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The estimate procedures after a preliminary multivariate nonparametric two-sample test

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§ 1. Introduction.

The sometimes pool procedures, which have been developed in the normal theory by Bancroft [3], Kitagawa [5] and many other authors, have been also discussed in the univariate nonparametric cases by the author [7], [8] and [9]. Moreover Asano-Sato [1] and Sato [2] have attempted some multivariate extensions in the normal theory. We are now in the situation to consider the multivariate extensions in the nonparametric case along the line of the previous papers of the author.

Our concern in this paper is the estimate procedure for the location parameters—(A) median vector and (B) shift vector—. The preliminary nonparametric test is performed by a multivariate two-sample statistic of Wilcoxon type which has been proposed by Sugiura [6]. As for the estimates of the location vectors, we use those of Hodges-Lehmann type [4] based on some rank tests.

(A) Let O_{n_1} : $\{(X_{ki},\ k=1,\dots,p)\}$ $i=1,\dots,n_1$ be a random sample of size n_1 from a continuous p-variate distribution $F(\boldsymbol{x}-\boldsymbol{\theta}_1)$ with continuous marginal distribution $F_k\ (x-\theta_{k1})$ of the k th component X_k and continuous joint marginal distribution $F_{ij}\ (x-\theta_{i1}, y-\theta_{j1})$ of the i th and j th components where $\boldsymbol{X}'=(X_1,\dots,X_p)$, $\boldsymbol{\theta}_1=(\theta_{11},\dots,\theta_{p1})$. We also assume that $F_k\ (x)$ with density $f_k\ (x)$ be symmetrical about origin for all k. We consider the case where there exists another random sample O_{n_2} : $\{(Y_{kj},\ k=1,\dots,p)\}$ $j=1,\dots n_2$ of size n_2 from the distribution $F(\boldsymbol{y}-\boldsymbol{\theta}_2)$ Where $\boldsymbol{Y}'=(Y_1,\dots,Y_p)$, $\boldsymbol{\theta}_2=(\theta_{12},\dots,\theta_{p2})$.

We here assumed that the distributions of the random vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$ are of same type except only median vector. We shall apply sometimes pool methods to estimate the median vector $\boldsymbol{\theta}_1$.

(B) Let O_{n_1} : $\{X_{kj},\ k=1,\dots,p)\}$ $j=1,\dots,n_1$, O_{n_2} : $\{(Y_{kj},\ k=1,\dots,p)\}$ $j=1,\dots,n_2$ and O_{n_3} : $\{(Z_{kj},\ k=1,\dots,p)\}$ $j=1,\dots,n_3$ be respectively three random samples of size n_i $i=1$, 2, 3 from the continuous distributions $F(\boldsymbol{x})$, $F(\boldsymbol{y}-\boldsymbol{\Delta}_1)$ and $F(\boldsymbol{z}-\boldsymbol{\Delta}_2)$ where $\boldsymbol{\Delta}_j=(\Delta_{1j},\dots,\Delta_{pj})$ $j=1,2$.

Our purpose is to estimate the value of the shift vector $\boldsymbol{\Delta}_1$ based on the samples O_{ni}. Sometimes pool procedure is also applied.

§ 2. A preliminary test and the estimates.

We first deal with the problem of the type (A).

Definition 2.1.

(2.1) $$W = 12\lambda(1-\lambda)N\sum_{i=1}^{p}\sum_{j=1}^{p}\tau^{ij}\ (\boldsymbol{U}_i-\tfrac{1}{2})\ (\boldsymbol{U}_j-\tfrac{1}{2})$$

where

(2.2) $$\boldsymbol{U}_i = U(X_i, Y_i) = (n_1n_2)^{-1}\sum_{\alpha=1}^{n_1}\sum_{\beta=1}^{n_2}\Phi(X_{i\alpha}, Y_{i\beta})$$

$$\Phi(x,y) = \begin{cases} 1 & \text{for} \quad x<y \\ 0 & \text{otherwise} \end{cases}$$

(2.3) $$\tau_{ij} = \begin{cases} 1 & i=j \\ 12\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_i(x)F_j(y)dF_{ij}(x,y)-3 & i\neq j \end{cases}$$

$$\left\| \tau^{ij} \right\| = \left\| \tau_{ij} \right\|^{-1} i,j=1,\ldots,p\ ,\quad N=n_1+n_2,\ n_1/N=\lambda$$

(2.4) $$W = 12\lambda\ (1-\lambda)N\sum_{i=1}^{p}\sum_{j=1}^{p}\hat{\tau}^{ij}\ (\boldsymbol{U}_i-\tfrac{1}{2})(\boldsymbol{U}_j-\tfrac{1}{2})$$

where $\left\| \hat{\tau}^{ij} \right\| = \left\| \hat{\tau}_{ij} \right\|^{-1}\ i, j=1,\ldots,p$

(2.5) $$\hat{\tau}_{ij} = \begin{cases} 1 & i=j \\ 12\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_i^{(N)}(x)F_j^{(N)}(y)dF_{ij}^{(N)}(x,y)-3 & i\neq j \end{cases}$$

is a consistent estimator of τ_{ij} where $F_k^{(N)}(x)$ and $F_{ij}^{(N)}(x,y)$ be respectively the sample distribution of the k th component of $(\boldsymbol{X},\boldsymbol{Y})$ and the i th and j th components.

The statistic $\hat{W}$ is used to test the hypothesis $H:\boldsymbol{\theta}_1=\boldsymbol{\theta}_2$ in the preliminary step. The following theorem has been proved by Sugiura [6]

Theorem 2.1.

Under H (or $K:\boldsymbol{\theta}_2=\boldsymbol{\theta}_1+\gamma/\sqrt{N}$), the statistics $\hat{W}$ and W are asymptotically equivalent and their distribution is asymptotically central (or non-central) χ^2-distribution with the degree of freedom p.

