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# Automorphisms of Some Surfaces and Equivariant Line Bundles

**Dedicated to Professor Tatsuji Kudo on his 60th birthday**

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**In §1 it is proved that any elliptic surface without exceptional curve admits a canonical involution, which is an extension of the involution in [7]. Since a general elliptic curve admits the unique non trivial involutive isomorphism, then we will call this a canonical one. By making use of a lemma in III [2], it is easy to construct the involution, but in order to find invariant divisors, we make it concretely. Non singular surfaces of degree 4 in $P^3$ are K3 surfaces and one of them is a singular K3 surface. We deduce an information about the homotopical cell structure of a K3 surface. Automorphisms of this surface are constructed in §2. Some of them translate a global section to another section and others do not preserve the elliptic structure. In the last section some remarks are given about elliptic modular surfaces which are singular K3 surfaces.**

## §1. Some generalities

Let $\Phi: S\to P$ be an elliptic surface without any exceptional curve over a non singular algebraic curve free from multiple singular fibres. The period of a generic fibre can be represented as $(\omega(u), 1)$, where Im $\omega(u)>0$. Denote by $J$ the elliptic modular function, then the function $\mathfrak{j}(u)=J(\omega(u))$ is meromorphic in $P$. Denote by $P'$ the set of points in $P$ over which fibres are regular. The restriction $S|P'\to P'$ is a differentiable torus bundle and the locally constant sheaf $G'=\cup\{H_1(\Phi^{-1}(u), Z); u\in P'\}$ can be extended to a sheaf $G$ over $P$. Each element $\beta$ of $\pi_1(P')$ induces a transformation $\omega(u)\to(a_\beta\omega(u)+b_\beta)(c_\beta\omega(u)+d_\beta)^{-1}$. According to the notation in II [2], the surface $S$ belongs to $\mathfrak{F}(\mathfrak{j}, G)$ and by a result in II [2], the family $\mathfrak{F}(\mathfrak{j}, G)$ admits a basic element $B$. Let $U'$ be the universal covering manifold of $P'$, then the restriction $B'=B|P'$ is given by $B'=U'\times C/\mathfrak{g}$, where $C$ is the complex number field and $\mathfrak{g}$ is the group of transformations, $\pi_1(P')\times Z\times Z$($Z$ is the group of integers) such that

$$g(\beta, n_1, n_2)(\tilde{u}, \zeta)=(\beta\tilde{u}, (\zeta+n_1\omega(\tilde{u})+n_2)(c_\beta\omega(\tilde{u})+d_\beta)^{-1})$$

for each $(\tilde{u}, \zeta)\in U'\times C$.
For the transformation $\rho: (\tilde{u}, \zeta)\to(\tilde{u}, -\zeta)$, the diagram

$$\begin{array}{ccc} (\tilde{u}, \zeta) & \xrightarrow{g(\beta, n_1, n_2)} & (\beta\tilde{u}, (\zeta + n_1\omega(\tilde{u}) + n_2)(c_\beta\omega(\tilde{u}) + d_\beta)^{-1} \\ \downarrow \rho & & \downarrow \rho \\ (\tilde{u}, -\zeta) & \xrightarrow{g(\beta, -n_1, -n_2)} & (\beta\tilde{u}, (-\zeta - n_1\omega(\tilde{u}) - n_2)(c_\beta\omega(\tilde{u}) + d_\beta)^{-1} \end{array}$$

is commutative, then the involution $\rho$ determines an involution $(\rho)$: $B' \to B'$, which preserves the zero section 0. By the lemma 10.4 III [2], it can be seen that the involution $(\rho)$ is extendible to an involution $[\rho]$: $B \to B$. But in order to find invariant divisors, we extend $(\rho)$ conceretely according to the construction in § 8, II [2].

