Mem. Fac. Sci., Shimane Univ.,
13, pp. 17–22 Dec. 20, 1979

Remarks on Solvability of Lie Triple Algebras

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(Received September 8, 1979)

The class of Lie triple algebras contains Lie algebras and Lie triple systems as special classes. Therefore, the study of Lie triple algebras may need to generalize various concepts in the theory of Lie algebras and Lie triple systems to those in Lie triple algebras. In this paper, we try to introduce the concept of solvability of Lie triple algebras (§1), although their detailed properties are not yet found satisfactorily. We show that both of the standard enveloping Lie algebra and inner derivation algebra of a Lie triple algebra $\mathfrak{g}$ are solvable if $\mathfrak{g}$ is solvable. In §2, the existence of the radical of Lie triple algebras is shown and some results about semi-simple Lie triple algebras are given, under the solvability introduced in §1. The general theory of Lie triple algebras seems to play an important role in the theory of analytic homogeneous systems since it is proved in [1] that every connected and simply connected analytic homogeneous system is characterized by its tangent Lie triple algebra.

§1. Solvability of Lie triple algebras

A *Lie triple algebra* (or a *general Lie triple system* in [3], [4]) $\mathfrak{g}$ is a vector space over an arbitrary field with a bilinear map denoted by XY of $\mathfrak{g}\times\mathfrak{g}$ into $\mathfrak{g}$ and a trilinear map denoted by $D(X, Y)Z$ of $\mathfrak{g}\times\mathfrak{g}\times\mathfrak{g}$ into $\mathfrak{g}$ satisfying the following axioms: (i) $XX=0$, (ii) $D(X, X)=0$, (iii) $\mathfrak{S}\{(XY)Z+D(X, Y)Z\}=0$ ($\mathfrak{S}$ denotes the cyclic sum with respect to X, Y, Z.), (iv) $\mathfrak{S}D(XY, Z)=0$, (v) $D(X, Y)(ZW)=(D(X, Y)Z)W+Z(D(X, Y)W)$, (vi) $[D(X, Y), D(Z, W)]=D(D(X, Y)Z, W)+D(Z, D(X, Y)W)$ for X, Y, Z, W in $\mathfrak{g}$, where $D(X, Y)$ is an endomorphism of $\mathfrak{g}$ called an *inner derivation* of $\mathfrak{g}$. In the following we assume that $\mathfrak{g}$ is a finite dimensional Lie triple algebra over a field of characteristic zero. The *standard enveloping Lie algebra* of $\mathfrak{g}$ is the Lie algebra $\mathfrak{A}=\mathfrak{g}\oplus D(\mathfrak{g}, \mathfrak{g})$[1)] with the bracket operation; $[X, Y]=XY+D(X, Y)$, $[U, X]=UX=-[X, U]$, $[U, V]=UV-VU$ for $X, Y\in\mathfrak{g}$ and $U, V\in D(\mathfrak{g}, \mathfrak{g})$, where $D(\mathfrak{g}, \mathfrak{g})$ denotes the Lie algebra of endomorphisms of $\mathfrak{g}$ consisting of all inner derivations and is called the *inner derivation algebra* of $\mathfrak{g}$. By definition of the bracket operation of $\mathfrak{A}$, $D(\mathfrak{g}, \mathfrak{g})$ is contained in $\mathfrak{A}$ as a Lie subalgebra. A Lie triple algebra $\mathfrak{g}$ is said to be *reduced to Lie algebra* if $D(\mathfrak{g}, \mathfrak{g})=0$. If $\mathfrak{g}$ is reduced to Lie algebra, then $\mathfrak{g}$ is a Lie algebra with $[X, Y]=XY$. A Lie triple algebra $\mathfrak{g}$ is said to be *reduced to Lie triple system* if $\mathfrak{g}\mathfrak{g}=0$. In this case, $\mathfrak{g}$ is a Lie triple system with $[X, Y, Z]=D(X, Y)Z$.

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1) For two vector spaces A and B, we denote by $A\oplus B$ the direct sum of A and B and by $A+B$ the vector space spanned by $A\cup B$.