We use the following $\tilde{\theta}_j$ or $\hat{\theta}_j$ as the estimate of θ_j based on O_{n_j},

(2.6)
$$\tilde{\boldsymbol{\theta}}_1 = \tilde{\boldsymbol{X}} = (\tilde{X}_1,\ldots,\tilde{X}_p) \qquad \tilde{\boldsymbol{\theta}}_2 = \tilde{\boldsymbol{Y}} = (\tilde{Y}_1,\ldots,\tilde{Y}_p)$$
$$\hat{\boldsymbol{\theta}}_1 = \hat{\boldsymbol{X}} = (\hat{X}_1,\ldots,\hat{X}_p) \qquad \hat{\boldsymbol{\theta}}_2 = \hat{\boldsymbol{Y}} = (\hat{Y}_1,\ldots,\hat{Y}_p)$$

where

$$\tilde{X}_k = \text{med}\{X_{k\alpha},\ 1\leq\alpha\leq n_1\}$$

(2.7) $$\hat{X}_k = \text{med}\{(X_{k\alpha}+X_{k\beta})/2,\ 1\leq\alpha\leq\beta\leq n_1\}$$

and $\tilde{Y}_k$, $\hat{Y}_k$ are also defined similarly from O_{n_2}.
Then we may define the sometimes pool estimate of $\boldsymbol{\theta}_1$ as follows,

Definition 2.2.

(2.8)
$$\tilde{\boldsymbol{\theta}} = \begin{cases} \lambda\tilde{\boldsymbol{\theta}}_1 + (1-\lambda)\tilde{\boldsymbol{\theta}}_2 & \text{when } W \leqq x_p^2(\alpha) \\ \tilde{\boldsymbol{\theta}}_1 & \text{otherwise} \end{cases}$$

$$\hat{\boldsymbol{\theta}} = \begin{cases} \lambda\hat{\boldsymbol{\theta}}_1 + (1-\lambda)\hat{\boldsymbol{\theta}}_2 & \text{when } W \leqq x_p^2(\alpha) \\ \hat{\boldsymbol{\theta}}_1 & \text{otherwise} \end{cases}$$

where $x_p^2(\alpha)$ means the upper α-percent point of χ^2-distribution with the degree of freedom p.

When $\boldsymbol{\theta}_2 = \boldsymbol{\theta}_1 +$ constant vector, the W-test rejects the hypothesis H almost certainly because the W-test is consistent and hence $\tilde{\boldsymbol{\theta}}$ or $\hat{\boldsymbol{\theta}}$ becomes equivalent to the never pool estimate $\tilde{\boldsymbol{\theta}}_1$ or $\hat{\boldsymbol{\theta}}_1$. Thus we may consider only the case $K : \boldsymbol{\theta}_2 = \boldsymbol{\theta}_1 + \boldsymbol{\gamma}/\sqrt{N}$.

Secondly we consider the shift problem of the type (B). The test atatistic in the preliminary step

$$W' = 12\lambda_{23}(1-\lambda_{23}) N_{23} \sum_{i=1}^{p}\sum_{j=1}^{p} \tau_{ij} (U_i - \tfrac{1}{2})(U_j - \tfrac{1}{2})$$

where U_i denotes $U(Y_i, Z_i)$ and $n_i + n_j = N_{ij}$, $n_i / N_{ij} = \lambda_{ij}$ The following estimates are used for the shift vectors

(2.9) $\quad \hat{\boldsymbol{\Delta}}_j = (\hat{\Delta}_{1j}, \ldots, \hat{\Delta}_{pj}) \qquad j = 1,2.$

where

(2.10) $\quad \hat{\Delta}_{k_1} = \text{med}(Y_{k\beta} - X_{k\alpha}) \qquad 1 \leqq \alpha \leqq n_1,\ 1 \leqq \beta \leqq n_2$

$\qquad \hat{\Delta}_{k_2} = \text{med}(Z_{k\alpha} - X_{k\alpha}) \qquad 1 \leqq \alpha \leqq n_3$

Definition 2.3.

Sometimes pool estimate $\hat{\boldsymbol{\Delta}}$ for $\boldsymbol{\Delta}_1$ is defined by the following

(2.11)
$$\hat{\boldsymbol{\Delta}} = \begin{cases} \lambda_{23}\hat{\boldsymbol{\Delta}}_1 + (1-\lambda_{23})\hat{\boldsymbol{\Delta}}_2 & \text{when } W \leqq x_p^2(\alpha) \\ \hat{\boldsymbol{\Delta}}_1 & \text{otherwise} \end{cases}$$

We also discuss only the case $\boldsymbol{\Delta}_2 = \boldsymbol{\Delta}_1 + \boldsymbol{\gamma}/\sqrt{N}$.

§ 3. The asymptotic properties of $\tilde{\boldsymbol{\theta}}$ and $\boldsymbol{\theta}$.

3.1. Some Lemmas.

Definition 3.1. We define the following random vectors,

(3.1) $\quad \tilde{\boldsymbol{\xi}}' = (2\sqrt{n_1} f_k(o)(X_k - \theta_{k_1}) \qquad k=1,\ldots,p)$

(3.2) $\quad \tilde{\boldsymbol{\eta}}' = (2\sqrt{n_2} f_k(o)(Y_k - \theta_{k_2}) \qquad k=1,\ldots,p)$

(3.3) $\quad U' = (\sqrt{sN}(U_k - \mu_{X_k, Y_k}), \ k=1,\ldots,p)$

where $\mu_{X_k, Y_k} = EU(X_k, Y_k)$, $s = 12\lambda(1-\lambda)$

(3.4) $\quad \tilde{\boldsymbol{p}} = -\sqrt{1-\lambda}\ \tilde{\boldsymbol{\xi}} + \sqrt{\lambda}\ \tilde{\boldsymbol{\eta}}$

(3.5) $\quad \tilde{\boldsymbol{q}} = \sqrt{\lambda}\ \tilde{\boldsymbol{\xi}} + \sqrt{1-\lambda}\ \tilde{\boldsymbol{\eta}}.$

Lemma 3.1. Under H or K,

(i) $Cov\ (2\sqrt{n_1} f_i\ (o)\ \tilde{X}_i,\ 2\sqrt{n_1} f_j\ (o)\ \tilde{X}_j)$

