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Construction of Finite Commutative Semigroups[1)]

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§ 1. Introduction.

A semigroup G with kernel K is called a *nilpotent semigroup* if it satisfies the following condition (C. 1) :

(C. 1) $G \supset G^2 \supset G^3 \supset \ldots\ldots \supset G^p = K$ for some positive integer p.[2)]

In particular, we shall call G a *nilsemigroup* if it is a nilpotent semigroup and its kernel is a subgroup of G. Further, a semigroup M with zero 0 is called a *generalized nilsemigroup* if $M^* = M \setminus \{0\}$ constitutes a nilsubsemigroup of M.[3)] Let T be a semigroup, and let 0 be a symbol not representing any element of T. Extend the given binary operation in T to one in $T \cup \{0\}$ by defining $00=0$ and $a0=0a=0$ for every a in T. It is easy to see that $T \cup \{0\}$ becomes a semigroup with zero element 0 with respect to this binary operation. We speak of the passage from the semigroup T to the semigroup $T \cup \{0\}$ as "the addition of a zero element to T", and denote the semigroup $T \cup \{0\}$ by T^0. It is clear that if T is a nilsemigroup then T^0 is a generalized nilsemigroup, and conversely that every generalized nilsemigroup can be obtained by adding a zero element to a nilsemigroup. As defined by Tamura [5], a semigroup is called a *z-semigroup* if it has a zero element, 0, but has no idempotent except 0. In particular, for a finite semigroup S it can be easily proved by the Corollary to Lemma 2 of Tamura [4] that S is a z-semigroup if and only if it satisfies the following condition (C. 2) :

$$(\text{C. } 2) \begin{cases} (1) & S \text{ has a zero element } 0, \\ (2) & S \supset S^2 \supset S^3 \supset \ldots\ldots \supset S^p = \{0\} \text{ for some positive integer } p. \end{cases}$$

That is, S is a z-semigroup if and only if it is a nilsemigroup with zero.[4)]

Let A and B be commutative semigroups, B having a zero element 0. Then, sometimes an ideal extension C of A by B in the sense of Clifford [1] can be commutative.[5)] If C is commutative, then C is called a *commutative ideal extension* of A by B. It should be noted that there exists at least one commutative ideal extension of A by B. If G is a finite [commutative] nilsemigroup having a kernel K, then the Rees factor semigroup G/K of G mod K is a finite [commutative] z-semigroup.[6)] Therefore, we can say that G is a [commutative] ideal extension of a finite [commutative] group by a finite [commutative]

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1) An abstract of a part of this paper has been appeared in Yamada [9].

2) $A \supset B$ means "B is a proper subset of A".

3) $A \setminus B$ means the set $\{x : x \in A, x \notin B\}$.

4) In the case where S is commutative, this is easily proved without the Corollary to Lemma 2 of [4].

5) Clifford [1] used the term "extension" for the term "ideal extension". The term "ideal extension" was introduced by Clifford & Preston [2].

6) The term "Rees factor semigroup" is due to Clifford & Preston [2]. However, this concept was firstly introduced by Rees [3].

z-semigroup.

In this paper, we shall present a method of constructing all possible finite commutative semigroups by using the concepts introduced above.

§ 2. The structure of finite commutative semigroups.

Let S be a finite commutative semigroup. Since S is finite, the set I of idempotents of S is not empty and is a commutative idempotent subsemigroup (i. e., a subsemilattice) of S. Define an ordering $\leqq$ in I as follows : $e \leqq f$ if and only if $ef=fe=e$. Then, I becomes a partially ordered set. Since I is a finite semilattice, there exists a maximal element (not necessarily unique), say e_0. Put $I_1=I\setminus\{e_0\}$. If I_1 is not empty, I_1 is also a finite subsemilattice of S. Hence there exists a maximal element of I_1, say e_1. Put $I_2=I_1\setminus\{e_1\}$. Repeating this process, we can obtain a sequence of semilattices, $I=I_0\supset I_1\supset \dots \supset I_m=\{e_n\}$. By the definition of I_i, it is clear that each I_i is an ideal of I. Let $S_i=\{x : x^u \in I_i$ for some positive integer $u\}$. Then, S_i is a subsemigroup of S, and $S=S_0\supset S_1\supset S_2\supset \dots \supset S_m$.

Further, we have

Lemma 1.

(1) *Each S_i is an ideal of S, and for $1\leqq i\leqq m$ the Rees factor semigroup S_{i-1}/S_i of S_{i-1} mod S_i is a finite commutative generalized nilsemigroup.*

(2) *S_m is a finite commutative nilsemigroup.*

Proof. Take elements x and y from S_i and S respectively. There exist positive integers r, k such that $x^r \in I_i$ and $y^k \in I$. Put $x^r=f$ and $y^k=g$. Then $(xy)^{rk}=x^{rk}y^{rk}=fg\in I_i I \subseteq I_i$. Hence $xy\in S_i$. Therefore, S_i is an ideal of S. Next, we shall show that $G_{i-1}=S_{i-1}\setminus S_i$ is a finite commutative nilsemigroup. Take elements a and b from G_{i-1}. Then, there exist positive integers u, v such that $a^u \in I_{i-1}$ and $b^v \in I_{i-1}$. If $a^u \in I_i$ or $b^v \in I_i$, then $a \in S_i$ or $b\in S_i$. Accordingly, $a\notin G_{i-1}$ or $b\notin G_{i-1}$, which contradicts to our assumption. Hence, $a^u=b^v=e_{i-1}\in I_{i-1}\setminus I_i$. Since $(ab)^{uv}=e_{i-1}$, the element ab is contained in $S_{i-1}\setminus S_i$. Hence $ab\in G_{i-1}$. This proved that G_{i-1} is a semigroup. Let $T_{i-1}=\{xe_{i-1} : x\in G_{i-1}\}$. Then, T_{i-1} is a subgroup of G_{i-1} as well as an ideal of G_{i-1}. Therefore, G_{i-1} has the group T_{i-1} as its kernel. It is clear that G_{i-1} satisfies the condition (C. 1). Thus, G_{i-1} is a nilsemigroup. Since G_{i-1} is a finit commutative nilsemigroup, S_{i-1}/S_i is a finite commutative generalized nilsemigroup. The second part of the lemma can be verified by a similar method.