If $\mathfrak{g}\mathfrak{g}=0$ and $D(\mathfrak{g}, \mathfrak{g})=0$, $\mathfrak{g}$ is called an *Abelian* Lie triple algebra. An *ideal* $\mathfrak{h}$ of a Lie triple algebra $\mathfrak{g}$ is a Lie triple subalgebra of $\mathfrak{g}$ satisfying $\mathfrak{g}\mathfrak{h}\subset\mathfrak{h}$ and $D(\mathfrak{g}, \mathfrak{h})\mathfrak{g}\subset\mathfrak{h}$. If $\mathfrak{h}$ is an ideal of $\mathfrak{g}$, then by (iii) of the axioms for Lie triple algebras, $D(\mathfrak{g}, \mathfrak{g})\mathfrak{h}\subset\mathfrak{h}$ is assured, that is, $\mathfrak{h}$ is an *invariant* Lie triple subalgebra of $\mathfrak{g}$.

For an ideal $\mathfrak{h}$ of $\mathfrak{g}$ we define Lie triple subalgebras $\mathfrak{h}^{(i)}$ $(i=0, 1, 2,\dots)$ inductively as follows:

$$\mathfrak{h}^{(0)}=\mathfrak{h}, \quad \mathfrak{h}^{(1)}=\mathfrak{h}\mathfrak{h}+D(\mathfrak{g}, \mathfrak{h})\mathfrak{h},$$

$$\mathfrak{h}^{(i+1)}=\mathfrak{h}^{(i)}\mathfrak{h}^{(i)}+D(\mathfrak{h}, \mathfrak{h})\mathfrak{h}^{(i)}+D(\mathfrak{g}, \mathfrak{h}^{(i)})\mathfrak{h}^{(i)}.$$

PROPOSITION 1. *For each ideal $\mathfrak{h}$ of $\mathfrak{g}$, $\mathfrak{h}^{(i)}$'s form a chain $\mathfrak{h}=\mathfrak{h}^{(0)}\supset\mathfrak{h}^{(1)}\supset\cdots\supset\mathfrak{h}^{(i)}\supset\mathfrak{h}^{(i+1)}\supset\cdots$ of invariant Lie triple subalgebras of $\mathfrak{g}$ such that each $\mathfrak{h}^{(i+1)}$ is an ideal of $\mathfrak{h}^{(i)}$ and that the quotient Lie triple algebra $\mathfrak{h}^{(i)}/\mathfrak{h}^{(i+1)}$ is Abelian.*

PROOF. The relation $\mathfrak{h}^{(i+1)}\subset\mathfrak{h}^{(i)}$ is clear from their construction. Since $\mathfrak{h}^{(i)}\mathfrak{h}^{(i+1)}\subset\mathfrak{h}^{(i)}\mathfrak{h}^{(i)}\subset\mathfrak{h}^{(i+1)}$ and $D(\mathfrak{h}^{(i+1)}, \mathfrak{h}^{(i)})\mathfrak{h}^{(i)}\subset D(\mathfrak{g}, \mathfrak{h}^{(i)})\mathfrak{h}^{(i)}\subset\mathfrak{h}^{(i+1)}$ hold, we see that $\mathfrak{h}^{(i+1)}$ is an ideal of $\mathfrak{h}^{(i)}$ and that each $\mathfrak{h}^{(i)}$ is a Lie triple subalgebra of $\mathfrak{g}$. By induction on i it is easy to show that each $\mathfrak{h}^{(i)}$ is an invariant subalgebra. The quotient Lie triple algebra $\mathfrak{h}^{(i)}/\mathfrak{h}^{(i+1)}$ is Abelian because of the relation $\mathfrak{h}^{(i)}\mathfrak{h}^{(i)}+D(\mathfrak{h}^{(i)}, \mathfrak{h}^{(i)})\mathfrak{h}^{(i)}\subset\mathfrak{h}^{(i+1)}$. *q.e.d.*

For an ideal $\mathfrak{h}$ of $\mathfrak{g}$, set $\mathfrak{B}=\mathfrak{h}\oplus D(\mathfrak{g}, \mathfrak{h})$. Then $\mathfrak{B}$ is an ideal of the standard enveloping Lie algebra $\mathfrak{A}=\mathfrak{g}\oplus D(\mathfrak{g}, \mathfrak{g})$ of $\mathfrak{g}$ generated by $\mathfrak{h}$. Now, put $\mathfrak{B}_i=\mathfrak{h}^{(i)}+D(\mathfrak{h}, \mathfrak{h})+D(\mathfrak{g}, \mathfrak{h}^{(i)})$ for $i=0, 1, 2,\dots$.