$\sim Cov\ (2\sqrt{n_2} f_i\ (o)\ \tilde{Y}_i,\ 2\sqrt{n_2} f_j\ (o)\ \tilde{Y}_j)$

$$\sim \rho_{ij} = \begin{cases} 4F_{ij}\ (o,o) - 1 & i \neq j \\ 1 & i = j \end{cases}$$

(ii) $\text{Cov}\ (\sqrt{sN}\ U_i,\ \sqrt{sN}\ U_j) \sim \tau_{ij}$

$$= \begin{cases} 12 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F_i\ (x)\ F_j\ (y)\ dF_{ij}\ (x,y) - 3 & i \neq j \\ 1 & i = j \end{cases}$$

(iii) $\text{Cov}\ (2\sqrt{n_1}\ f_i\ (o)\ \tilde{X}_i,\ \sqrt{sN}\ U_j) \sim -\sqrt{3(1-\lambda)}\ \omega_{ij}/2$

where

$$\omega_{ij} = \begin{cases} 2 - 4\text{P}_o\ (o < X_i, X_j < Y_j) & i \neq j \\ 1 & i = j \end{cases}$$

P_o expresses the probability under $\theta_j = \mathbf{0}$

(iv) $\text{Cov}\ (2\sqrt{n_2}\ f_1\ (o)\ \tilde{Y}_i,\ \sqrt{sN}\ U_j) \sim \sqrt{3\lambda}\ \omega_{ij}/2$

Proof.

For (i) , let

$$\text{h}\ (X) = n_1^{-1} \sum_{j=1}^{n_1} s(X_j), \quad \text{where} \quad s\ (x) = \begin{cases} 1 & \text{for} \quad x \geqq o \\ o & \text{otherwise} \end{cases}$$

Then

$$P[\sqrt{n_1}\ (\tilde{X}_i - \theta_{i1}) < a,\ \sqrt{n_1}\ (\tilde{X}_j - \theta_{j1}) < b]$$

$$= P_o\ [h\ (X_1 - a/\sqrt{n_1}) < 1/2,\ h\ (X_j - b/\sqrt{n_1}) < 1/2]$$

$$\sim P_o\ [2\sqrt{n_1}\ \{h\ (X_i - a/\sqrt{n_1}) - E_o h\ (X_i - a/\sqrt{n_1})\} < 2af_i\ (o),$$

$$2\sqrt{n_1}\ \{h\ (X_j - b/\sqrt{n_1}) - E_o h\ (X_j - \text{b}/\sqrt{n_1})\} < 2bf_j\ (o)]$$

$$\sim \int_{-\infty}^{2af_i\ (o)} \int_{-\infty}^{2bf_j\ (o)} n\ (\boldsymbol{x} : \mathbf{0},\ \Sigma)\ dx_i\ dx_j$$

where $n\ (\boldsymbol{x}\,;\mathbf{0},\ \boldsymbol{\Sigma})$ means the multivariate normal density of the random vector $\boldsymbol{x}$ with mean vector $\mathbf{0}$ and covariance matrix $\boldsymbol{\Sigma}$. To derive $\boldsymbol{\Sigma}$, we compute as follows,

$$n_1 E_o\ h\ (X_i - a/\sqrt{n_1}) h(X_j - b/\sqrt{n_1}) = n_1^{-1} E_o \sum_{\alpha=1}^{n_1} s\ (X_{i\alpha} - a/\sqrt{n_1}) \sum_{\beta=1}^{n_1} s\ (X_{j\beta} - b/\sqrt{n_1})]$$

$$\sim P_o\ (o < X_i,\ o < X_j) + (n_1 - 1)\ P_o\ (\theta < X_i)\ P_o\ (\theta < X_j)$$

where E_o and Cov_o mean the covariance and expectation respectively under $\boldsymbol{\theta}_j = \mathbf{0}$.
Hence we get

$$\text{Cov}_o\ (2\sqrt{n_1}\ h\ (X_i - a/\sqrt{n_1}),\ 2\sqrt{n_1}\ h\ (X_j - b/\sqrt{n_1}) \sim 4F_{ij}\ (o,o) - 1$$

Thus it follows that

(3.6) $\boldsymbol{\Sigma} = \| \rho_{ij} \| \quad i, j = 1,\ldots,p$

For (iii),

$$P[\sqrt{n_1}(\widetilde{X}_i - \theta_{i1}) < a, \sqrt{N}\ (U_j - \mu_{X_j, Y_j}) < b]$$

$$\sim P_o[2\sqrt{n_1}\{h(X_i - a/\sqrt{n_1}) - E_o h(X_i - a/\sqrt{n_1})\} < 2af_i(o), \sqrt{sN}(U_j - \mu_{X_j,Y_j}) < \sqrt{s}\, b]$$

$$\sim \int_{-\infty}^{2af_i(o)} \int_{-\infty}^{\sqrt{s}b} n(\boldsymbol{x}; \boldsymbol{0}, \boldsymbol{\Sigma}_1)\, dx_i\, dx_j$$

where

$$\boldsymbol{\Sigma}_1 = \left\| \begin{matrix} 1 & -\sqrt{3(1-\lambda)}\ \omega_{ij}/2 \\ -\sqrt{3(1-\lambda)}\ \omega_{ij}/2 & 1 \end{matrix} \right\|$$

In fact,

$$\mathrm{Cov}_o(2\sqrt{n_1}\ h(X_i - a/\sqrt{n_1}), \sqrt{sN}U_j)$$

$$\sim 2\sqrt{n_1 Ns}\left[(n_1^2 n_2)^{-1} E_o\left\{\sum_\alpha s(X_{i\alpha} - a/\sqrt{n_1}) \sum_{\beta,\gamma} \phi(X_{j\beta}, Y_{j\gamma})\right\} - 1/4\right]$$

$$\sim 4\sqrt{3(1-\lambda)}\ [P_o(0 < X_i, X_j < Y_j) - 1/4]$$

$$= -\sqrt{3(1-\lambda)}\ \omega_{ij}/2.$$

The remainings are easily shown by the same techniques.

Corollary 3.1.