If A is a finite commutative semigroup and if B is a finite commutative generalized nilsemigroup, for convenience we shall call "a commutative ideal extension of A by B" simply "an elementary extension".

Then,

Theorem 1.

(1) *Every finite commutative semigroup can be built up from a finite commutative nilsemigroup by repeating elementary extensions finitely many times successively.*

Conversely,

(2) *a semigroup which is built up from a finite commutative nilsemigroup by repeating elementary extensions finitely many times successively is a finite commutative semigroup.*

Proof. The first part of the theorem follows from Lemma 1 and the definition of elementary extensions. The second part is obvious.

By Theorem 1, the problem of constructing all possible finite commutative semigroups is reduced to the following two problems.

(P. I) Determining all possible commutative ideal extensions of A by B, for a given finite commutative semigroup A and for a given finite commutative generalized nilsemigroup B.

More generally,

(P. I)* Determining all possible commutative ideal extensions of A by B, for a given commutative semigroup A and for a given commutative semigroup B with zero in which the set of non-zero elements is a subsemigroup.

(P. II) Construction of finite commutative nilsemigroups.

The problem (P. I)*(hence, the problem (P. I)) was solved by Tamura [7] ; that is,

Theorem 2. *Let A be a commutative semigroup, and B a commutative semigroup with zero 0 in which the set $B^*=B\setminus\{0\}$ is a subsemigroup. Let $\mathscr{T}(A)$ be the semigroup of translations of A. Let φ be a homomorphism of B^* into $\mathscr{T}(A)$. Then, $A+B^*$ becomes a commutative ideal extension of A by B if multiplication $\circ$ in C is defined as follows :*

$$\text{(M. 1)} \quad x\circ y=\begin{cases} xy & \text{if } x,\ y\in A \text{ or } x,\ y\in B^*, \\ \lambda_x(y) & \text{if } y\in A \text{ and } x\in B^*, \\ \lambda_y(x) & \text{if } x\in A \text{ and } y\in B^*, \end{cases}$$

where $\lambda_u=\varphi(u)$.

Further, every commutative ideal extension of A by B is found in this fashion.[7]

Proof. See Theorem 4 of [7].

Remarks. 1. In Theorem 2, $\mathscr{T}(A)$ contains the identity mapping τ on A. Now, consider the mapping $\varphi: B^*\longrightarrow\mathscr{T}(A)$ defined by $\varphi(x)=\tau$, $x\in B^*$. Then, the mapping φ is clearly a homomorphism. Therefore, this φ gives a commutative ideal extension of A by B. Hence, there exists at least one commutative ideal extension of A by B.

2. Tamura [6], [7], [8] studied also the problem of construction of finite commutative semigroups in connection with the problem of constuction of finite semigroups. We shall discuss this problem by taking a process which is somewhat different from the process taken in [6], [7] and [8].

By Theorem 2, we need only to consider the problem (P. II). So, we shall deal with this problem in the following paragraphs.

§ 3. Finite commutative nilsemigroups.

The following was proved by Clifford [1] :

Theorem 3. *Let S and T be semigroups, T having a zero element 0. Let $T^*=T\setminus\{0\}$ and $\Sigma=T^*+S$. Then, a partial homomorphism ξ of the partial groupoid T^* into S determines an ideal extension $\Sigma(\circ)$ of S by T as follows :*

$$\text{(M. 2)} \quad a\circ b=\begin{cases} ab & \text{if } a,b\in S \text{ or if } a,b\in T^* \text{ and } ab\neq 0, \\ \xi(a)\xi(b) & \text{if } a,b\in T^* \text{ and } ab=0, \\ \xi(a)\,b & \text{if } a\in T^* \text{ and } b\in S, \\ a\,\xi(b) & \text{if } b\in T^* \text{ and } a\in S. \end{cases}$$

If S has an identity, then every ideal extension of S by T is found in this fashion.[8]

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7) For sets M and N, $M+N$ means the class sum of M and N. In (M. 1), xy means the product of x, y in A or B^*, corresponding to the case $x,y\in A$ or $x,y\in B^*$ respectively.

8) Clifford [1] used the term "a ramified homomorphism" for the term "a partial homomorphism". The term "a partial homomorphism" is due to Clifford & Preston [2]. For the definition of partial groupoids, see also [2].

Now, assume that both S and T of Theorem 3 are commutative. Then, the binary operation $\circ$ given by (M. 2) is also commutative. Therefore, in this case every ideal extension of S by T is necessarily commutative.

Hence, we have

Lemma 2. *If S is a commutative semigroup with identity, and if T is a commutative semigroup with zero, then every ideal extension of S by T is a commutative ideal extension.*

Now, as was seen in § 1, a finite commutative nilsemigroup G having a kernel K is an ideal extension of the finite commutative group K by the finite commutative z-semigroup G/K. Conversely, it follows from Lemma 2 that an ideal extension of a finite commutative group by a finite commutative z-semigroup is a finite commutative nilsemigroup. Therefore, the problem (P. II) is reduced to the following three problems :

(P. III) Construction of finite commutative groups.

(P. IV) Construction of finite commutative z-semigroups.

(P. V) Determining all possible ideal extensions of A by B, for a given finite commutative group A and for a given finite commutative z-semigroup B.

The problem (P. III) is clear from the theory of groups, while the problem (P. V) can be also solved as a special case of Theorem 3; that is,

Corollary. *Let S be a finite commutative group, and T a finite commutative z-semigroup. Let 0 be the zero element of T. Let $T^*=T\backslash\{0\}$ and $\Sigma=T^*+S$. Take a partial homomorphism ξ of the partial groupoid T^* into S, and define multiplication $\circ$ in Σ by* (M. 2). *Then, $\Sigma(\circ)$ becomes an (commutative) ideal extension of S by T. Further, every (commutative) ideal extension of S by T is obtained by this method.*

Now, we need only to consider the problem (P. IV) which takes the most complicated and important part of this paper. We shall solve this problem in the following § 4 and § 5.

Remark. Tamura [8] solved the problem of construction of finite z-semigroups (not necessarily commutative) by using the concept of the decompositions of certain finite free z-semigroups. We shall solve the problem (P. IV) by another method which is essentially quite differ from the method used in [8].

