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Almost Periodic Functions on Topological Semigroups

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In this paper, an almost periodic function on a topological semigroup will be characterized by ε-almost periods. A weakly almost periodic function will be characterized by a Banach space and its dual space.

§1. Introduction

Let S be a *topological semigroup* with *identity* e, equipped with a *Hausdorff topology* $\mathscr{T}$ in which multiplication is *separately continuous*, that is, for each $s \in S$ the maps $x \to sx$ and $x \to xs$ are continuous. Let $C(S)$ be the space of all bounded continuous functions on S with the *norm* $\|f\| = \sup_{x \in S} |f(x)|$ and let $\mathscr{U}$ be the *system of neighborhoods* of e. For $f \in C(S)$ and $s \in S$, let us put $f_s = f(xs)$ and ${}_sf(x) = f(sx)$. For $f \in C(S)$ we denote $O_R(f) = \{f_y: y \in S\}$ and $O_L(f) = \{{}_yf: y \in S\}$. We say that a function $f \in C(S)$ is *almost periodic* (resp. *weakly almost periodic*) if $O_R(f)$ or $O_L(f)$ is relatively compact in the norm (resp. weak) topology on $C(S)$. Note that $O_R(f)$ is relatively compact in the weak (resp. norm) topology if and only if $O_L(f)$ is so in the weak (resp. norm) topology (cf. [2]). A subset A of S is *right* (resp. *left*) *totally bounded* if, for every $V \in \mathscr{U}$ there exists a finite subset $\{y_1, \dots, y_n\}$ of A such that

$$\bigcup_{i=1}^{n} y_i V \supset A \quad (\text{resp. } \bigcup_{i=1}^{n} V y_i \supset A).$$

The topology $\mathscr{T}$ on S is called *right* (resp. *left*) *translation invariant* if $\mathscr{U}x = \{Vx: V \in \mathscr{U}\}$ (resp. $x\mathscr{U} = \{xV: V \in \mathscr{U}\}$) is the system of neighborhoods of x for every $x \in S$. We shall say that $\mathscr{T}$ is *translation invariant* if it is left and right translation invariant. In case $\mathscr{T}$ is right translation invariant we shall say that a function $f \in C(S)$ is *right quasi-uniformly continuous* on A if, for every $\varepsilon > 0$ there exists $V \in \mathscr{U}$ such that

$$|f(\eta x) - f(x)| < \varepsilon \qquad \text{for} \quad \eta \in V \quad \text{and} \quad x \in A.$$

In case $\mathscr{T}$ is left translation invariant, the *left quasi-uniform continuity* of f is similarly defined. In case $\mathscr{T}$ is translation invariant, a function $f \in C(S)$ is called *quasi-uniformly continuous* if it is right and left quasi-uniformly continuous on A.

REMARK. If S is in particular a topological group, $\mathscr{T}$ is *translation invariant* and a quasi-uniformly continuous function on S is *uniformly continuous*.

We shall prove in § 2 that an almost periodic function on a topological semigroup is characterized by a family of totally bounded subsets of S. The existence of an "*ε-almost period*" in every translated set of a totally bounded set will be shown in § 3 in the case where S is a commutative group. Note that a topological semigroup does not have such a uniform structure as in [1]. We shall give in § 4 some examples of almost periodic functions in case S is an ordered space or metric space. A characterization of a weakly almost periodic function on S will be given in § 5.

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§ 2. Almost periodic functions on topological semigroups

Let $\mathscr{A}$ be a family of subsets of S. We define the following several conditions. We shall say that a function $f \in C(S)$ satisfies condition (B) with respect to $\mathscr{A}$ if, for every $\varepsilon > 0$ there is $A_\varepsilon \in \mathscr{A}$ such that

$$A_\varepsilon(f, y) = \{\sigma \in A_\varepsilon : \|f_{yx} - f_{\sigma x}\| < \varepsilon \quad \text{for all} \quad x \in S\} \neq \emptyset \qquad \text{for each} \quad y \in S.$$

We shall also say that a function $f \in C(S)$ satisfies condition (B_R) with respect to $\mathscr{A}$ if, for every $\varepsilon > 0$ there is $A_\varepsilon \in \mathscr{A}$ such that

$$A_\varepsilon^R(f, y) = \{\tau \in S : y \in A_\varepsilon \tau \quad \text{and} \quad \|f_\tau - f\| < \varepsilon\} \neq \emptyset \qquad \text{for each} \quad y \in S.$$