(3.7) $E(\widetilde{\boldsymbol{q}}\boldsymbol{U}') = \boldsymbol{0}$

(3.8) $E(\widetilde{\boldsymbol{q}}\widetilde{\boldsymbol{p}}') = \boldsymbol{0}$

(3.9) $E(\widetilde{\boldsymbol{p}}\boldsymbol{U}') = \frac{\sqrt{3}}{2} \| \omega_{ij} \| \quad i,j=1,2,\ldots,p = \Omega$

Proof. Using Lemma 3.1,

$$E(\widetilde{\boldsymbol{q}}\boldsymbol{U}') = \sqrt{\lambda}\ E(\widetilde{\boldsymbol{\xi}}\boldsymbol{U}') + \sqrt{1-\lambda}\, E(\widetilde{\boldsymbol{\eta}}\boldsymbol{U}')$$

$$= \| 0 \| = \boldsymbol{0}$$

$$E(\widetilde{\boldsymbol{p}}\boldsymbol{U}) = -\sqrt{1-\lambda}\ E(\widetilde{\boldsymbol{\xi}}\boldsymbol{U}') + \sqrt{\lambda}\ E(\widetilde{\boldsymbol{\eta}}\boldsymbol{U}')$$

$$= \| \sqrt{3}\ (1-\lambda)\,\omega_{ij}/2 + \sqrt{3}\,\lambda\,\omega_{ij}/2 \| = \| \sqrt{3}\ \omega_{ij}/2 \|$$

Similarly, we also get $E(\widetilde{\boldsymbol{q}}\widetilde{\boldsymbol{p}}') = \boldsymbol{0}$

Definition 3.2. We define the following random vectors,

(3.10) $\hat{\boldsymbol{\xi}} = [\sqrt{12n_1}\, g_k\ (X_k - \theta_{k1}), \quad k=1,\ldots,p]$

(3.11) $\hat{\boldsymbol{\eta}} = [\sqrt{12n_2}\, g_k\ (Y_k - \theta_{k2}), \quad k=1,\ldots,p]$

where

$$g_k = \int_{-\infty}^{\infty} f_k^2\ (x)\, dx$$

(3.12) $\hat{\boldsymbol{q}} = \sqrt{\lambda}\ \hat{\boldsymbol{\xi}} + \sqrt{1-\lambda}\ \hat{\boldsymbol{\eta}}$

(3.13) $\hat{\boldsymbol{p}} = -\sqrt{1-\lambda}\ \hat{\boldsymbol{\xi}} + \sqrt{\lambda}\ \hat{\boldsymbol{\eta}}$

Lemma 3.2.

(i) $\mathrm{Cov}\,(\sqrt{12n_1}\, g_i\, \hat{X}_i,\ \sqrt{12n_1}\, g_j\, \hat{X}_j) \sim \mathrm{Cov.}\,(\sqrt{12n_2}\, g_i\, \hat{Y}_i,\ \sqrt{12n_2}\, g_j\, \hat{Y}_j)$
$\sim \tau_{ij}$

(ii) $\mathrm{Cov}\,(\sqrt{12n_1}\, g_i\, \hat{X}_i,\ \sqrt{sN}\ U_j) \sim -\sqrt{1-\lambda}\ \tau_{ij}$

(iii) $\mathrm{Cov}\,(\sqrt{12n_2}\ g_i\ \hat{Y}_i,\ \sqrt{sN}\ U_j) \sim \sqrt{\lambda}\ \tau_{ij}$

Proof. We shall prove only (ii). Let

$$V(X_i) = \binom{n_1+1}{2}^{-1} \sum_{1\leqq\alpha\leqq\beta\leqq n_1} \psi\ (X_{i\alpha}, X_{i\beta})$$

$$\psi\,(x, y) = \begin{cases} 1 & \text{for } x+y>0 \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$P[\sqrt{n_1}\ (\hat{X}_i - \theta_{i1}) < a,\ \sqrt{N}\ (U_j - \mu_{X_jY_j}) < u]$$
$$= P_o\,[\sqrt{n_1}\ \{V(X_i - a/\sqrt{n_1}) - E_o\, V(X_i - a/\sqrt{n_1})\} < 2ag_i, \sqrt{N}(U_j - \mu_{X_jY_j}) > u]$$

Now,

$$\mathrm{Cov}_o\ [\sqrt{n_1}\ V(X_i - \mathrm{a}/\sqrt{n_1}),\ \sqrt{N}\,U_j] \sim \sqrt{Nn_1}\left[E_o\,\{2\,(n_1^3\ n_2)^{-1} \sum_{\alpha,\beta} \phi\ (X_{j\alpha},\ Y_{j\beta})\right.$$
$$\left.\sum_{\gamma,\delta} \psi\ (X_{i\gamma}, X_{i\delta}\} - \tfrac{1}{4}\right] \sim \sqrt{Nn_1}\ (n_1n_2)^{-1}\Big[-2n_2P_o\ (X_{i\gamma} + X_{i\delta} > o, X_{j\alpha} < Y_{j\beta}) + 2n_2P_o$$
$$(X_{i\gamma} + X_{i\delta} > o,\ X_{j\alpha} < Y_{j\beta}\Big]\ \delta \sim -\ \tau_{ij}/6\ \sqrt{\lambda}.$$

where $\alpha \neq \gamma,\ \delta,\ \beta \neq \gamma$,

Hence we get

$$\mathrm{Cov}\,(\sqrt{12n_1}\ g_i\ \hat{X}_i,\ \sqrt{sN}\ U_j) \sim \mathrm{Correlation}_o\ [\sqrt{n_1}\ V(X_i - a/\sqrt{n_1}),\ \sqrt{N}\ U_j]$$
$$\sim (-\tau_{ij}/6\sqrt{\lambda})\ \sqrt{s}\ \sqrt{3} = -\sqrt{1-\lambda}\,\tau_{ij}.$$

Corollary 3.2.