(i) In the neighbourhoods of fibres of type $I_1$, using the representation of $x$, $y$ by Weierstrass' functions $\mathfrak{p}(\zeta)$, $\mathfrak{p}'(\zeta)$ in p. 592, II [2], $\rho$ induces an involution $(x, y) \to (x, -y)$, then the formula (8.40) in II [2] is invariant and so we have an involuiton $[\rho]$, which leaves invariant fibres of type $I_1$. For fibres of type $I_b$, it induces $(\tau, w) \to (\tau, w^{-1})$, where $w = \exp 2\pi i\zeta$, $\tau = \exp 2\pi i\tilde{u}$, so the covering map given by (8.44) in II [2] admits a natural lift $[\rho]$: $B \to B$. Thus each fibre of type $I_b$ is $[\rho]$-invariant.

(ii) For fibres of type IV*, by the relation $(2_3)$, p. 591–592 [2], we have

$$-\left(\frac{1}{3}\eta + \frac{2}{3}\right) = \left(\frac{2}{3}\eta + \frac{1}{3}\right) \bmod Z[\eta] + Z,$$

then fixed points by the cyclic group $\mathfrak{c}$ and so divisors $\{\Theta_{11}, \Theta_{12}\}$, $\{\Theta_{21}, \Theta_{22}\}$ are non invariant under the action $[\rho]$, and they are mutually transformed into others by $[\rho]$. For fibres of type IV the situation is quite similar ($(2_3)$ p. 592–593 [2]).

(iii) For fibres of other types, by the checking in each case, we see that they are all $[\rho]$-invariant.

Throughout this paper we treate elliptic surfaces without any exceptional curve. By Theorem 11.1, III [2], any elliptic surface free from multiple singular fibres can be obtaned by the pasting method from a basic element. Then by the results in § 14, [2], [7], we have

PROPOSITION 1. *Any elliptic surface free from multiple singular fibres admits a non trivial involution.*

REMARK. The involution constructed in [7] is a special one of the canonical involution.

By the result in § 4, [3], any elliptic surface is obtained by logarithmic transformations from an elliptic surface without multiple fibres. The transformation (33) (36) in § 4 [3] is compatible with the involution

$$(\zeta - \gamma(\sigma))(c\omega(\sigma^m) + d)^{-1} \longrightarrow (-\zeta - \gamma(\sigma))(c\omega(\sigma^m) + d)^{-1},$$

$$(\sigma, w))_j \longrightarrow ((\sigma, w^{-1}))_j$$

respectively. Then we have

COROLLARY. *Any elliptic surface admits a non trivial involution.*

## §2. Hypersurfaces in $P^3$

We consider a non singular hypersurface of degree 4 in the three dimensional projective space $P^3$,

$$S: \Sigma_{\mu_0+\mu_1+\mu_2+\mu_3=4} t_\mu z_0^{\mu_0} z_1^{\mu_1} z_2^{\mu_2} z_3^{\mu_3} = 0,$$

where $\mu=(\mu_0, \mu_1, \mu_2, \mu_3)$ is a sequence of non negative integers and $t$ is a complex number for each $\mu$. Let $E\to P^3$ be the bundle given by transition functions $\{e_{ij}= z_j/z_i\}$. Then the divisor $[S]$ determines the line bundle $E^{-4}\to P^3$. We have the following exact sequence of sheaves of germs of holomorphic sections,

$$0 \longrightarrow O(E^{-4}) \longrightarrow O(\mathbf{1}) \longrightarrow O_S(\mathbf{1}) \longrightarrow 0,$$

where $\mathbf{1}$ is the trivial line bundle over $P^3$ and $O_S$ is the structure sheaf of $S$. By this we have an exact sequence of cohomology groups,

$$H^1(P^3, O) \longrightarrow H^1(S, O_S(\mathbf{1})) \longrightarrow H^2(P^3, O(E^{-4})).$$