Condition (B_L) with respect to $\mathscr{A}$ is defined by replacing $A_\varepsilon \tau$ and f_τ by τA_ε and ${}_\tau f$ respectively. We shall say that a function $f \in C(S)$ satisfies condition (B_{RR}) with respect to $\mathscr{A}$ if, for every $\varepsilon > 0$ there is $A_\varepsilon \in \mathscr{A}$ such that

$$A_\varepsilon^{RR}(f, y) = \{\sigma \in A_\varepsilon y : \|f_\sigma - f\| < \varepsilon\} \neq \emptyset \qquad \text{for each} \quad y \in S.$$

Conditions (B_{RL}), (B_{LR}) and (B_{LL}) with respect to $\mathscr{A}$ are defined by replacing the pair $(A_\varepsilon y, f_\sigma)$ by $(A_\varepsilon y, {}_\sigma f)$, $(yA_\varepsilon, f_\sigma)$ and $(yA_\varepsilon, {}_\sigma f)$ respectively.

We have the following:

THEOREM 1. *If f is almost periodic, then there exists a family $\mathscr{A} = \{A_\varepsilon\}$ of finite subsets of S such that f satisfies condition* (B) *with respect to $\mathscr{A}$.* (*cf.* [4])

PROOF. Let $f \in C(S)$ be an almost periodic function on S. Then $O_R(f)$ is relatively compact in the norm topology on $C(S)$, whence it is totally bounded. For any $\varepsilon > 0$, there exists a finite subset $\{x_1, \dots, x_n\}$ such that

$$\bigcup_{k=1}^{n} U(f_{x_k}, \varepsilon/4) \supset O_R(f), \text{ where } U(f, \varepsilon) = \{g \in C(S) : \|g - f\| < \varepsilon\}.$$

Note that $f_{yx_i} \in O_R(f)$ for every $y \in S$ and $i \in Z_n = \{1, \dots, n\}$. Let J be the set of all mappings of Z_n into Z_n, and put

$$V_j=\{y\in S\colon \|f_{yx_i}-f_{x_{j(i)}}\|<\varepsilon/4 \quad \text{for all} \quad i\in Z_n\} \qquad \text{for} \quad j\in J.$$

Then $\bigcup_{j\in J} V_j=S$. Choose one element y_j from each V_j, and let A_ε be the set $\{y_j\}$. Clearly A_ε is a finite set. Let $y\in S$. Then there exists $j\in J$ such that $y\in V_j$, so that

$$\|f_{yx_i}-f_{y_jx_i}\|\leqq \|f_{yx_i}-f_{x_{j(i)}}\|+\|f_{x_{j(i)}}-f_{y_jx_i}\|<\varepsilon/2.$$

Let $x\in S$ and choose $x_i\in U(f_x, \varepsilon/4)$. Then we have

$$\|f_{yx}-f_{y_ix}\|\leqq \|f_{yx}-f_{yx_i}\|+\|f_{yx_i}-f_{y_jx_i}\|+\|f_{y_jx_i}-f_{y_jx}\|<\varepsilon.$$

Namely f satisfies condition (B) with respect to $\mathscr{A}=\{A_\varepsilon\}$.

THEOREM 2. *Assume that the topology $\mathscr{T}$ of S is right (resp. left) translation invariant and let $\mathscr{A}=\{A_\varepsilon\}$ be a family of subsets of S. If a function $f\in C(S)$ satisfies condition* (B) *with respect to $\mathscr{A}$ and if f is right (resp. left) quasi-uniformly continuous on each A_ε, then f is right (resp. left) quasi-uniformly continuous on S.*

PROOF. For any $\varepsilon>0$, there is $V\in\mathscr{U}$ such that

$$|f(\eta\rho)-f(\rho)|<\varepsilon/3 \quad \text{for} \quad \eta\in V \quad \text{and} \quad \rho\in A_{\varepsilon/3}$$

by the quasi-uniform continuity of f on $A_{\varepsilon/3}$. Let $y\in S$. Then there exists $y_0\in A_{\varepsilon/3}(y)$ by condition (B) with respect to $\mathscr{A}$, that is,

$$|f_{yx}(z)-f_{y_0x}(z)|<\varepsilon/3 \qquad \text{for every} \quad x,\ z\in S.$$