(3.14) $E\,(\hat{\boldsymbol{q}}\boldsymbol{U}') = \boldsymbol{0}$

(3.15) $E\,(\hat{\boldsymbol{q}}\hat{\boldsymbol{p}}') = \boldsymbol{0}$

(3.16) $E\,(\hat{\boldsymbol{p}}\boldsymbol{U}) = \boldsymbol{T}\ (= \|\tau_{ij}\|)$

(3.17) $E\,(\hat{\boldsymbol{p}}\hat{\boldsymbol{p}}') = \boldsymbol{T}$

Proof. We prove only (3.17), for the remainings are similar.

$$E(\hat{\boldsymbol{p}}\hat{\boldsymbol{p}}') = (1-\lambda)\,E\,(\hat{\boldsymbol{\xi}}\,\hat{\boldsymbol{\xi}}') + \lambda\ E(\hat{\boldsymbol{\eta}}\hat{\boldsymbol{\eta}}') - \sqrt{\lambda\,(1-\lambda)}\ [E(\hat{\boldsymbol{\xi}}\,\hat{\boldsymbol{\eta}}') + E(\hat{\boldsymbol{\eta}}\,\hat{\boldsymbol{\xi}}')]$$
$$= \|\ \tau_{ij}\ \|$$

where we used $E\,(\hat{\boldsymbol{\xi}}\hat{\boldsymbol{\eta}}') = E\,(\hat{\boldsymbol{\eta}}'\hat{\boldsymbol{\xi}}) = \boldsymbol{0}$.

Lemma 3.3. If $|\,\boldsymbol{T}\,| \neq 0$, the random vector $\hat{\boldsymbol{p}}$ is asymptotically equivalent to the random vector $\boldsymbol{U}$ with probability 1.

Proof. From Corollary 3.2, the covariance matrix of the random vector $(\hat{\boldsymbol{p}},\ \boldsymbol{U})$ is given by

$$\left\|\begin{matrix} \boldsymbol{T} & \boldsymbol{T} \\ \boldsymbol{T} & \boldsymbol{T} \end{matrix}\right\|$$

with rank p under $|T| \neq 0$. Hence we have the following relation with probability 1

$$A\hat{p} = BU$$

where A, B are some nonsingular matrix of order p. By multiplying $\hat{p}'$ and taking the expectation, we get the result.

3.2. The mean and the mean square error.

We first discuss about the estimate based on the sample median· Denoting that

(3.18) $\tilde{F} = \| \delta_{ij} / 2\sqrt{N}\ f_i\ (o) \| \quad i,j=1,\ldots,p$

$$\delta_{ij} = 1\ (o) \qquad \text{for } i=j \text{ (otherwise)}$$

$$\tilde{M} = \lambda\theta_1 + (1-\lambda)\,\theta_2,\ G = \sqrt{(1-\lambda)/\lambda},$$

then we get

(3.19) $\lambda\tilde{X} + (1-\lambda)\,\tilde{Y} = \tilde{M} + \tilde{F}\tilde{q}$

$$= \theta_1 + (1-\lambda)\,(\theta_2 - \theta_1) + \tilde{F}\tilde{q}$$

(3.20) $X = \theta_1 + Fq - GFp.$

Theorem 3.1. The mean vector $E\,(\tilde{\theta})$ and mean square error M. S. E. $(\tilde{\theta})$ of the estimate $\tilde{\theta}$ are given by the following

(3.21) $E\,(\tilde{\theta}) = \theta_1 + O\,(1/\sqrt{N}\,)$

(3.22) $\text{M.S.E.}\,(\tilde{\theta}) = \boldsymbol{\Pi} + (1-\lambda)\boldsymbol{\gamma}\boldsymbol{\gamma}' P[W < x_p^2\ (\alpha)\,]/N + G^2 I_{D_2}\,(\tilde{F}\tilde{p}\tilde{p}'\tilde{F}')$

where

$$\boldsymbol{\Pi} = \| \rho_{ij} / 4Nf_i\ (o)\, f_j\ (o) \|$$

$$I_{D_2}\,(\tilde{F}\tilde{p}\tilde{p}'\tilde{F}') = \underset{D_2}{\int\cdots\int} n\,(\tilde{p},\ U\,;\,\mathbf{0},\ \tilde{R})\ \tilde{F}\tilde{p}\tilde{p}'\tilde{F}'\,d\tilde{p}\ dU$$

$\tilde{R}$ = covariance matrix of $(\tilde{p},\ U)$

$$= \begin{Vmatrix} \boldsymbol{\Sigma} & \boldsymbol{\Omega} \\ \boldsymbol{\Omega} & \boldsymbol{T} \end{Vmatrix}$$

$$D_1\ : -\infty < \tilde{p} < \infty,\ \ W < x_p^2\,(\alpha),\ D_2 : -\infty < \tilde{p} < \infty,\ \ W > x_p^2\,(\alpha).$$

Proof. We first notice that the asymptotic distribution of the random vector $(\tilde{q},\ \tilde{p},\ U)$ is normal and the covariance matrix is given by the form

$$\begin{Vmatrix} \boldsymbol{\Sigma} & \mathbf{0} \\ \mathbf{0} & \tilde{R} \end{Vmatrix}$$

In fact. the asymptotic normality is established from the theory of U statistics and the covariance matrix is derived from Lemma 3 1 and Corollary 3.1. Thus the random vector $\tilde{q}$ is asymptotically independent on the random vector $(\tilde{p},\ U)$. From Definition 2 2 and (3.19), (3.20), we may write $E\,(\tilde{\theta})$ as

$$E\,(\theta) = \underset{\substack{-\infty<\tilde{q}<\infty \\ D_1}}{\int\cdots\int} (\tilde{M} + \tilde{F}\tilde{q})\ n\,(\tilde{q}\,;\,\mathbf{0},\ \boldsymbol{\Sigma})\ n\,(\tilde{p},\ U\,;\,\mathbf{0},\ \tilde{R})\ d\tilde{q}\ d\tilde{p}\ dU$$

$$+ \underset{\substack{-\infty<\tilde{q}<\infty \\ D_2}}{\int\cdots\int} (\theta_1 + \tilde{F}\tilde{q} - G\tilde{F}\tilde{p})\ n\,(\tilde{q}\,;\,\mathbf{0},\ \boldsymbol{\Sigma})\ n\,(\tilde{p},\ U\,;\,\mathbf{0},\ \tilde{R})\ d\tilde{q}\ d\tilde{p}\ d\ U.$$