§ 4. Commutative z-semigroups of order n.

Let S be a finite commutative z-semigroup, and let 0 be its zero element.

Then, S satisfies the following condition :

(Z) $S\supset S^2\supset S^3\supset\ldots\ldots\supset S^p=\{0\}$ for some positive integer p.

In the case of $p=1$ or $p=2$, S satisfies the following

(Z. 1) $S=\{0\}$

or (Z. 2) $S^2=\{0\}$, that is, $xy=0$ for all $x, y\in S$,

respectively.

Such a semigroup S is called a *trivial z-semigroup* or a *null semigroup*, corresponding to $p=1$ or $p=2$.

Remark. Let M be a set consisting of n elements, and let e be an element of M. If we define multiplication $\circ$ in M by

(M. 3) $x\circ y=e$ for all $x, y\in M$,

then M becomes a null semigroup of order n. Hence, of course M is a commutative z-

semigroup of order n. Thus, we have the following result : *For any positive integer n, there exists a commutative z-semigroup of order n.*

Let G be a semigroup with zero 0. The subset A of G, where $A=\{x : x\in G, xy=yx=0$ for all $y\in G\}$, is a subsemigroup of G. We shall call A the *annihilator* of G.

Lemma 3. *The annihilator of a non-trivial, finite commutative z-semigroup has a non-zero element.*

Proof. Let S be a non-trivial, finite commutative z-semigroup, and let 0 be its zero element. Then, there exists a positive integer $p\geqq 2$ such that $S\supset S^2\supset\ldots\ldots\supset S^p=\{0\}$. Take an element x from the set $S^{p-1}\backslash S^p$. Then, $x\neq 0$ and $xy\in S^p=\{0\}$ for all $y\in S$. Hence, x is a non-zero element contained in the annihilator of S.

Remark. Following Tamura [4], Lemma 3 is generalized as follows : *The annihilator of a non-trivial, finite z-semigroup has a non-zero element.* We can prove this proposition by a similar method to that of the proof of Lemma 3 if we use the fact that a finite z-semigroup satisfies the condition (Z), even if it is non-commutative.

Lemma 4. *Let S be a commutative z-semigroup of order $n+1$ ($n\geqq 1$). Let 0 be the zero element of S, and let u be a non-zero element contained in the annihilator of S. Then the set $\{0, u\}$ is both a null subsemigroup and an ideal of S, and the Rees factor semigroup $D=S/\{0, u\}$ of S mod $\{0, u\}$ is a commutative z-semigroup of order n. Further, in this case S is a commutative ideal extension of a null semigroup of order 2 by a commutative z-semigroup of order n.*

Proof. Ovbious.

Conversely, we have

Lemma 5. *A commutative ideal extension of a null semigoup of order 2 by a commutative z-semigroup of order n is a commutative z-semigroup of order $n+1$.*

Proof. Let Z be a commutative z-semigroup of order n. Let $\mathbf{0}$ be the zero element of Z. Let $U=\{u, 0\}$ be a null semigroup, where 0 is the zero element of U. Let $Z^*=Z\backslash\{\mathbf{0}\}$ and $T=U+Z^*$. Suppose that T is a commutative ideal extension of U by Z under a binary operation $\circ$. For any element x of $T(\circ)$, $x\circ 0=x0\circ 0\subseteq U\circ 0=\{0\}$, *i. e.*, $x\circ 0=0$. Hence 0 is the zero element of $T(\circ)$. Next, there exists a positive integer p such that $Z\supset Z^2\supset\ldots\ldots\supset Z^p=\{\mathbf{0}\}$. Therefore, $T(\circ)^2=U^2\cup U\circ Z^*\cup Z^*(\circ)^2\subseteq U\cup Z^2\backslash\{\mathbf{0}\}$. In general, for $1\leqq m\leqq p$ $T(\circ)^m\subseteq U\cup Z^m\backslash\{\mathbf{0}\}$. Hence, especially $T(\circ)^p\subseteq U$ and hence $T(\circ)^{p+1}=\{0\}$. Since the order of $T(\circ)$ is $n+1$, $T(\circ)$ is a commutative z-semigroup of order $n+1$.

Remark. For any given null semigroup N of order 2 and for any commutative z-semigroup Z of order n, existence of a commutative ideal extension of N by Z is proved by the following example : Let $N=\{0, u\}$, where 0 is the zero element of N. Let $\mathbf{0}$ be the zero element of Z, and put $S=Z\backslash\{\mathbf{0}\}+\{0, u\}$. Then S becomes a commutative ideal extension of N by Z if multiplication $\circ$ in S is defined as follows :

$$x\circ y=\begin{cases} xy & \textit{if } x,y\in Z\backslash\{\mathbf{0}\} \text{ and } xy\neq\mathbf{0},\\ 0 & \text{otherwise.}\end{cases}$$

Combining Lemmas 4 and 5, we have

Theorem 4. *A commutative z-semigroup of order $n+1$ ($n\geqq 1$) is a commutative ideal extension of a null semigroup of order 2 by a commutative z-semigroup of order n, and vice-versa.*

Now, we consider the problem :

(P. VI) Construct all possible commutative z-semigroups of order n for a given positive integer n.

For $n=1$ or 2 this problem is easily solved, since a commutative z-semigroup of order 1 or 2 is a trivial z-semigroup or a null semigroup respectively.

Hence, the problem (P. VI) is reduced to the following problem :

(P. VII) We assume that we can construct all possible commutative z-semigroups of order m $(m\geqq 2)$. Construct all possible commutative z-semigroups of order $m+1$.

Further, by Theorem 4 the problem (P. VII) is reduced to the following problem :

(P. VIII) Construct all possible commutative ideal extensions of a given null semigroup of order 2 by a given commutative z-semigroup of order m $(m\geqq 2)$.

We shall deal with this problem (P. VIII) in the next paragraph.

Remark. Lemmas 4, 5 and Theorem 4 still hold, even if we substitute the terms "z-semigroup" and "ideal extension" for the terms "commutative z-semigroup" and "commutative ideal extension" respectively. Accordingly, by a similar process to the process used above the problem of constructing all finite z-semigroups is reduced to the following problem :

(P. IX) Construct all possible ideal extensions of a given null semigroup of order 2 by a given z-semigroup of order m $(m\geqq 2)$.