Now, set $x=e$. Then

$$|f(zy)-f(zy_0)|<\varepsilon/3 \qquad \text{for} \quad z\in S.$$

For $\eta\in V$, we have

$$\begin{aligned}|f(\eta y)-f(y)|&\leqq |f(\eta y)-f(\eta y_0)|+|f(\eta y_0)-f(y_0)|+|f(y_0)-f(y)|\\&<\varepsilon/3+\varepsilon/3+\varepsilon/3=\varepsilon.\end{aligned}$$

Therefore, f is right quasi-uniformly continuous.

Note that in case $\mathscr{T}$ is right (resp. left) translation invariant, a continuous function is right (resp. left) quasi-uniformly continuous on any finite subset of S. Then, by Theorem 1 and Theorem 2 an almost periodic function f is right (resp. left) quasi-uniformly continuous on S.

THEOREM 3. *Let $\mathscr{T}$ be right (resp. left) translation invariant. If $f\in C(S)$ is right (resp. left) quasi-uniformly continuous, if there exists a family $\mathscr{A}=\{A_\varepsilon\}$ of left (resp. right) totally bounded sets of S and if f satisfies condition* (B) *with respect to $\mathscr{A}$, then f is almost periodic.*

PROOF. For each $\varepsilon>0$ and $y\in S$, let $\rho\in A_{\varepsilon/2}(f, y)$. Then we have

$$|_{\rho}f(x)-{}_{y}f(x)|<\varepsilon/2 \qquad \text{for} \quad x\in S.$$

Since f is right quasi-uniformly continuous on S, there exists $W\in\mathscr{U}$ such that

$$|f(\eta x)-f(x)|<\varepsilon/2 \qquad \text{for} \quad \eta\in W \quad \text{and} \quad x\in S.$$

For $x\in S$ we have

$$|_{\eta y}f(x)-{}_{y}f(x)|=|f(\eta yx)-f(yx)|\leqq\sup_{z\in S}|f(\eta z)-f(z)|\leqq\varepsilon/2.$$

Since $A_{\varepsilon/2}$ is left totally bounded, there exists a finite subset $\{y_1,\dots, y_n\}$ of $A_{\varepsilon/2}$ such that $\bigcup_{i=1}^{n} Wy_i\supset A_{\varepsilon/2}$. There exist $i\in Z_n$ and $\eta\in W$ such that $\rho=\eta y_i$. It follows that

$$\|_{y}f-{}_{y_i}f\|=\sup_{x\in S}|_{y}f(x)-{}_{y_i}f(x)|\leqq\sup_{x\in S}|_{y}f(x)-{}_{\rho}f(x)|+\sup_{x\in S}|_{\rho}f(x)-{}_{y_i}f(x)|\leqq\varepsilon.$$

Hence $O_L(f)$ is totally bounded and the function f is almost periodic on S.

§3. Almost periodic functions on groups

Replacing the inequality $\|_{\tau}f-f\|<\varepsilon$ in condition $(\mathrm{B_L})$ by the inequality $\|_{\tau}f-f\|\leqq\varepsilon$, we define condition $(\mathrm{B'_L})$.

LEMMA 1. *Let S be an algebraic group, $f\in C(S)$ and let $\mathscr{A}=\{A_\varepsilon\}$ be a family of subsets of S.*

(a) *If f satisfies condition* (B) *with respect to $\mathscr{A}$, then f satisfies conditions* $(\mathrm{B_R})$ *and* $(\mathrm{B'_L})$ *with respect to $\mathscr{A}$.*

(b) *f satisfies condition* $(\mathrm{B_R})$ *(resp.* $(\mathrm{B_L})$*) with respect to $\mathscr{A}$ if and only if f satisfies condition* $(\mathrm{B_{RR}})$ *(resp.* $(\mathrm{B_{LL}})$*) with respect to the family $\mathscr{A}^{-1}=\{A_\varepsilon^{-1}\}$, where $A_\varepsilon^{-1}=\{x^{-1}: x\in A_\varepsilon\}$.*