LEMMA 1. *The chain $\mathfrak{B}=\mathfrak{B}_0\supset\mathfrak{B}_1\supset\cdots\supset\mathfrak{B}_i\supset\mathfrak{B}_{i+1}\supset\cdots$ is a chain of Lie subalgebras of $\mathfrak{A}$ such that each $\mathfrak{B}_{i+1}$ is an ideal of $\mathfrak{B}_i$. In fact the following relations hold for $i=0, 1, 2,\dots$;*

$$[\mathfrak{B}_i, \mathfrak{B}_i]\subset\mathfrak{B}_{i+1}. \tag{1}$$

Moreover, if the i-th derived ideal of the Lie algebra $\mathfrak{B}$ is denoted by $\mathfrak{B}^{(i)}$, the following relations hold for $i=0, 1, 2,\dots$;

$$\mathfrak{B}^{(i)}\subset\mathfrak{B}_i. \tag{2}$$

PROOF. The inclusion relation $\mathfrak{B}_{i+1}\subset\mathfrak{B}_i$ is clear from the definition of $\mathfrak{B}_i$'s. The relation (1) is shown as follows:

$$\begin{aligned}[\mathfrak{B}_i, \mathfrak{B}_i]\subset{}&\mathfrak{h}^{(i)}\mathfrak{h}^{(i)}+D(\mathfrak{h}, \mathfrak{h})\mathfrak{h}^{(i)}+D(\mathfrak{g}, \mathfrak{h}^{(i)})\mathfrak{h}^{(i)}\\ &+D(\mathfrak{h}^{(i)}, \mathfrak{h}^{(i)})+[D(\mathfrak{h}, \mathfrak{h}), D(\mathfrak{h}, \mathfrak{h})]\\ &+[D(\mathfrak{h}, \mathfrak{h}), D(\mathfrak{g}, \mathfrak{h}^{(i)})]+[D(\mathfrak{g}, \mathfrak{h}^{(i)}), D(\mathfrak{g}, \mathfrak{h}^{(i)})]\end{aligned}$$

$$\subset \mathfrak{h}^{(i+1)}+D(\mathfrak{h},\mathfrak{h})+D(\mathfrak{g}, D(\mathfrak{h},\mathfrak{h})\mathfrak{h}^{(i)})+D(\mathfrak{g}, D(\mathfrak{g},\mathfrak{h}^{(i)})\mathfrak{h}^{(i)})$$

$$\subset \mathfrak{h}^{(i+1)}+D(\mathfrak{h},\mathfrak{h})+D(\mathfrak{g},\mathfrak{h}^{(i+1)})=\mathfrak{B}_{i+1}.$$

The relation (2) is shown by induction on i: For $i=1$, we get $\mathfrak{B}^{(1)}=[\mathfrak{B},\mathfrak{B}]=[\mathfrak{h}\oplus D(\mathfrak{g},\mathfrak{h}), \mathfrak{h}\oplus D(\mathfrak{g},\mathfrak{h})]\subset\mathfrak{h}\mathfrak{h}+D(\mathfrak{g},\mathfrak{h})\mathfrak{h}+D(\mathfrak{h},\mathfrak{h})+[D(\mathfrak{g},\mathfrak{h}), D(\mathfrak{g},\mathfrak{h})]\subset\mathfrak{h}^{(1)}+D(\mathfrak{h},\mathfrak{h})+D(\mathfrak{g},\mathfrak{h}^{(1)})=\mathfrak{B}_1$. Suppose that $\mathfrak{B}^{(i)}\subset\mathfrak{B}_i$ holds. Then $\mathfrak{B}^{(i+1)}=[\mathfrak{B}^{(i)},\mathfrak{B}^{(i)}]\subset[\mathfrak{B}_i,\mathfrak{B}_i]\subset\mathfrak{B}_{i+1}$ is obtained by (1). *q.e.d.*

