR_0=0$ and $dR_0=R_0$. Hence $R=R_0$ and hence $|R|=1$. This is a contradiction.

Remark 2. In the above theorem, it also follows that $\Lambda(L)\times R_0$ is not the injective hull of $L\times R$ as a right $L\times R$-system.

Theorem 9. *Any direct product $\prod_{\alpha\in\Delta} S_\alpha$ of right self-injective semigroups S_α with 0 and 1 is itself right self-injective.*

Proof. Put $\prod_{\alpha\in\Delta} S_\alpha=T$. Let M_T, N_T be right T-systems and η, ξ an injective T-homomorphism of N_T to M_T, a T-homomorphism of N_T to T, respectively. Then it will be shown that there exists a T-homomorphism $\psi: M_T\to T$ such that $\psi\eta=\xi$: For each $\alpha\in\Delta$, take $e_\alpha\in T$ such that the α-component of e_α is 1 and all the other components of e_α are 0. Then $\xi(N_Te_\alpha)\subseteq Te_\alpha$, $\eta(N_Te_\alpha)\subseteq M_Te_\alpha$ and $Te_\alpha\cong S_\alpha$ (as semigroups) for all $\alpha\in\Delta$. Since Te_α is a right self-injective semigroup, there exists a Te_α-homomorphism $\psi_\alpha: M_Te_\alpha\to Te_\alpha$ such that $\psi_\alpha\eta\mid N_Te_\alpha=\xi\mid N_Te_\alpha$ for each $\alpha\in\Delta$. Define a mapping $\psi: M_T\to T$ by $\psi(m)e_\alpha=\psi_\alpha(me_\alpha)$ for all $m\in M_T$ and for all $\alpha\in\Delta$. Clearly, ψ is well defined. For any $m\in M_T$ and for any $t\in T$, $\psi(mt)e_\alpha=\psi_\alpha(mte_\alpha)=\psi_\alpha(me_\alpha)te_\alpha=\psi(m)te_\alpha$ for all $\alpha\in\Delta$. Thus $\psi(mt)=\psi(m)t$ for all $m\in M_T$ and for all $t\in T$, that is, ψ is a T-homomorphism. For any $n\in N_T$ $(\psi\eta)(n)e_\alpha=\psi(\eta(n))e_\alpha=\psi_\alpha(\eta(n)e_\alpha)=\psi_\alpha(\eta(ne_\alpha))=\xi(ne_\alpha)=\xi(n)e_\alpha$ for all $\alpha\in\Delta$. This implies that $\psi\eta=\xi$. Therefore we have that $T=\prod_{\alpha\in\Delta} S_\alpha$ is right self-injective. The theorem follows.

Corollary 5. *Any direct product of self-injective semigroups is itself self-injective.*

The following result gives examples of commutative self-injective semigroups which are not semilattices of groups.

THEOREM 10. *Let* $S=\langle x\rangle$ *be a cyclic semigroup. Then* $S_0^{1\,*)}$ *is a self-injective semigroup if and only if* $|S|$ *is finite.*

PROOF. The "only if" part: Suppose that $S=\langle x\rangle$ is an infinite cyclic semigroup. Then we have $x^m \neq x^n$ for each pair of distinct integers m, n. Define a mapping ϕ: $x^2S_0^1 \to S_0^1$ by $\phi(x^2s)=xs$ for all $s\in S_0^1$. Then ϕ is an S_0^1-homomorphism. Since S_0^1 is self-injective, ϕ can be extended to an S_0^1-homomorphism ψ: $S_0^1 \to S_0^1$. Then we have $\psi(1)x^2=x$. This implies that $x=x^n$ $(n\geqq 2)$. This is a contradiction. Therefore $|S|$ is finite.

The "if" part: Let $I(S_0^1)$ be the injective hull of S_0^1. Suppose that there is $m\in I(S_0^1)\backslash S_0^1$. Since $I(S_0^1)$ is an essential extension of S_0^1, it easily follows that $m1=m$. Since $|S|$ is finite, there exists an integer n such that $mx^n\in S_0^1$ and $mx^{n-1}\notin S_0^1$, where $x^0=1$ if necessary. So we can assume without a loss of generality that $m\notin S_0^1$ and $mx\in S_0^1$. By Lemma 3, $mx\neq 0$. Since $|mxS_0^1|<|S_0^1|$, $mx=x^n$ for some positive integer n. Define a mapping η: $S_0^1 \cup mS_0^1 \to I(S_0^1)$ by $\eta(s)=s$ for all $s\in S_0^1$ and $\eta(ms)=x^{n-1}s$ for all $s\in S_0^1$. Then η is an S_0^1-homomorphism and is not injective. But $\eta\,|\,S_0^1=1_{S_0^1}$. This contradicts that S_0^1 is essential in $I(S_0^1)$. The proof of the theorem is complete.

In contrast with Ring theory, it is natural to ask whether the semigroup of a complete $n\times n$-matrix ring over a field is self-injective or not. The following example shows that the answer is in the negative:

EXAMPLE. Let Q be the field of rational integers and $M_2(Q)$ the complete 2×2-matrix ring over Q. For brevity, let S denote the multiplicative semigroup of the ring $M_2(Q)$. Let $S=\left\{\begin{bmatrix} a & c \\ b & d\end{bmatrix};\ a,\ b,\ c \text{ and } d\in Q\right\}$, $f=\begin{bmatrix} 2 & 2 \\ -1 & -1\end{bmatrix}$, $e_1=\begin{bmatrix} 1 & 0 \\ 0 & 0\end{bmatrix}$ and $e_2=\begin{bmatrix} 0 & 0 \\ 0 & 1\end{bmatrix}$. Then f, e_1 and e_2 are idempotents and $fS\cap(e_1S\cup e_2S)=0$. Here suppose that S is right self-injective. Then there exists an S-homomorphism ψ: $S\to S$ such that $\psi(f)=0$, $\psi(e_1)=e_1$ and $\psi(e_2)=e_2$. Put $\psi(1)=\begin{bmatrix} x & y \\ u & v\end{bmatrix}$. Since $\begin{bmatrix} x & y \\ u & v\end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0\end{bmatrix}=\begin{bmatrix} x & 0 \\ u & 0\end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 0\end{bmatrix}$ and $\begin{bmatrix} x & y \\ u & v\end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1\end{bmatrix}=\begin{bmatrix} 0 & y \\ 0 & v\end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 0 & 1\end{bmatrix}$, we have $\begin{bmatrix} x & y \\ u & v\end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1\end{bmatrix}$. This contradicts that $\psi(f)=\psi(1)f=0$. Therefore S is not right self-injective. Dually S is not left self-injective.

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*) S_0^1 is the semigroup obtained from S by adjoining with zero and identity.

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