By noticing the relation $E(\tilde{q}) = \mathbf{0}$, we get

$$E(\tilde{\theta}) = \theta_1 + (1-\lambda)(\theta_2-\theta_1) P[W<x_p^2(\alpha)] - G \int_{D_2} \cdots \int \tilde{F}\tilde{p}\, n(\tilde{p}, U; \mathbf{0}, \tilde{R})\, d\tilde{p}\, dU.$$

Secondly, using the identity

$$\tilde{\theta} - \theta_1 = \begin{cases} \tilde{F}\tilde{q} + (1-\lambda)\gamma/\sqrt{N} & \text{for } W < x_p^2(\alpha) \\ \tilde{F}\tilde{q} - G\tilde{F}\tilde{p} & \text{otherwise} \end{cases}$$

$$E[(\tilde{\theta}-\theta_1)(\tilde{\theta}-\theta_1)']$$
$$= \underset{\substack{-\infty<\tilde{q}<\infty \\ D_1}}{\int \cdots \int} \{\tilde{F}\tilde{q}\tilde{q}\tilde{F}' + \frac{1-\lambda}{N}\gamma\gamma' + \frac{1-\lambda}{\sqrt{N}}\gamma\tilde{q}\tilde{F}' + \tilde{F}\tilde{q}\gamma')\}$$
$$\times n(\tilde{q}; \mathbf{0}, U)\, n(\tilde{p}, U; \mathbf{0}, \tilde{R})\, d\tilde{q}\, d\tilde{p}\, dU$$
$$+ \underset{\substack{-\infty<\tilde{q}<\infty \\ D_2}}{\int \cdots \int} \{\tilde{F}\tilde{q}\tilde{q}'\tilde{F}' - G\tilde{F}\tilde{q}\tilde{q}'\tilde{F}' - G\tilde{F}\tilde{p}\tilde{q}'\tilde{F}' + G^2\tilde{F}\tilde{p}\tilde{p}'\tilde{F}'\}$$
$$\times n(\tilde{q}; \mathbf{0}, \Sigma)\, n(\tilde{p}, U; \mathbf{0}, \tilde{R})\, d\tilde{q}\, d\tilde{p}\, dU$$
$$= \tilde{F}E(\tilde{q}\tilde{q}')\tilde{F}' + (1-\lambda)\gamma\gamma' P[W < x_p^2(\alpha)]/N + G^2 I_{D_2}\{\tilde{F}\tilde{p}\tilde{p}'\tilde{F}'\}$$

where we also used $E(\tilde{q}) = \mathbf{0}$ and $E(\tilde{q}\tilde{p}') = E(\tilde{q})E(\tilde{p}') = \mathbf{0}$ in the last computation. Moreover we notice that

$$\tilde{F}E(\tilde{q}\tilde{q}')\tilde{F}' = \tilde{\Pi}$$

means the covariance matrix of the "always pool estimate" $\lambda\tilde{X} + (1-\lambda)\tilde{Y}$ under $\theta_i = \mathbf{0}$. When the estimators $\hat{X}$ and $\hat{Y}$ are used instead of $\tilde{X}$ and $\tilde{Y}$, we may the similar results.

Theorem 3.2. The mean vector and the mean square error of the sometimes pool estimate $\hat{\theta}$ are given by the following

(3.23) $\quad E(\hat{\theta}) = \theta_1 + 0(1/\sqrt{N})$

(3.24) $\quad \text{M.S.E.}(\hat{\theta}) = \hat{\Pi} + \frac{1-\lambda}{N}\gamma\gamma' P[W<x_p^2(\alpha)] + G^2 I_{W>x_p^2(\alpha)}[\hat{F}UU'\hat{F}']$

where

$$\hat{\Pi} = \|\tau_{ij}/12Ng_i\, g_j\|,\ \hat{F} = \|\delta_{ij}/\sqrt{12N}\ g_i\|,\ i, j = 1, \ldots, p$$

Proof. We first notice that the statistics $\hat{p}$ and U are equivalent as shown in Lemma 3.3 and $\hat{q}$ and U are asymptotically distributed as $n(q; \mathbf{0}, T)\, n(U; \mathbf{0}, T)$. From the above facts, we may derive (3.23) and (3.24) by the similar technique to Theorem 3.1.

§ 4. The estimator for shift vector.

In this section we discuss about the sometimes pool estimate for shift vector. The sometimes pocl estimate Δ has been defined as as follows,

$$\hat{\Delta} = \begin{cases} \lambda_{23}\hat{\Delta}_1 + (1-\lambda_{23})\hat{\Delta}_2 & \text{for } W' \leqq x_p^2(\alpha) \\ \hat{\Delta}_1 & \text{otherwise} \end{cases}$$

where

$$\hat{\boldsymbol{\Delta}}_i = (\hat{\Delta}_{ki}\,;\, k = 1,2,\ldots,p) \qquad i=1,2$$

$$\Delta_{k_1} = \text{med}\,(Y_{k\beta}-X_{k\alpha}) \quad , \quad \hat{\Delta}_{k_2} = \text{med}(Z_{k\gamma} - X_{k\alpha})$$

and we shall denote W' by W without confusion.

Lemma 4.1.

(i) $\text{Cov}\,(\sqrt{s_{12}N_{12}}\, g_i\, \hat{\Delta}_{i\alpha},\ \sqrt{s_{12}N_{12}}\, g_j\, \hat{\Delta}_{j\alpha}) \sim \tau_{ij}$ for $\alpha = 1,2$

(ii) $\text{Cov}\,(\sqrt{s_{12}N_{12}}\, g_i\, \hat{\Delta}_{i_1},\ \sqrt{s_{13}N_{13}}\, g_j\, \hat{\Delta}_{j_2}) \sim \sqrt{(1-\lambda_{12})(1-\lambda_{13})}\ \tau_{ij}$

(iii) $\text{Cov}\,(\sqrt{s_{12}N_{12}}\, g_i\, \hat{\Delta}_{i_1},\ \sqrt{s_{23}N_{23}}\ U\,(Y_i, Z_j)) \sim -\sqrt{\lambda_{12}(1-\lambda_{23})}\ \tau_{ij}$

(iv) $\text{Cov}\,(\sqrt{s_{13}N_{13}}\, g_i\, \hat{\Delta}_{i_2},\ \sqrt{s_{23}N_{23}}\ U(Y_j, Z_j)) \sim \sqrt{\lambda_{13}\lambda_{23}}\ \tau_{ij}$

Proof. We prove (i) and (ii). The proof for (iii) and (iv) are similar.