§ 5. *C*-factors of a finite commutative *z*-semigroups.

Let N be a null semigroup of order 2, and put $N=\{0, z\}$, where 0 is the zero element of N. Let T be a finite commutative z-semigroup having $\mathbf{0}$ as its zero element. Let $T^*=T\setminus\{\mathbf{0}\}$, and let $S=N+T^*$. Let $\Omega=\{(x, y) : xy=\mathbf{0},\ x, y\in T\}$. Then, any subset Λ of Ω satisfying the following condition (C) is an ideal of the direct product $T\times T$:

$$(C)\begin{cases} (1) & (t, \mathbf{0})\in\Lambda \text{ for any } t\in T, \\ (2) & (tv, w)\in\Lambda \text{ implies } (t, vw)\in\Lambda, \\ (3) & (v, w)\in\Lambda \text{ implies } (w, v)\in\Lambda. \end{cases}$$

For, let $(x, y)\in\Lambda$ and $(t_1, t_2)\in T\times T$. Then, $(x, y)(t_1, t_2)=(xt_1, yt_2)$. Since $(t_1, \mathbf{0})=(t_1, xyt_2)\in\Lambda$, (xyt_2, t_1) is contained in Λ. Hence $(yt_2, xt_1)\in\Lambda$, and hence $(xt_1, yt_2)\in\Lambda$. Therefore, $(x, y)(t_1, t_2)\in\Lambda$. Thus, Λ is an ideal of $T\times T$. Such a Λ is called a *commutative extension factor* (abbrev. *C-factor*) of T. It is easy to see that Ω itself is the greatest C-factor of T.

Under this definition, we have

Theorem 5. *Let Λ be a C-factor of T, and define multiplication $\circ$ in S by the following :*

$$(\text{M. }4)\quad x\circ y=\begin{cases} xy & \text{if } x, y\in T^*,\ xy\neq\mathbf{0} \text{ or if } x, y\in N, \\ 0 & \text{if } x\in N \text{ or } y\in N, \\ 0 & \text{if } (x, y)\in\Lambda,\ x, y\in T^* \text{ and } xy=\mathbf{0}, \\ z & \text{if } (x, y)\notin\Lambda,\ x, y\in T^* \text{ and } xy=\mathbf{0}. \end{cases}$$

Then, $S(\circ)$ becomes a commutative ideal extension of N by T. Further, every commutative ideal extension of N by T is found in this fashion.

Proof. At first, we shall prove thet $S(\circ)$ is a commutative semigroup. Since clearly $S(\circ)$ is closed and commutative, we need only to prove that $S(\circ)$ is associative, *i. e.*, $a\circ(b\circ c)=(a\circ b)\circ c$ for all $a, b, c\in S(\circ)$. If at least one of the elements a, b, c is an element of N, then $a \circ (b \circ c)=0$ and $(a \circ b) \circ c=0$. Hence, in this case $a \circ (b \circ c)=(a \circ b) \circ c$ is satisfied. Further, if $a, b, c \in T^*$ and $abc \neq \mathbf{0}$, then $a \circ (b \circ c)=a(bc)=(ab)c=(a \circ b) \circ c$. Therefore, in this case $a \circ (b \circ c)=(a \circ b) \circ c$ is also satisfied. Hence, we assume that $a, b, c \in T^*$ and $abc=\mathbf{0}$.

Case 1. $\underline{ab=\mathbf{0},\ bc=\mathbf{0}.}$ $(a \circ b) \circ c=0 \circ c$ or $z \circ c=0$ and $a \circ (b \circ c)=a \circ 0$ or $a \circ z=0$. Hence $(a \circ b) \circ c=a \circ (b \circ c)$.

Case 2. $\underline{ab \neq \mathbf{0}.}$

Subcase (*i*). $(ab, c) \in \Lambda$. $(a \circ b) \circ c=ab \circ c=0$. Since (ab, c) is contained in Λ, (a, bc) is also contained in Λ. If $bc \neq \mathbf{0}$, then $a \circ (b \circ c)=a \circ bc=0$. If $bc=\mathbf{0}$, then $a \circ (b \circ c)=a \circ 0$ or $a \circ z=0$. Hence, in both cases we have $a \circ (b \circ c)=0$. Therefore, $(a \circ b) \circ c=a \circ (b \circ c)$.

Subcase (*ii*). $(ab, c) \notin \Lambda$. At first, $(ab, c) \notin \Lambda$ implies $(a, bc) \notin \Lambda$. Next, $bc \neq \mathbf{0}$. For, if $bc=\mathbf{0}$ then $(a, \mathbf{0})=(a, bc) \in \Lambda$. This contradicts to $(a, bc) \notin \Lambda$. Hence, $bc \neq \mathbf{0}$. Now, $a \circ (b \circ c)=a \circ (bc)=z=ab \circ c=(a \circ b) \circ c$. Therefore, $(a \circ b) \circ c=a \circ (b \circ c)$.

Case 3. $\underline{bc \neq \mathbf{0}.}$ In this case, we can prove associativity $a \circ (b \circ c)=(a \circ b) \circ c$ by a similar method to that of the proof of the case 2.

Thus, in any case associativity $a \circ (b \circ c)=(a \circ b) \circ c$ is satisfied.

It is easy to see that $S(\circ)$ satisfies the following

(1) $x \circ y=xy$ if $x, y \in N$ or if $x, y \in T^*$, $xy \neq \mathbf{0}$,

and (2) $N \circ S \subseteq N$ (in fact, $N \circ S=\{0\}$).