LEMMA 2. *For each ideal $\mathfrak{h}$ of $\mathfrak{g}$, i-th derived ideal $D^{(i)}(\mathfrak{h},\mathfrak{h})$ of the Lie algebra $D(\mathfrak{h},\mathfrak{h})$ satisfies the following;*

$$D^{(i)}(\mathfrak{h},\mathfrak{h})\subset\sum_{0\le k\le\frac{i}{2}} D(\mathfrak{h}^{(k)},\mathfrak{h}^{(i-k)}),\qquad i=1,2,\dots. \tag{3}$$

PROOF. The following relations hold for non-negative integers k, l and m;

$$D(\mathfrak{h}^{(k)},\mathfrak{h}^{(l)})\mathfrak{h}^{(m)}\subset\mathfrak{h}^{(m+1)}. \tag{4}$$

In fact, the left hand side of (4) is contained in $D(\mathfrak{h},\mathfrak{h})\mathfrak{h}^{(m)}$. We show the relation (3) by induction on i: For $i=1$, $D^{(1)}(\mathfrak{h},\mathfrak{h})=[D(\mathfrak{h},\mathfrak{h}), D(\mathfrak{h},\mathfrak{h})]\subset D(\mathfrak{h}, D(\mathfrak{h},\mathfrak{h})\mathfrak{h})\subset D(\mathfrak{h},\mathfrak{h}^{(1)})$. Suppose that (3) holds for a positive integer i. By using (4) and axioms for Lie triple algebra we get the following relations;

$$D^{(i+1)}(\mathfrak{h},\mathfrak{h})\subset[\sum_{0\le k\le\frac{i}{2}} D(\mathfrak{h}^{(k)},\mathfrak{h}^{(i-k)}), \sum_{0\le m\le\frac{i}{2}} D(\mathfrak{h}^{(m)},\mathfrak{h}^{(i-m)})]$$

$$\subset \sum_{0\le k\le\frac{i}{2}}\sum_{0\le m\le\frac{i}{2}}\{D(D(\mathfrak{h}^{(k)},\mathfrak{h}^{(i-k)})\mathfrak{h}^{(m)},\mathfrak{h}^{(i-m)})$$

$$+D(\mathfrak{h}^{(m)}, D(\mathfrak{h}^{(k)},\mathfrak{h}^{(i-k)})\mathfrak{h}^{(i-m)})\}$$

$$\subset \sum_{0\le m\le\frac{i}{2}}\{D(\mathfrak{h}^{(m+1)},\mathfrak{h}^{(i-m)})+D(\mathfrak{h}^{(m)},\mathfrak{h}^{(i-m+1)})\}$$

$$\subset \sum_{0\le k\le\frac{1}{2}(i+1)} D(\mathfrak{h}^{(k)},\mathfrak{h}^{(i-k+1)}).$$

Thus (3) is shown. *q.e.d.*

LEMMA 3. *If $\mathfrak{h}$ and $\mathfrak{m}$ are ideals of $\mathfrak{g}$, then $\mathfrak{h}\cap\mathfrak{m}$ and $\mathfrak{h}+\mathfrak{m}$ are ideals of $\mathfrak{g}$ satisfying the following inclusion relations for $i=0,1,2,\dots$:*

$$(\mathfrak{h}\cap\mathfrak{m})^{(i)}\subset\mathfrak{h}^{(i)}\cap\mathfrak{m}^{(i)}, \tag{5}$$

$$(\mathfrak{h}+\mathfrak{m})^{(i)}\subset\mathfrak{h}^{(i)}+\mathfrak{m}^{(i)}+\mathfrak{h}\cap\mathfrak{m}. \tag{6}$$