Since the canonical bundle of $P^3$ is equal to $E^{-4}\to P^3$, then by Theorem 3 and Theorem 2 in [4]. We have

$$H^1(P^3, O)=H^2(P^3, O(E^{-4}))=0 \quad \text{and} \quad H^1(S, O_S)=0.$$

On the other hand we have an exact sequence associated with the embedding $S\subset P^3$,

$$0 \longrightarrow T(S) \longrightarrow T(P^3)|S \longrightarrow \nu \longrightarrow 0,$$

where we mean by $T(\ \ )$ the tangent bundle and by $\nu$ the normal bundle of the embedding. Since $\nu=[S]\,|S=E^{-4}|S$, then the canonical bundle of $S$ is trivial. Thus the surface $S$ is a $K3$ surface. The second Betti number of a $K3$ surface is 22, and its homology basis and the intersection numbers are known (for example [5]). Then by the duality between intersection numbers and cup products, and 12.2 (b) [10], it can be seen that the surface admits the following cell decomposition,

$$S\simeq K=(\vee_{i=1}^{22} S_i^2) \cup_\beta e^4,$$

where $\vee_{i=1}^{22} S_i^2$ denotes the bouquet of 2-spheres and $\beta\colon \partial e^4=S^3\to \vee_{i=1}^{22} S_i^2$ is the homotopy boundary of the 4-cell and its homotopy class is given by

$$[\beta]=\Sigma\varepsilon_{ij}[\iota_i, \iota_j], \quad \text{a sum of Whitehead products,}$$

and $\varepsilon_{ij}=\pm 1$ or zero, especially $\varepsilon_{ii}=-1$ for all $i=1,\dots, 20$.

By [6], there exists a non singular hypersurface of degree 4 in $P^3$ such that the group of automorphisms is an infinite group. For a K3 surface $S$ and its automorphisms $f$, $g$ if $f$ is homotopic to $g$, then the homomorphism

$$(g^{-1}f)_*: H_2(S, Z) \longrightarrow H_2(S, Z)$$

is the identity, so by the proposition 2 in §2 [8], $g^{-1}f$ is the identity automorphism of $S$, thus $f=g$. Then we have

PROPOSITION 2. *The set consisting of homotopy classes of homotopy equivalences $K \to K$ is an infinite set.*

REMARK 1. By a result in [1], we have another cell decomposition,

$$(M_3 \cup e_1^2 \cup \cdots \cup e_r^2) \cup e^4,$$

where $M_3$ is a plane curve of genus 3 which is given by the equation $z_1^4+z_2^4+z_3^4=0$.

Now we consider a typical K3 surface with the Picard number 20, which is given by the equation,

$$S_0: z_0^4+z_1^4+z_2^4+z_3^4=0 \quad \text{in} \quad P^3.$$

Let us set

$$t=\frac{z_0^2+iz_1^2}{z_2^2-iz_3^2}=-\frac{z_2^2+iz_3^2}{z_0^2-iz_1^2}, \ i=\sqrt{-1},$$

then we have an elliptic structure of $S_0$ over the projective plane $P$ which is given by

$$\Phi: S_0 \ni [z_0, z_1, z_2, z_3] \longrightarrow t \in P.$$

Singular fibres of $\Phi$ are given as follows:

$$\left.\begin{array}{ll} t=0 & \cdots [z_0, \pm\sqrt{i}\,z_0, z_2, \pm\sqrt{i}\,z_2] \\ t=\infty & \cdots [z_0, \pm\sqrt{-i}\,z_0, z_2, \pm\sqrt{-i}\,z_2] \\ t=1 & \cdots [z_0, z_1, \pm\sqrt{i}\,z_1, \pm\sqrt{i}\,z_0] \\ t=-1 & \cdots [z_0, z_1, \pm\sqrt{-i}\,z_1, \pm\sqrt{-i}\,z_0] \\ t=i & \cdots [z_0, z_1, \pm\sqrt{-i}\,z_0, \pm\sqrt{i}\,z_1] \\ t=-i & \cdots [z_0, z_1, \pm\sqrt{i}\,z_0, \pm\sqrt{-i}\,z_1]. \end{array}\right\} \qquad \text{(L)}$$