Therefore, $S(\circ)$ is a commutative ideal extension of N by T. Next, we shall prove that every commutative ideal extension of N by T can be obtained by the method given in the theorem. Let $S(\circledcirc)$ be any commutative ideal extension of N by T. Let $\Lambda_1=\{(x, y): x \circledcirc y=0,\ (x, y) \in \Omega,\ x, y \in T^*\}$ and $\Lambda_2=\{(x, y): x, y \in T$, and $x=\mathbf{0}$ or $y=\mathbf{0}\}$, and put $\Lambda=\Lambda_1 \cup \Lambda_2$. Then, Λ is a C-factor of T. This is proved as follows: Clearly, Λ satisfies the conditions (1) and (3) of (C). Let $(t_1t_2, a) \in \Lambda$. Then, $(t_1t_2, a) \in \Lambda_1$ or $\in \Lambda_2$. If $(t_1t_2, a) \in \Lambda_1$, then $t_1t_2 \circledcirc a=(t_1 \circledcirc t_2) \circledcirc a=0$. Hence, $t_1 \circledcirc (t_2 \circledcirc a)=0$. If $t_2 \circledcirc a=0$ or z, then $t_2a=\mathbf{0}$ and hence $(t_1, t_2a) \in \Lambda_2$. If $t_2 \circledcirc a \neq 0, z$, then $t_2a \in T^*$ and hence $(t_1, t_2a) \in \Lambda_1$. Next, suppose that $(t_1t_2, a) \in \Lambda_2$. If $t_2a=\mathbf{0}$, then $(t_1, t_2a)=(t_1, \mathbf{0}) \in \Lambda_2$. If $t_2a \neq \mathbf{0}$, then $t_1t_2=\mathbf{0}$ since $(t_1t_2, a) \in \Lambda_2$ and $a \neq \mathbf{0}$. Hence, we have $t_1 \circledcirc t_2a=t_1 \circledcirc (t_2 \circledcirc a)=(t_1 \circledcirc t_2) \circledcirc a=0 \circledcirc a$ or $z \circledcirc a=0$. Therefore, $(t_1, t_2a) \in \Lambda_1$. In any case, $(t_1t_2, a) \in \Lambda$ implies $(t_1, t_2a) \in \Lambda$. Thus, Λ satisfies the condition (2) of (C). Therefore, Λ is a C-factor of T. Now, let $S(\circ)$ be the commutative ideal extension of N by T that is determined by the C-factor Λ and the multiplication $\circ$ defined by (M. 4). Then, it is easy to see that $S(\circ)=S(\circledcirc)$, *i. e.*, $x \circ y=x \circledcirc y$ for all $x, y \in S$.

By Theorem 5, the problem of determining all commutative ideal extensions of N by T is reduced to the problem of finding all C-factors of T. Next, we shall consider this problem.

Theorem 6. *Let* $\Gamma=\{(t_1t_2, t_3): t_1, t_2, t_3 \in T,\ t_1t_2t_3=\mathbf{0}\} \cup \{(t_1, t_2t_3): t_1, t_2, t_3 \in T,\ t_1t_2t_3=\mathbf{0}\}$.

Then,

(1) Γ *is a* C-*factor of* T,

(2) $\Gamma=\Omega \setminus \{(x, y): x, y$ *are prime elements of* $T\}$,

(3) *if* $\Omega \supseteq \Lambda \supseteq \Gamma$ *and if* Λ *satisfies the condition* (3) *of* (C), *then* Λ *is a* C-*factor of* T.[9)]

9) Let G be a semigroup such that $G \setminus G^2$ is not empty. Then, every element of $G \setminus G^2$ is called a *prime element*.

Proof. (1) Obvious.

(2) $\Gamma \subseteq \Omega \setminus \{(x, y) : x, y$ are prime elements of $T\}$ is clear. Take (u, v) from the set $\Omega \setminus \{x, y) : x, y$ are prime elements of $T\}$. Then, u or v is not a prime element. Hence, $u=u_1u_2$ or $v=v_1v_2$. If $u=u_1u_2$, then $(u_1u_2, v)=(u, v) \in \Gamma$. If $v_1v_2=\mathrm{v}$, then $(u, v_1v_2)=(u, v) \in \Gamma$. Therefore, in any case $(u, v) \in \Gamma$.

(3) Suppose that $\Omega \supseteq \Lambda \supseteq \Gamma$ and Λ satisfies the condition (3) of (C). Since $\Lambda \supseteq \Gamma$, Λ satisfies the condition (1) of (C). Let (t_1t_2, t_3) be an element of Λ. Since $(t_1t_2, t_3) \in \Omega$ and since t_1t_2 is not a prime element, by (2) of the theorem (t_1t_2, t_3) is contained in Γ. Since Γ is a C-factor of T, (t_1, t_2t_3) is also contained in Γ, and hence in Λ. Thus, Λ satisfies the condition (2) of (C). Therefore, Λ is a C-factor of T.

If a sequence $\mathfrak{S}=\{t, t_0, t_1, t_2, \ldots\ldots, t_r\}$ of elements of T, where r is an even integer $\geqq 2$, satisfies

(1) $(t, t_0) \in \Omega$

and (2) $t=t_1t_2,\ t_0t_1=t_3t_4,\ t_2t_3=t_5t_6, \ldots\ldots, t_{r-4}t_{r-3}=t_{r-1}t_r$ ($t=t_1t_2$ in the case of $r=2$),

then $\mathfrak{S}$ is called a *(t, t_0)-chain (in T)*.

Further, in this case the ordered set (t_{r-2}, t_{r-1}, t_r) is called *the final part* of $\mathfrak{S}$. It should be noted that for a given $(t, t_0) \in \Omega$ such a (t, t_0)-chain is not necessarily unique even if it exists.

Lemma 6. (1) *If (t_0, t_1, t_2) is the final part of a $(t, \mathbf{0})$-chain, $\{t, t_0, t_1, t_2\}$, then $(t_0t_1, t_2)=(\mathbf{0}, t_2)$ and $(t_2, t_0t_1)=(t_2, \mathbf{0})$.*

(2) *If (t_{r-2}, t_{r-1}, t_r) is the final part of a $(t, \mathbf{0})$-chain, $\{t, t_0, t_1, \ldots\ldots, t_r\}$, and if $r \geqq 4$, then (t_{r-2}, t_{r-1}, t_r) is also the final part of some $(\mathbf{0}, t')$-chain.*

Proof. (1) Since (t_0, t_1, t_2) is the final part of the $(t, \mathbf{0})$-chain $\{t, t_0, t_1, t_2\}$, $t_0=\mathbf{0}$. Hence $(t_0t_1, t_2)=(\mathbf{0}, t_2)$ and $(t_2, t_0t_1)=(t_2, \mathbf{0})$.