PROOF. It is easy to see that $\mathfrak{h}\cap\mathfrak{m}$ and $\mathfrak{h}+\mathfrak{m}$ are ideals of $\mathfrak{g}$. We show (5) and

(6) by induction on i as follows: For $i=1$, (5) is valid as $(\mathfrak{h}\cap\mathfrak{m})^{(1)}=(\mathfrak{h}\cap\mathfrak{m})(\mathfrak{h}\cap\mathfrak{m})+D(\mathfrak{g},\mathfrak{h}\cap\mathfrak{m})(\mathfrak{h}\cap\mathfrak{m})\subset\mathfrak{h}^{(1)}\cap\mathfrak{m}^{(1)}$. If $(\mathfrak{h}\cap\mathfrak{m})^{(i)}\subset\mathfrak{h}^{(i)}\cap\mathfrak{m}^{(i)}$, then $(\mathfrak{h}\cap\mathfrak{m})^{(i+1)}\subset(\mathfrak{h}^{(i)}\cap\mathfrak{m}^{(i)})(\mathfrak{h}^{(i)}\cap\mathfrak{m}^{(i)})+D(\mathfrak{h}\cap\mathfrak{m},\mathfrak{h}\cap\mathfrak{m})(\mathfrak{h}^{(i)}\cap\mathfrak{m}^{(i)})+D(\mathfrak{g},\mathfrak{h}^{(i)}\cap\mathfrak{m}^{(i)})(\mathfrak{h}^{(i)}\cap\mathfrak{m}^{(i)})\subset\mathfrak{h}^{(i+1)}\cap\mathfrak{m}^{(i+1)}$. Hence (5) is proved. As for the relation (6) we see that $(\mathfrak{h}+\mathfrak{m})^{(1)}\subset\mathfrak{h}\mathfrak{h}+\mathfrak{m}\mathfrak{m}+\mathfrak{h}\mathfrak{m}+D(\mathfrak{g},\mathfrak{h})\mathfrak{h}+D(\mathfrak{g},\mathfrak{m})\mathfrak{m}+D(\mathfrak{g},\mathfrak{m})\mathfrak{h}+D(\mathfrak{g},\mathfrak{h})\mathfrak{m}\subset\mathfrak{h}^{(1)}+\mathfrak{m}^{(1)}+\mathfrak{h}\cap\mathfrak{m}$, since $\mathfrak{h}$ and $\mathfrak{m}$ are ideals of $\mathfrak{g}$. If (6) is valid for an integer i, then $(\mathfrak{h}+\mathfrak{m})^{(i+1)}\subset(\mathfrak{h}^{(i)}+\mathfrak{m}^{(i)}+\mathfrak{h}\cap\mathfrak{m})(\mathfrak{h}^{(i)}+\mathfrak{m}^{(i)}+\mathfrak{h}\cap\mathfrak{m})+D(\mathfrak{h}+\mathfrak{m},\mathfrak{h}+\mathfrak{m})(\mathfrak{h}^{(i)}+\mathfrak{m}^{(i)}+\mathfrak{h}\cap\mathfrak{m})+D(\mathfrak{g},\mathfrak{h}^{(i)}+\mathfrak{m}^{(i)}+\mathfrak{h}\cap\mathfrak{m})(\mathfrak{h}^{(i)}+\mathfrak{m}^{(i)}+\mathfrak{h}\cap\mathfrak{m})\subset\mathfrak{h}^{(i+1)}+\mathfrak{m}^{(i+1)}+\mathfrak{h}\cap\mathfrak{m}$, since $\mathfrak{h}^{(i)}\subset\mathfrak{h}$ and $\mathfrak{m}^{(i)}\subset\mathfrak{m}$. *q.e.d.*

An ideal $\mathfrak{h}$ of $\mathfrak{g}$ is *solvable* if $\mathfrak{h}^{(i)}=0$ for some integer i.