PROOF. (a) First we show that $A_\varepsilon(f, y)\neq\emptyset$ implies $A_\varepsilon^R(f, y)\neq\emptyset$. Let $\sigma\in A_\varepsilon(f, y)$. Then

$$\sup_{z\in S}|f(zy)-f(z\rho)|=\|f_y-f_\rho\|<\varepsilon.$$

Setting $\sigma^{-1}y=\tau$, we have

$$y=\sigma\tau\in A_\varepsilon\tau \quad \text{and} \quad \|f_\tau-f\|<\varepsilon.$$

Next we shall show that f satisfies condition $(\mathrm{B'_L})$ with respect to $\mathscr{A}$. It is easy to see that for each $x\in S$ and $\sigma\in A_\varepsilon(f, y)$, $\sup_{z\in S}|f(zyx)-f(z\sigma x)|<\varepsilon$. Then it yields the following

$$\sup_{z\in S}\|_{zy}f-{}_{z\sigma}f\|=\sup_{z\in S}\sup_{x\in S}|_{zy}f(x)-{}_{z\sigma}f(x)|=\sup_{x\in S}\sup_{z\in S}|f(zyx)-f(z\sigma x)|\leqq\varepsilon$$

Setting $y\sigma^{-1}=\tau$, similarly we have $y=\tau\sigma\in\tau A_{\varepsilon}$ and $\|{}_{\tau}f-f\|<\varepsilon$.

(b) This is obvious.

LEMMA 2. *Let S be a topological group. A subset A of S is right totally bounded if and only if A^{-1} is left totally bounded.*

PROOF. Let A be right totally bounded. For each $V\in\mathscr{U}$ we can choose $W\in\mathscr{U}$ such that $W^{-1}\subset V$. There exists a finite subset $\{y_1,\dots, y_n\}$ of A such that $\bigcup_{i=1}^{n} y_iW\supset A$. Then we have $\{y_1^{-1},\dots, y_n^{-1}\}\subset A^{-1}$ and

$$\bigcup_{i=1}^{n} Vy_i^{-1}\supset\bigcup_{i=1}^{n} W^{-1}y_i^{-1}=\bigcup_{i=1}^{n}(y_iW)^{-1}=(\bigcup_{i=1}^{n} y_iW)^{-1}\supset A^{-1}.$$

Thus A^{-1} is left totally bounded.

We introduce the following conditions for $f\in C(S)$:

($\mathrm{I_a}$) f is almost periodic.

($\mathrm{I_b}$) There exists a family $\mathscr{A}$ of finite subsets of S and f satisfies condition (B) with respect to $\mathscr{A}$.

($\mathrm{I_c}$) (resp. ($\mathrm{I_d}$)) There exists a family $\mathscr{A}$ of right (resp. left) totally bounded subsets of S and f satisfies condition (B) with respect to $\mathscr{A}$.

($\mathrm{II_b}$) (resp. ($\mathrm{II'_b}$)) There exists a family $\mathscr{A}$ of finite subsets of S and f satisfies condition ($\mathrm{B_R}$) (resp. ($\mathrm{B_L}$)) with respect to $\mathscr{A}$.

($\mathrm{II_c}$) (resp. ($\mathrm{II'_c}$)) There exists a family $\mathscr{A}$ of right (resp. left) totally bounded subsets of S and f satisfies condition ($\mathrm{B_R}$) (resp. ($\mathrm{B_L}$)) with respect to $\mathscr{A}$.

($\mathrm{III_b}$) (resp. ($\mathrm{III'_b}$)) There exists a family $\mathscr{A}$ of finite subsets of S and f satisfies condition ($\mathrm{B_{RR}}$) (resp. ($\mathrm{B_{LL}}$)) with respect to $\mathscr{A}$.

($\mathrm{III_c}$) (resp. ($\mathrm{III'_c}$), ($\mathrm{III_d}$), ($\mathrm{III_e}$)) There exists a family $\mathscr{A}$ of right (resp. left, left, right) totally bounded subsets of S and f satisfies condition ($\mathrm{B_{RR}}$) (resp. ($\mathrm{B_{LL}}$), ($\mathrm{B_{RL}}$), ($\mathrm{B_{LR}}$)) with respect to $\mathscr{A}$.