(2) Since $\{t, t_0, t_1, \ldots\ldots, t_r\}$ is a $(t, \mathbf{0})$-chain, $t_0=\mathbf{0}$ and $t=t_1t_2$, $t_0t_1=t_3t_4$, $t_2t_3=t_5t_6, \ldots\ldots, t_{r-4}t_{r-3}=t_{r-1}t_r$. Hence, we have the series $\mathbf{0}=t_3t_4$, $t_2t_3=t_5t_6, \ldots\ldots, t_{r-4}t_{r-3}=t_{r-1}t_r$. Now, put $t'_{k-2}=t_k$ for $k \geqq 3$ and $t'_0=t_2$. Then, we get the series $\mathbf{0}=t'_1t'_2$, $t'_0t'_1=t'_3t'_4, \ldots\ldots, t'_{r-6}t'_{r-5}=t'_{r-3}t'_{r-2}$. Hence, $(t'_{r-4}, t'_{r-3}, t'_{r-2})=(t_{r-2}, t_{r-1}, t_r)$ is the final part of the $(\mathbf{0}, t'_0)$-chain $\{\mathbf{0}, t_2, t_3, \ldots\ldots, t_r\}$.

Theorem 7. *The least C-factor Λ_0 of T is as follows :*
$\Lambda_0=\{(v, \mathbf{0}) : v \in T\} \cup \{\mathbf{0}, w) : w \in T\} \cup \{(t_{r-2}t_{r-1}, t_r) : (t_{r-2}, t_{r-1}, t_r)$ is the final part of a $(\mathbf{0}, t)$-or $(t, \mathbf{0})$-chain for some $t \in T\} \cup \{(t_r, t_{r-2}t_{r-1}) : (t_{r-2}, t_{r-1}, t_r)$ is the final part of a $(\mathbf{0}, t)$-or $(t, \mathbf{0})$-chain for some $t \in T\}=\{(v, \mathbf{0}) : v \in T\} \cup \{(\mathbf{0}, w) : w \in T\} \cup \{(t_{r-2}t_{r-1}, t_r) : (t_{r-2}, t_{r-1}, t_r)$ is the final part of a $(\mathbf{0}, t)$-chain for some $t \in T\} \cup \{(t_r, t_{r-2}t_{r-1}) : (t_{r-2}, t_{r-1}, t_r)$ is the final part of a $(\mathbf{0}, t)$-chain for some $t \in T\}$.

Proof. Let $\Delta=\{(v,\mathbf{0}) : v \in T\} \cup \{(\mathbf{0}, w) : w \in T\} \cup \{(t_{r-2}t_{r-1}, t_r) : (t_{r-2}, t_{r-1}, t_r)$ is the final part of a $(\mathbf{0}, t)$-chain for some $t \in T\} \cup \{(t_r, t_{r-2}t_{r-1}) : (t_{r-2}, t_{r-1}, t_r)$ is the final part of a $(\mathbf{0}, t)$-chain for some $t \in T\}$. Since the last equality follows from Lemma 6, we shall only show that Δ is the least C-factor of T. Let Λ be any C-factor of T. Then, it is clear that $\Lambda \supseteq \{(v, \mathbf{0}) : v \in T\} \cup \{\mathbf{0}, w) : w \in T\}$. Next, we prove that for a (u, v)-chain $\{u, v, t_1, t_2, \ldots\ldots, t_r\}$, where $(u, v) \in \Lambda$, both $(t_{r-2}t_{r-1}, t_r)$ and $(t_r, t_{r-2}t_{r-1})$ are contained in Λ. Since $\{u, v, t_1, t_2, \ldots\ldots, t_r\}$ is a (u, v)-chain, we obtain the series $u=t_1t_2$, $vt_1=t_3t_4$, $t_2t_3=t_5t_6, \ldots\ldots, t_{r-4}t_{r-3}=t_{r-1}t_r$. Now, the elements $(t_1t_2, v)=(u, v)$, $(t_2, t_1v)=(t_2,$

$t_3t_4)$, $(t_2t_3,\ t_4)=(t_5t_6,\ t_4),\ldots\ldots$, and the elements $(v,\ t_1t_2)=(v,\ u)$, $(t_1v,\ t_2)=(t_3t_4,\ t_2)$, $(t_4,\ t_2t_3)=(t_4,\ t_5t_6)$, ... are all contained in Λ. In particular, $(t_{r-2}t_{r-1},\ t_r)$ and $(t_r,\ t_{r-1}t_{r-2})$ are contained in Λ. Hence, if $(t_{r-2},\ t_{r-1},\ t_r)$ is the final part of a $(\mathbf{0},\ t)$-chain, then both $(t_{r-2}t_{r-1},\ t_r)$ and $(t_r,\ t_{r-1}t_{r-2})$ are contained in Λ, since Λ contains the set $\{(\mathbf{0},\ w) : w\in T\}$. Therefore, $\Delta\subseteq\Lambda$. Next, we shall prove that Δ is a C-factor of T. It is clear that Δ satisfies the conditions (1) and (3) of (C). Also, (2) of (C), *i. e.*, "$(xy,\ z)\in\Delta$ implies $(x,\ yz)\in\Delta$" is proved as follows :

Case 1. $(xy,z)=(\mathbf{0},\ w),\ w\in T.$ Since $z=w$ and $xy=\mathbf{0}$, $(z,\ y,\ x)$ is the final part of the $(\mathbf{0},\ w)$-chain $\{\mathbf{0},\ w,\ y,\ x\}$. Hence, $(x,\ yz)\in\Delta$.

Cass 2. $(xy,\ z)=(v,\ \mathbf{0}),\ v\in T.$ Since $z=\mathbf{0}$ and $xy=v$, $(x,\ yz)=(x,\ \mathbf{0})\in\Delta$.