All of these fibres are of type $I_4$. For other $t$ than in the list (L), we have

$$z_0^2=\frac{1}{2}\left(t-\frac{1}{t}\right)z_2^2-\frac{1}{2}\left(t+\frac{1}{t}\right)iz_3^2, \quad z_1^2=-\frac{i}{2}\left(t+\frac{1}{t}\right)z_2^2-\frac{1}{2}\left(t-\frac{1}{t}\right)z_3^2,$$

and so set $a=\frac{1}{2}\left(t-\frac{1}{t}\right)$, $b=\frac{1}{2}\left(t+\frac{1}{t}\right)$, then we have

$$z_0^2 = az_2^2 - ibz_3^2, \quad z_1^2 = -ibz_2^2 - az_3^2.$$

Further set $\lambda = \left(\frac{z_3}{z_2}\right)^2$, $\mu = \frac{z_0 z_1 z_3}{z_2^3}$, then the fibre over $t$ is given by the equation,

$$\mu^2 = \lambda(a - ib\lambda)(-ib - a\lambda), \tag{E}$$

hence the surface $S_0$ is an elliptic surface. The second Chern class of a K3 surface is 24, then there does not exist other singular fibre.

Next we seek global sections of the elliptic structure $\Phi$. Set $z_1 = \alpha z_0$, $z_3 = \alpha z_2$, then we have

$$z_0^2 + iz_1^2 = (1 + i\alpha^2) z_0^2 = t(z_0^2 - iz_1^2) = t(1 - i\alpha^2) z_0^2,$$

$$t(z_2^2 - iz_3^2) = t(1 - i\alpha^2) z_2^2 = -(z_2^2 + iz_3^2) = -(1 + i\alpha^2) z_2^2,$$

and so

$$\frac{z_0^2}{z_1^2} \frac{1 + i\alpha^2}{t(1 - i\alpha^2)} = -\frac{z_0^2 t(1 - i\alpha^2)}{z_2^2(1 + i\alpha^2)}.$$

Thus we have $t^2 = \frac{(1 + i\alpha^2)^2}{(1 - i\alpha^2)^2}$, $t = \pm i\frac{1 + i\alpha^2}{1 - i\alpha^2}$, and $\alpha = \pm\sqrt{\frac{1 \mp t}{1 + it}}$. On the other hand, by the equation $z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0$, we have

$$(1 + \alpha^4)(z_0^4 + z_2^4) = 0,$$

hence we obtain 16 sections given by $[z_0, \alpha z_0, (\exp n\pi i/4) z_0, \alpha(\exp n\pi i/4) z_0]$, $n = \pm 1$, $\pm 3$. In the case $t = +i$, we have $\alpha = \infty$ and take $[0, 1, 0, \exp n\pi i/4]$ as sections.

The second algebraic homology basis is given by 18 divisors in the list (L) and a generic fibre, and a global section. Then by Theorem 1.1 in [9], the surface $S_0$ is a singular K3 surface, i.e. the Picard number is 20. By Lemma 10.3, 10.4 III [2], the above 16 sections determine 16 automorphisms of the surface $S_0$. In these automorphisms, the ones corresponding to the translations by $\exp n\pi i/4$ keep the divisors invariant, but the others by $\alpha$'s do not. By the proposition 1, we have

PROPOSITION 2. *The surface $S_0$ admits* 32 *automorphisms which preserve the elliptic structure. The equivariant Picard number is smaller than* 20.

REMARK 2. The symmetric group $S_4$ acts on $P^3$, and the surface $S_0$ is invariant by this action, but the action does not preserve the elliptic structure. Further the surface $S_0$ admits actions given by

$$[z_0, z_1, z_2, z_3] \longrightarrow [i^{\varepsilon_0} z_0, i^{\varepsilon_1} z_1, i^{\varepsilon_2} z_2, i^{\varepsilon_3} z_3],$$

where $\varepsilon_j = 0, 1, 2, 3$ and $j = 0, 1, 2, 3$. Set $\varepsilon = \varepsilon(\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3)$. These actions give a transformation group of order $4^3$. The ε-type action does not preserve the elliptic structure. For example, let $\Phi_1: S_0 \to P$ be the elliptic structure given by