THEOREM 4. (a) *Let S be an algebraic group. Then we have*

$$\begin{array}{ccccc}
 & & (\mathrm{III'_b}) & \Longrightarrow & (\mathrm{III'_c}) \\
 & & \Updownarrow & & \\
 & & (\mathrm{II'_b}) & \Longrightarrow & (\mathrm{II'_c}) \\
 & & \Uparrow & & \\
(\mathrm{I_a}) & \Longrightarrow & (\mathrm{I_b}) & & \\
 & & \Downarrow & & \\
 & & (\mathrm{II_b}) & \Longrightarrow & (\mathrm{II_c}) \\
 & & \Updownarrow & & \\
 & & (\mathrm{III_b}) & \Longrightarrow & (\mathrm{III_c}).
\end{array}$$

(b) *Let S be an algebraic group, and let $\mathscr{T}$ be right (resp. left) translation invariant. Then we have*

($\mathrm{III_d}$) (resp. ($\mathrm{III_e}$)) *and f is right (resp. left) quasi-uniformly continuous* $\Rightarrow$($\mathrm{I_a}$).

(c) *Let S be an Abelian group, and let $\mathscr{T}$ be translation invariant. Then we have*

$(I_a)\Leftrightarrow(I_b)\Leftrightarrow(I_c)$ *and f is quasi-uniformly continuous* $\Leftrightarrow(II_b)\Leftrightarrow(III_b)\Leftrightarrow(III_c)$ *and f is quasi-uniformly continuous.*

(d) *Let S be a topological group. Then we have*

$$\begin{array}{ccccl} & & (III'_b) & \Longrightarrow & (III'_c) \\ & & \Updownarrow & & \Updownarrow \\ & & (II'_b) & \Longrightarrow & (II'_c) \\ & & \Uparrow & & \Uparrow \\ (I_a) & \Longleftrightarrow & (I_b) & \Longleftrightarrow & (I_c)\ (\textit{or}\ (I_d))\ \textit{and } f \textit{ is uniformly continuous on } S \\ & & \Downarrow & & \Downarrow \\ & & (II_b) & \Longrightarrow & (II_c) \\ & & \Updownarrow & & \Updownarrow \\ & & (III_b) & \Longrightarrow & (III_c). \end{array}$$

§4. Ordered topological semigroups and metric semigroups

Let P be a subsemigroup of S such that $e\in P$ and $xP=Px$ for every $x\in S$. For $a, b\in S$ we write $a\succeq b$ if $a\in Pb$. Then this relation "$\succeq$" is a preorder in S which is compatible with the multiplication, that is,

(1) $a\succeq a$ for every $a\in S$.

(2) $a\succeq b,\ b\succeq c\Rightarrow a\succeq c$

(3) $a\succeq b\Rightarrow ac\succeq bc,\ ca\succeq cb$ for every $c\in S$.

Suppose that S is an *upper* and *lower directed set*, that is, for every $a, b\in S$ there exist $c, d\in S$ such that $c\succeq a\succeq d$ and $c\succeq b\succeq d$. For $a, b\in S$ we put $[a, b]=\{x\in S: b\succeq x\succeq a\}$. We say that S satisfies condition (R*) (resp. (L*)) if $[x, y]$ is right (resp. left) totally bounded for every $x, y\in S$ with $y\succeq x$.

THEOREM 5. *Assume that S satisfies condition* (L*) (*resp.* (R*)), *that $\mathscr{T}$ is right* (*resp. left*) *translation invariant and that f is a bounded right* (*resp. left*) *quasi-uniformly continuous function. Then f is almost periodic if and only if, for every $\varepsilon>0$, there exist $\lambda(\varepsilon), \mu(\varepsilon)\in S$ such that $\lambda(\varepsilon)\succeq\mu(\varepsilon)$ and f satisfies condition* (B) *with respect to the family $\mathscr{A}=\{[\mu(\varepsilon), \lambda(\varepsilon)]\}$.*

PROOF. If f is almost periodic, then by Theorem 1 there exists a family $\{A_\varepsilon\}$ of finite subsets of S such that f satisfies condition (B) with respect to $\{A_\varepsilon\}$. Since A_ε is a finite set, there exist $\mu(\varepsilon), \lambda(\varepsilon)\in S$ such that $\lambda(\varepsilon)\succeq\mu(\varepsilon)$ and $[\mu(\varepsilon), \lambda(\varepsilon)]\supset A_\varepsilon$. It is clear that f satisfies condition (B) with respect to the family $\{[\mu(\varepsilon), \lambda(\varepsilon)]\}$. Since $[\mu(\varepsilon), \lambda(\varepsilon)]$ is left totally bounded by the assumption, we can see from Theorem 5 that f is almost periodic.