Case 3. $(xy,\ z)=(t_{r-2}t_{r-1},\ t_r)$, where $(t_{r-2},\ t_{r-1},\ t_r)$ is the final part of a $(\mathbf{0},\ t)$-chain $\{\mathbf{0},\ t,\ t_1,\ t_2,\ldots\ldots,\ t_r\}$. Since $\{\mathbf{0},\ t,\ t_1,\ t_2,\ldots\ldots,\ t_r\}$ is a $(\mathbf{0},\ t)$-chain, we obtain the series $\mathbf{0}=t_1t_2,\ tt_1=t_3t_4,\ \ldots\ldots,\ t_{r-4}t_{r-3}=t_{r-1}t_r$. Putting $y=t_{r+1}$ and $x=t_{r+2}$, we get the series $\mathbf{0}=t_1t_2,\ tt_1=t_3t_4,\ldots\ldots,\ t_{r-4}t_{r-3}=t_{r-1}t_r,\ t_{r-2}t_{r-1}=t_{r+1}t_{r+2}$. Therefore, $\{\mathbf{0},\ t,\ t_1,\ t_2,\ldots\ldots,\ t_r,\ t_{r+1},\ t_{r+2}\}$ is a $(\mathbf{0},\ t)$-chain in T. Hence, $(t_{r+2},\ t_r\,t_{r+1})=(x,\ yz)\in\Delta$.

Case 4. $(xy,\ z)=(t_r,\ t_{r-1}t_{r-2})$, *where* $(t_{r-2},\ t_{r-1},\ t_r)$ *is the final part of a* $(\mathbf{0},t)$-*chain* $\{\mathbf{0},\ t,\ t_1,\ t_2,\ldots\ldots,\ t_r\}$. In this case, $xy=t_r$ and $\mathbf{0}=t_1t_2,\ tt_1=t_3t_4,\ t_2t_3=t_5t_6,\ldots\ldots,t_{r-4}t_{r-3}=t_{r-1}t_r$.

Subcase (i) $r=2$. $\mathbf{0}=t_1t_2=t_1xy$. Putting $t_1y=t_1'$ and $x=t_2'$, we obtain the $(\mathbf{0},\ t)$-chain $\{\mathbf{0},\ t,\ t_1',\ t_2'\}$. Since $(t,\ t_1',\ t_2')$ is the final part of $\{\mathbf{0},\ t,\ t_1',\ t_2'\}$, $(t_2',\ tt_1')=(x,\ yz)\in\Delta$.

Subcase (ii) $r>2$, *i. e.* $r\geqq4$. Since $t_{r-4}t_{r-3}=t_{r-1}t_r$ ($t_{r-4}=t$ in the case of $r=4$), $xyt_{r-1}=t_{r-4}t_{r-3}$. Putting $yt_{r-1}=t_{r-1}'$ and $x=t_r'$, we have $t_{r-4}t_{r-3}=t_{r-1}'\,t_r'$. Hence, $(t_{r-2},\ t_{r-1}',\ t_r')$ is the final part of the $(\mathbf{0},\ t)$-chain $\{\mathbf{0},\ t,\ t_1,\ldots,\ t_{r-2},\ t_{r-1}',\ t_r'\}$. Therefore, $(t_r',\ t_{r-1}'t_{r-2})=(x,\ yz)\in\Delta$. Thus, Δ satisfies the condition (2) of (C).

Further, we have the following

Theorem 8. *Let* Δ_0 *be a* C-*factor of* T *and let* $(u,\ v)$ *be an element of* Ω. *Then, the* C-*factor* Δ *of* T *generated by* $\{\Delta_0,\ (u,\ v)\}$, *that is, the least* C-*factor containing* Δ_0 *and* $(u,\ v)$ *is as follows :*

$\Delta=\Delta_0\cup\{(u,\ v)\}\cup\{v,\ u)\}\cup\{(t_{r-2}t_{r-1},\ t_r) : (t_{r-2},\ t_{r-1},\ t_r)$ *is the final part of a* $(u,\ v)$-*or* $(v,\ u)$-*chain*$\}\cup\{(t_r,\ t_{r-2}t_{r-1}) : (t_{r-2},\ t_{r-1},\ t_r)$ *is the final part of a* $(u,\ v)$-*or* $(v,\ u)$-*chain*$\}$.

Proof. Let $\Lambda=\Delta_0\cup\{(u,\ v)\}\cup\{(v,\ u)\}\cup\{(t_{r-2}t_{r-1},\ t_r) : (t_{r-2},\ t_{r-1},\ t_r)$ is the final part of a $(u,\ v)$-or $(v,\ u)$-chain$\}\cup\{(t_r,\ t_{r-2}t_{r-1}) : ((t_{r-2},\ t_{r-1},\ t_r)$ is the final part of a $(u,\ v)$-or $(v,\ u)$-chain$\}$. At first, we shall prove that Λ is a C-factor of T. It is clear that Λ satisfies the conditions (1) and (3) of (C). Also, (2) of (C), *i. e.*, "$(xy,\ z)\in\Lambda$ implies $(x,\ yz)\in\Lambda$" is proved as follows :

Case 1. $(xy,\ z)\in\Delta_0$. Since Δ_0 is a C-factor of T, $(x,\ yz)\in\Delta_0\subseteq\Lambda$.

Case 2. $(xy,\ z)=(u,\ v)$. In this case, $xy=u$ and $z=v$. Therefore, $\{u,\ v,\ y,\ x\}$ is a $(u,\ v)$-chain. Hence, $(v,\ y,\ x)$ is the final part of the $(u,\ v)$-chain $\{u,\ v,\ y,\ x\}$ and hence $(x,\ yv)=(x,\ yz)\in\Lambda$.

Case 3. $(xy,\ z)=(v,\ u)$. By a similar method to that of the proof of the case 2, we have $(x,\ yz)\in\Lambda$.

Case 4. $(xy,\ z)=(t_{r-2}t_{r-1},\ t_r)$, *where* $(t_{r-2},\ t_{r-1},\ t_r)$ *is the final part of a* $(u,\ v)$-*chain*

$\{u, v, t_1, t_2, \ldots\ldots, t_r\}$. Since $\{u, v, t_1, t_2, \ldots\ldots, t_r\}$ is a (u, v)-chain, we obtain the series $u=t_1t_2$, $vt_1=t_3t_4$, $t_2t_3=t_5t_6, \ldots\ldots, t_{r-4}t_{r-3}=t_{r-1}t_r$. Putting $y=t_{r+1}$ and $x=t_{r+2}$, we get also the series $u=t_1t_2$, $vt_1=t_3t_4$, $t_2t_3=t_5t_6, \ldots\ldots, t_{r-4}t_{r-3}=t_{r-1}t_r$, $t_{r-2}t_{r-1}=t_{r+1}t_{r+2}$. Hence, (t_r, t_{r+1}, t_{r+2}) is the final part of the (u, v)-chain $\{u, v, t_1, t_2, \ldots\ldots, t_r, t_{r+1}, t_{r+2}\}$. Therefore, $(t_{r+2}, t_rt_{r+1})=(x, yz)\in\Lambda$.