$$t_1 = \frac{z_0^2 + iz_1^2}{z_2^2 + iz_3^2} = -\frac{z_2^2 - iz_3^2}{z_0^2 - iz_1^2},$$

then we have the commutative diagram,

$$\begin{array}{ccc} S_0 & \xrightarrow{\varepsilon(1,0,0,0)} & S_0 \\ \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \Phi_1} \\ P & \xrightarrow{\varepsilon_*(1,0,0,0)} & P \end{array},$$

where $\varepsilon_*(1, 0, 0, 0)$ is given by the mapping $t \to \frac{1}{t}$.

## §3. Some elliptic modular surfaces

**1** Let $\Gamma(4)$ be the principal congruence subgroup of level 4 in $SL(2, Z)$. Consider the action of $\Gamma(4)$ on the upper half plane. There are 6 cusp points. By the aspect of the representation of isotropy groups, we have a singular K3 surface $B_{\Gamma(4)}$ [9]. It has 6 singular fibres of type $I_4$ and it is an elliptic surface, and admits 16 global sections, $s_m(\exp \pi iz/2) = ((\exp \pi i(m_1 z + m_2)/2))_0 \in W_0$ [9]. The automorphisms induced by the translations of $m_1$ do not leave divisors invariant, but the ones corresponding to the variation of $m_2$ leave them invariant. These automorphisms do not include the canonical one in §1.

**2** let $\Gamma_0'(7)$ be the subgroup of $SL(2, Z)$ given by $\Gamma_0'(7) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, Z): c \equiv 0 \bmod 7, \left(\frac{a}{7}\right) = 1\right\}$, where $\left(\frac{a}{7}\right)$ denotes the Legendre symbol. Then the corresponding elliptic modular surface $B_{\Gamma_0'(7)}$ is a singular K3 surface with singular fibres $I_1$, $I_7$, IV*, IV*. The canonical automorphism $[\rho]$ does not leave divisors in the fibres of type IV* invariant as we have seen in §1.

REMARK. By the formula (E) in §2, the involution $\iota$: $[z_0, z_1, z_2, z_3] \to [z_0, z_1, -z_2, z_3]$ gives an involution $(\lambda, \mu) \to (\lambda, -\mu)$ of a generic fibre. A generic fibre is an elliptic curve and a general elliptic curve admits unique involutive isomorphism.

## References

[1] R. Bott, On a theorem of Lefschetz, Mich. Math. J. **6** (1959), 211–216.
[2] K. Kodaira, On compact complex analytic surfaces, II–III, Ann. of Math. **77** (1963), 563–626, **78** (1963), 1–40.
[3] K. Kodaira, On the structure of compact complex analytic surfaces, I, Amer. J. Math. **86** (1964), 751–798.

[4] K. Kodaira, On a differential-geometric method in the theory of analytic stacks, Proc. N.A.S. **39** (1953), 1268–1273.

[5] K. Kodaira, Theory of complex analytic surfaces, Lecture note, Univ. Tokyo, 1974, in Japanese.

[6] H. Matsumura and P. Monsky, On the automorphisms of hypersurfaces, J. Math. Kyoto Univ. **3**-3 (1964), 347–361.

[7] H. Matsunaga, On equivariant line bundles over some surfaces, Mem. Fac. Sci. Shimane Univ. **12** (1978), 15–18.

[8] I. I. Pjateckiĭ-Šapiro and I. R. Šafarevič, A Torelli theorem for algebraic surfaces of type K3, Math. USSR. Izvestija, **5**-3 (1971), 547–588.

[9] T. Shioda, On elliptic modular surfaces, J. Math. Soc. Japan, **24** (1) (1972), 20–59.

[10] J. H. C. Whitehead, On simply connected 4-dimensional polyhedra, Comm. Math. Helv. **22** (1949), 48–92.