COROLLARY 1. *Assume that S and f satisfy the same conditions as in Theorem*

4 and that S is an algebraic group. Then f is almost periodic if and only if, for every $\varepsilon>0$, there exists $\mu(\varepsilon)\in S$ such that $e\succeq\mu(\varepsilon)$ and f satisfies condition (B) *with respect to the family* $\{[\mu(\varepsilon), e]\}$.

COROLLARY 2. *Assume that S and f satisfy the same conditions as in Theorem 4 and that S is an Abelian group. Then f is almost periodic if and only if, for every $\varepsilon>0$, there exists $\mu(\varepsilon)\in S$ such that $e\succeq\mu(\varepsilon)$ and f satisfies condition* (B_{RR}) *with respect to the family* $\{[\mu(\varepsilon), e]\}$.

REMARK. Let f be a function on S and let $\varepsilon>0$. We call $\tau\in S$ a right (resp. left) ε-almost period of f if

$$\sup_{x\in S}|f(x\tau)-f(x)|<\varepsilon \text{ (resp. } \sup_{x\in S}|f(\tau x)-f(x)|<\varepsilon).$$

In case S is a topological semigroup as in Theorem 4 and is an Abelian group, we have shown that a quasi-uniformly continuous function f is almost periodic if and only if for every $\varepsilon>0$ there exists an interval $[\mu(\varepsilon), e]$ such that for all $y\in S$ $A_\varepsilon y=[\mu(\varepsilon)y, y]$ contains an ε-almost period τ (cf. [3]).

The following example shows that our characterization of an almost periodic function is effective in case S is not locally compact.

EXAMPLE. Set $S=Q^n=\{x=(x_1,\dots, x_n): x_i\in Q, \text{ for } i=1,\dots, n\}$, where Q denotes the usual additive group of the rational numbers. Then S is an Abelian group and is a topological semigroup equipped with the usual sum operation and the usual topology. Set $P=\{x=(x_1,\dots, x_n)\in S: x_i\geqq 0, \text{ for } i=1,\dots, n\}$. For every x, $y\in S$ such that $y\succeq x$, the interval $[x, y]$ is totally bounded. Therefore, a function $f\in C(S)$ is almost periodic if and only if f is uniformly continuous and for every $\varepsilon>0$ there exists an element $\lambda=(\lambda_1,\dots, \lambda_n)$ such that for every $y\in S$ there exists $\tau\in[0, \lambda]+y$ which satisfies $\sup_{x\in S}|f(x+\tau)-f(x)|<\varepsilon$.

Now we consider the case where S is a *metric semigroup*, that is, the topology $\mathscr{T}$ coincides with the topology induced by the *distance* d on S. We assume that the distance d is translation invariant, that is, $d(a, b)=d(ac, bc)=d(ca, cb)$ for all a, b and $c\in S$. Then in its topology $\mathscr{T}$ multiplication is *jointly continuous*, that is, the map $(x, y)\to xy$ is continuous. We shall say that a metric semigroup S satisfies condition $(*)$ if every closed ball $B(x, r)=\{y\in S: d(x, y)\leqq r\}$ is totally bounded.

We have

THEOREM 6. *Assume that S is a metric semigroup which satisfies condition* $(*)$, *that $\mathscr{T}$ is translation invariant and that f is uniformly continuous. A function $f\in C(S)$ is almost periodic if and only if for every $\varepsilon>0$ there exists $r(\varepsilon)>0$ such that f satisfies condition* (B) *with respect to the family* $\{B(e, r(\varepsilon))\}$.

PROOF. Assume that f is almost periodic. Then there exists a family $\{A_\varepsilon\}$ of finite subsets of S such that f satisfies condition (B) with respect to $\{A_\varepsilon\}$. Writing $r(\varepsilon)=\max\{d(e, x): x\in A_\varepsilon\}$, we can easily see that $A_\varepsilon\subset B(e, r(\varepsilon))$ and f satisfies condition (B) with respect to $\{B(e, r(\varepsilon))\}$.