(i) $P[\sqrt{N_{12}}\ (\hat{\Delta}_{i_1} - \Delta_{i_1}) < a,\ \sqrt{N_{12}}\ (\hat{\Delta}_{j_1} - \Delta_{j_1}) < b\,]$

$\sim P_o\,[\sqrt{N_{12}}\,\{U\,(X_i, Y_j - a/\sqrt{N_{12}}) - \mu_{X_iY_i}\} < a g_i, \sqrt{N_{12}}\,\{U\,(X_j, Y_j$

$-b/\sqrt{N_{12}}) - \mu_{X_jY_j}\} < b g_j\,]$

where $\mu_{X_iY_i} = EU\,(X_i, Y_i - a/\sqrt{N_{12}})$

Now,

$$\text{Cov}_o\,[\sqrt{N_{12}}\,U\,(X_i, Y_i - a/\sqrt{N_{12}}),\ \sqrt{N_{12}}\,U(X_j, Y_j - b/\sqrt{N_{12}})\,]$$

$$\sim N_{12}\,(n_1 n_2)^{-2}\, E_o\,\{\sum_{\gamma,\delta} \phi\ (X_{1\alpha}, Y_{i\beta}) \sum_{\alpha,\beta} \phi\ (X_{j\gamma}, Y_{j\delta})\} - N_{12}/4$$

$$\sim [\lambda_{12}\,(1-\lambda_{12})]^{-1}\,[P_o\,\{X_{i\alpha} < Y_{i\alpha}, X_{j\alpha} < Y_{j\delta},\ \beta \neq \delta\} - 1/4]$$

$$= \tau_{ij}/s_{12}.$$

Hence it holds that

$$\text{Cov}\,(\sqrt{s_{12}N_{12}}\, g_i\, \hat{\Delta}_{i_1}, \sqrt{s_{12}N_{12}}\, g_j\, \hat{\Delta}_{j_1}) \sim \text{Cov}_o\,(\sqrt{s_{12}N_{12}}\,U_i, \sqrt{s_{12}N_{12}}\,U_j)$$

$$\sim \tau_{ij}.$$

(ii) $P[\sqrt{N_{12}}\ (\hat{\Delta}_{i_1} - \Delta_{i_1}) < a,\ \sqrt{N_{13}}\ (\hat{\Delta}_{j_2} - \Delta_{j_2}) < b]$

$\sim P_o\,[\sqrt{N_{12}}\,\{U(X_i, Y_i - a/\sqrt{N_{12}}) - \mu_{X_iY_i}\} < a g_i,\ \sqrt{N_{13}}\,\{U(X_j, Z_j - b/\sqrt{N_{13}})$

$- \mu_{X_jZ_j}\} < b g_j\,]$

Now we get

$$\text{Cov}_o\,[\sqrt{N_{12}}\,U(X_i, Y_i - a/\sqrt{N_{12}}),\ \sqrt{N_{13}}\,U(X_j, Z_j - b/\sqrt{N_{13}})]$$

$$\sim \sqrt{N_{12}N_{13}}\ n_1^{-1}\,[P_o\,\{X_{i\alpha} < Y_{i\beta},\ X_{j\alpha} < Z_{j\delta}\,;\, \beta \neq \delta\} - 1/4]$$

$$= \tau_{ij}/12\sqrt{\lambda_{12}\lambda_{13}}$$

This leads to the results (ii).

Definition 4.1. We define the random vectors,

(4.1) $\boldsymbol{\xi}' = (\sqrt{s_{12}N_{12}}\, g_k\, (\hat{\Delta}_{k_1} - \Delta_{k_1}) \quad k=1,\ldots,p)$

(4.2) $\boldsymbol{\eta}' = (\sqrt{s_{13}N_{13}}\, g_k\, (\hat{\Delta}_{k_2} - \Delta_{k_2}) \quad k=1,\ldots,p)$

(4.3) $\boldsymbol{U}' = (\sqrt{s_{23}N_{23}}\ (U\,(Y_k, Z_k) - \mu_{Y_kZ_k}) \quad k=1,\ldots,p)$

(4.4) $\boldsymbol{q} = \sqrt{N_{12}\lambda_{23}/N}\,\boldsymbol{\xi} + \sqrt{N_{13}(1-\lambda_{23})/N}\ \boldsymbol{\eta}$

(45) $\mathbf{p} = -\sqrt{(1-\lambda_{23})/\lambda_{12}}\ \boldsymbol{\xi} + \sqrt{\lambda_{23}/\lambda_{13}}\ \boldsymbol{\eta}$

Theorem 4.1. The asymptotic joint distribution of the random vector $(\boldsymbol{q}, \boldsymbol{p}, \boldsymbol{U})$ is normal with mean vector $\mathbf{0}$ and covariance matrix

$$\begin{Vmatrix} \boldsymbol{T} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{T} & \boldsymbol{T} \\ \mathbf{0} & \boldsymbol{T} & \boldsymbol{T} \end{Vmatrix}$$

As a consequence, the statistic $\boldsymbol{q}$ is asymptotically independent on the statistic $(\boldsymbol{p}, \boldsymbol{U})$ and $\boldsymbol{p}$ is asymptotically equivalent to $\boldsymbol{U}$ with probability 1.