Case 5. $(xy, z)=(t_{r-2}t_{r-1}, t_r)$, *where* (t_{r-2}, t_{r-1}, t_r) *is the final part of a* (v, u)*-chain* $\{v, u, t_1, t_2, \ldots\ldots, t_r\}$. By a similar mothod to that of the proof of the case 4, we have $(x, yz)\in\Lambda$.

Case 6. $(xy, z)=(t_r, t_{r-2}t_{r-1})$, *where* (t_{r-2}, t_{r-1}, t_r) *is the final part of a* (u, v)*-chain* $\{u, v, t_1, t_2, \ldots\ldots, t_r\}$. In this case, $xy=t_r$ and $u=t_1t_2$, $vt_1=t_3t_4$, $t_2t_3=t_5t_6, \ldots\ldots,$ $t_{r-4}t_{r-3}=t_{r-1}t_r$.

subcase (i) $r=2$. Since $r=2$, $u=t_1xy$ and $z=vt_1$. Putting $t_1y=t_1'$ and $x=t_2'$, we have a (u, v)-chain $\{u, v, t_1', t_2'\}$. Since (v, t_1', t_2') is the final part of the (u, v)-chain $\{u, v, t_1', t_2'\}$, $(t_2', t_1'v)=(x, yz)\in\Lambda$.

Subcase (ii) $r>2$, *i. e.*, $r\geqq 4$. Since $t_{r-4}t_{r-3}=t_{r-1}t_r$ ($t_{r-4}=v$ in the case of $r=4$), $xyt_{r-1}=t_{r-4}t_{r-3}$. Putting $yt_{r-1}=t_{r-1}'$ and $x=t_r'$, we have $t_{r-1}'t_r'=t_{r-4}t_{r-3}$. Hence, $(t_{r-2}, t_{r-1}', t_r')$ is the final part of the (u, v)-chain $\{u, v, t_1, t_2, \ldots\ldots, t_{r-2}, t_{r-1}', t_r'\}$. Therefore, $(t_r', t_{r-1}'t_{r-2})=(x, yz)\in\Lambda$.

Case 7. $(xy, z)=(t_r, t_{r-2}t_{r-1})$, *where* (t_{r-2}, t_{r-1}, t_r) is the final part of a (v, u)*-chain* $\{v, u, t_1, t_2, \ldots\ldots, t_r\}$. By a similar method to that of the proof of the case 6, we have $(x, yz)\in\Lambda$. Thus Λ satisfies the condition (2) of (C), and hence Λ is a C-factor of T. Now, let Θ be any C-factor containing $\{\Delta_0, (u, v)\}$. Since Θ contains both (u, v) and (v, u), for the final part (t_{r-2}, t_{r-1}, t_r) of a (u, v)- or (v, u)-chain Θ contains both $(t_{r-2}t_{r-1}, t_r)$ and $(t_r, t_{r-2}t_{r-1})$. Hence Θ contains Λ. Therefore, Λ is the least C-factor containing $\{\Lambda_0, (u, v)\}$.

For any C-factor Λ of T and for any subset Ξ of Ω, let $\Gamma(\Lambda, \Xi)$ be the least C-factor of T containing Λ and Ξ. Put $\{\alpha_1, \alpha_2, \alpha_3, \ldots\ldots, \alpha_k\}=\Xi$, where $\alpha_j\in\Omega$. Then, we can easily prove the following relation :

$$\Gamma(\Gamma(\ldots.\Gamma(\Gamma(\Gamma(\Lambda,\{\alpha_1\}), \{\alpha_2\}), \{\alpha_3\}) \ldots\ldots, \{\alpha_{k-1}\}), \{\alpha_k\})=\Gamma(\Lambda, \Xi).$$

Now, by Theorems 7 and 8 we can obtain all C-factors of T. In fact : $\{\Gamma(\Lambda_0, \Sigma) : \Sigma\subseteq\Omega\setminus\Lambda_0$ is the totality of C-factors of T, where Λ_0 is the least C-factor of T.

Remark. In the case in which T is not necessarily commutative, we can also introduce the concept of E-factors of T as follows : A subset Π of Ω satisfying the condition

$$(\mathrm{E})\begin{cases} (1)\quad (t, \mathbf{0})\in\Pi \text{ and } (\mathbf{0}, t)\in\Pi \text{ for any } t\in T, \\ (2)\quad (tv, w)\in\Pi \text{ implies } (t, vw)\in\Pi, \text{ and } (t, vw)\in\Pi \text{ implies} \\ \quad\quad (tv, w)\in\Pi, \end{cases}$$

is called an *extension factor* (abbrev. *E-factor*) of T. It is clear that Ω itself is the greatest E-factor of T. Let $\Psi=\{(x, y) : x, y\in T\}$. Define multiplication $\circ$ [$\circledcirc$] in Ψ as follows : $(x, y)\circ(v, w)=(xv, wy)$ [$(x, y)\circledcirc(v, w)=(vx, yw)$]. Then, the resulting system $\Psi(\circ)$ [$\Psi(\circledcirc)$] becomes a semigroup. It is easy to see that any E-factor of T is a left ideal of $\Psi(\circ)$ and a right ideal of $\Psi(\circledcirc)$. Also, it is easily proved that both $\Psi(\circ)$ and $\Psi(\circledcirc)$

coincide with $T\times T$ if T is commutative. Hence, an E-factor of T is an ideal of $T\times T$ if T is commutative. Every C-factor of a finite commutative z-semigroup is an E-factor, but the converse is not true.

We have

Theorem. *An E-factor Λ of a finite commutative z-semigroup is a C-factor if and only if it satisfies the condition* (3) *of* (C).

Finally, we obtain the following extension theorem for the case in which T is not necessarily commutative :

Theorem. *Let Λ be an E-factor of T, and define multiplication $\circ$ in S by* (M. 4) *of Theorem* 5. *Then, the resulting system $S(\circ)$ becomes an ideal extension of N by T. Further, every ideal extension of N by T is found in this fashion.*

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