COROLLARY. *Assume that S and f satisfy the same conditions as in Theorem 6 and that S is an algebraic group. Then f is almost periodic if and only if for every $\varepsilon>0$ there exists $r(\varepsilon)>0$ such that f satisfies condition* (B_{RR}) *with respect to the family* $\{B(e, r(\varepsilon))\}$.

PROOF. By Theorem 4, it suffices to show that conditions (B_{LR}) and (B_{RR}) with respect to $\{B(e, r(\varepsilon))\}$ are equivalent. Since $d(yxy^{-1}, e)=d(yxy^{-1}, yy^{-1})=d(yx, ye)=d(x, e)\leqq r$ for every $x\in B(e, r)$, $B(e, r)$ also contains yxy^{-1}. Then $yx=(yxy^{-1})y\in B(e, r)y$ and $yB(e, r)\subset B(e, r)y$. Thus $yB(e, r)=B(e, r)y$. Therefore, conditions (B_{LR}) and (B_{RR}) with respect to $\{B(e, r(\varepsilon))\}$ are equivalent.

§5. Weakly almost periodic functions on topological semigroups

We shall give a characterization of a weakly almost periodic function on a topological semigroup. To this end, we consider a quartet (X, τ, ρ, x_0) of a Banach space X with norm $\| \ \|$, a mapping τ of S into the dual space X^* of X, a mapping ρ of S into X and $x_0\in X$. Assume that X^* is equipped with the weak* topology $\sigma(X^*, X)$.

THEOREM 7. *A function $f\in C(S)$ is weakly almost periodic if and only if there exists a quartet (X, τ, ρ, x_0) with the properties*:

(1) *τ is continuous and* $\sup_{t\in S}\|\tau t\|<+\infty$.

(2) *The range $\rho(S)$ is relatively compact in the weak topology.*

(3) $<\tau(ts), x_0>=<\tau t, \rho(s)>$ *for* $t, s\in S$.

(4) $f(t)=<\tau t, x_0>$.

PROOF. Assume that there exists a quartet (X, τ, ρ, x_0) with the properties (1)–(4). Define a mapping T of X into $C(S)$ by $(Tx)(t)=<\tau t, x>$. Then $|(Tx)(t)|=|<\tau t, x>|\leqq\|\tau t\|\,\|x\|$ and $\|Tx\|\leqq\sup_{t\in S}\|\tau t\|\,\|x\|$. Thus T is $\sigma(X, X^*)$-$\sigma(C(S), C(S)^*)$-continuous. Then we have

$$f_s(t)=f(ts)=<\tau(ts), x_0>=<\tau t, \rho(s)>=(T\rho(s))(t)$$

for $t\in S$ and $f_s=T\rho(s)$. Note that $O_R(f)=T\rho(S)$. Since $\rho(S)$ is relatively $\sigma(X, X^*)$-compact, $O_R(f)$ is relatively $\sigma(C(S), C(S)^*)$-compact.
Thus f is weakly almost periodic.

Next we assume that a function $f\in C(S)$ is weakly almost periodic. Let $X=W(S)$ be the Banach space of all weakly almost periodic functions on S and let $x_0=f\in W(S)$. Define a mapping τ of S into X^* by $<\tau t, g>=g(t)$ for $t\in S$ and $g\in X$. Then we have

$$\sup_{t\in S}\|\tau t\|=\sup_{t\in S}\sup_{\substack{g\in X\\ \|g\|\leqq 1}}|<\tau t, g>|=\sup_{t\in S}\sup_{\substack{g\in X\\ \|g\|\leqq 1}}|g(t)|\leqq 1.$$

We can easily see that τ is a continuous mapping of S into X^* in the weak* topology. Further we define a mapping ρ of S into X by $\rho(t)=f_t$. Then $\rho(S)$ is relatively $\sigma(X, X^*)$-compact and $<\tau(ts), x_0>=f(ts)=f_s(t)=<\tau t, \rho(s)>$. Thus we have $f(t)=<\tau t, f>=<\tau t, x_0>$. Namely the quartet (X, τ, ρ, x_0) satisfies the properties (1)–(4).

The "if" part of Theorem 7 was proved in a special case where $\tau t=\phi U_t$ ($\phi\in X^*$), $\rho t=U_t x_0$ (cf. Burckel [2]).

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