Proof. We first compute the covariance matrix of $\boldsymbol{q}$ by using Lemma 4.1. From Definition 4.1,

$$E(\boldsymbol{q}\boldsymbol{q}') = N_{12}\lambda_{23}N^{-1}\ E(\boldsymbol{\xi}\boldsymbol{\xi}') + N_{13}(1-\lambda_{13})N^{-1}\ E(\boldsymbol{\xi}\boldsymbol{\xi}') + \sqrt{N_{12}N_{13}\lambda_{23}(1-\lambda_{23})}/N\,[E(\boldsymbol{\xi}\boldsymbol{\eta}') + E(\boldsymbol{\eta}\boldsymbol{\xi}')].$$

Hence Lemma 4.1 leads to the identity

(4.6) $E(\boldsymbol{q}\boldsymbol{q}') = \|\ \tau_{ij}\ \|.$

Analogously we get $E(\boldsymbol{p}\boldsymbol{p}') = E(\boldsymbol{U}\boldsymbol{U}') = \boldsymbol{T}.$

Secondly we shall show that the random vector $\boldsymbol{q}$ is independent on both $\boldsymbol{p}$ and $\boldsymbol{U}$. In fact,

(4.7) $E(\boldsymbol{q}\boldsymbol{U}') = \| -\sqrt{N_{12}\lambda_{12}\lambda_{23}(1-\lambda_{23})}/N\,\tau_{ij} + \sqrt{N_{13}\lambda_{13}\lambda_{23}(1-\lambda_{23})}/N\,\tau_{ij}\,\| = \mathbf{0}$

$$E(\boldsymbol{q}\boldsymbol{p}') = -\sqrt{\lambda_{23}(1-\lambda_{23})N_{12}/N\lambda_{12}}\ E(\boldsymbol{\xi}\boldsymbol{\xi}') + \sqrt{\lambda_{23}(1-\lambda_{23})N_{13}/N\lambda_{13}}\ E(\boldsymbol{\eta}\boldsymbol{\eta}') + \lambda_{23}\sqrt{N_{12}/N\lambda_{13}}\ E(\boldsymbol{\xi}\boldsymbol{\eta}') - (1-\lambda_{23})\sqrt{N_{13}/N\lambda_{12}}\ E(\boldsymbol{\eta}\boldsymbol{\xi}')$$

By Lemma 4.1, we get $E(\boldsymbol{q}\boldsymbol{p}') = \mathbf{0}$.

Theorem 4.2. The asymptotic mean and means quare error is given by the following

(4.8) $E(\hat{\boldsymbol{\Delta}}) = \boldsymbol{\Delta}_1 + 0(1/\sqrt{N})$

(4.9) $$\text{M.S.E.}(\hat{\boldsymbol{\Delta}}) = \Pi + \frac{1-\lambda_{23}}{N}\gamma\gamma' P[W < x_p^2(\alpha)] + G^2 \underset{W > x_p^2(\alpha)}{\int \cdots \int} (\boldsymbol{F}\boldsymbol{U}\boldsymbol{U}'\boldsymbol{F}')\, n(\boldsymbol{U}; \mathbf{0}, \boldsymbol{T})\, d\boldsymbol{U}$$

where

(4.10) $\Pi = \|\ N\tau_{ij}/12n_1N_{23}\,g_i\,g_j\ \|$

(4.11) $\boldsymbol{F} = \|\ N\delta_{ij}/\sqrt{12n_1N_{23}}\,g_i\ \| \qquad i, j = 1, \ldots, p$

(4.12) $G = \sqrt{n_1n_3/n_2N}$

Proof. From (4.4), (4.5) and (4.11), we first get

$$\lambda_{23}\hat{\boldsymbol{\Delta}}_1 + (1-\lambda_{23})\hat{\boldsymbol{\Delta}}_2 = \boldsymbol{M} + \boldsymbol{F}\boldsymbol{q} = \boldsymbol{\Delta}_1 + (1-\lambda_{23})(\boldsymbol{\Delta}_2 - \boldsymbol{\Delta}_1) + \boldsymbol{F}\boldsymbol{q}$$

$$\hat{\boldsymbol{\Delta}}_1 = \boldsymbol{\Delta}_1 + \boldsymbol{F}\boldsymbol{q} - G\boldsymbol{F}\boldsymbol{p}$$

where

$$\boldsymbol{M} = \lambda_{23}\boldsymbol{\Delta}_1 + (1-\lambda_{23})\hat{\boldsymbol{\Delta}}_2$$

We may also express by Theorem 4.1 as follows,

$$E(\hat{\boldsymbol{\Delta}}) = \underset{\substack{-\infty<\mathbf{q}<\infty \\ W<x_p^2(\alpha)}}{\int\cdots\int} (\boldsymbol{M}+\boldsymbol{Fq})\, n(\boldsymbol{q};\boldsymbol{0},\ \boldsymbol{T})\, n(\boldsymbol{U};\boldsymbol{0},\ \boldsymbol{T})\, d\boldsymbol{q}\ d\boldsymbol{U}$$

$$+ \underset{\substack{-\infty<\mathbf{q}<\infty \\ W>x_p^2(\alpha)}}{\int\cdots\int} (\Delta_1+\boldsymbol{Fq}-G\boldsymbol{Fp})\, n(\boldsymbol{q};\boldsymbol{0},\ \boldsymbol{T})\, n(\boldsymbol{U};\boldsymbol{0},\ \boldsymbol{T})\, d\boldsymbol{q}\ d\boldsymbol{U}$$

$$= \boldsymbol{\Delta}_1 + (1-\lambda_{23})\, \gamma\, P[\, W<x_p^2(\alpha)]/\sqrt{N} - G \underset{W>x_p^2(\alpha)}{\int\cdots\int} (\boldsymbol{FU})\, n(\boldsymbol{U};\boldsymbol{0},\ \boldsymbol{T})\, d\boldsymbol{U}.$$

Analogously we may also compute the following

$$\text{M.S.E.}(\hat{\boldsymbol{\Delta}}) = \boldsymbol{\Pi} + \frac{1-\lambda_{23}}{N}\gamma\gamma'\, P[\, W<x_p^2(\alpha)]$$

$$+ G^2 \underset{W>x_p^2(\alpha)}{\int\cdots\int} (\boldsymbol{FUU'F'})\, n(\boldsymbol{U};\boldsymbol{0},\ \boldsymbol{T})\, d\boldsymbol{U}.$$

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