Proposition 2. *If $\mathfrak{h}$ is a solvable ideal of a Lie triple algebra $\mathfrak{g}$, then the Lie algebra $\mathfrak{B}=\mathfrak{h}\oplus D(\mathfrak{g},\mathfrak{h})$ generated by $\mathfrak{h}$ is a solvable ideal of the standard enveloping Lie algebra $\mathfrak{A}=\mathfrak{g}\oplus D(\mathfrak{g},\mathfrak{g})$ of $\mathfrak{g}$.*

Proof. If $\mathfrak{h}^{(i)}=0$ for some i, the Lie algebra $\mathfrak{B}_i$ in Lemma 1 is reduced to $D(\mathfrak{h},\mathfrak{h})$, and so the relation (2) in Lemma 1 implies $\mathfrak{B}^{(i)}\subset D(\mathfrak{h},\mathfrak{h})$. From Lemma 2 it follows that $D(\mathfrak{h},\mathfrak{h})$ is a solvable Lie algebra if $\mathfrak{h}$ is solvable. Thus $\mathfrak{B}^{(j)}=0$ for sufficiently large j. *q.e.d.*

Proposition 3. *If $\mathfrak{h}$ is a solvable ideal of $\mathfrak{g}$, then $D(\mathfrak{g},\mathfrak{h})$ is a solvable ideal of the Lie algebra $D(\mathfrak{g},\mathfrak{g})$ of inner derivations of $\mathfrak{g}$.*

Proof. Since $\mathfrak{h}$ is an invariant subalgebra of $\mathfrak{g}$, $[D(\mathfrak{g},\mathfrak{g}),D(\mathfrak{g},\mathfrak{h})]\subset D(\mathfrak{g},\mathfrak{h})$ follows directly from the axiom (vi) for Lie triple algebras, which shows that $D(\mathfrak{g},\mathfrak{h})$ is an ideal of $D(\mathfrak{g},\mathfrak{g})$. By induction on i, we can prove easily the following relation;

$$D^{(i)}(\mathfrak{g},\mathfrak{h})\subset D(\mathfrak{h},\mathfrak{h})+D(\mathfrak{g},\mathfrak{h}^{(i)}),\qquad i=0,1,2,\dots \tag{7}$$

Now, if $\mathfrak{h}$ is solvable, then (7) implies $D^{(i)}(\mathfrak{g},\mathfrak{h})\subset D(\mathfrak{h},\mathfrak{h})$ for some integer i and so, by Lemma 2, it is seen that $D(\mathfrak{g},\mathfrak{h})$ is a solvable Lie algebra. *q.e.d.*

Proposition 2 and Proposition 3 imply the following;

Theorem 1. *Assume that a Lie triple algebra $\mathfrak{g}$ is solvable. Then, its standard enveloping Lie algebra $\mathfrak{A}=\mathfrak{g}+D(\mathfrak{g},\mathfrak{g})$ is solvable and its inner derivation algebra $D(\mathfrak{g},\mathfrak{g})$ is a solvable Lie subalgebra of $\mathfrak{A}$.*

Remark 1. If $\mathfrak{g}$ is reduced to Lie algebra, i.e., $D(\mathfrak{g},\mathfrak{g})=0$, then $\mathfrak{g}^{(i)}$ is the i-th derived ideal of the Lie algebra $\mathfrak{g}=\mathfrak{A}$ and the fact that $\mathfrak{h}$ is a solvable Lie triple algebra ideal of $\mathfrak{g}$ means that it is a solvable ideal of the Lie algebra $\mathfrak{g}$.

Remark 2. Let $\mathfrak{g}$ be a Lie triple algebra which is reduced to Lie triple system, i.e., $\mathfrak{g}\mathfrak{g}=0$. In this case, the following relations hold for $\mathfrak{g}$ and its standard enveloping Lie algebra $\mathfrak{A}$: $\mathfrak{g}^{(1)}=D(\mathfrak{g},\mathfrak{g})\mathfrak{g}=[D(\mathfrak{g},\mathfrak{g})\mathfrak{g}]\subset[\mathfrak{A}^{(1)},\mathfrak{A}]\subset\mathfrak{A}^{(1)}$, $\mathfrak{g}^{(2)}=D(\mathfrak{g},\mathfrak{g})\mathfrak{g}^{(1)}=[D(\mathfrak{g},\mathfrak{g}),\mathfrak{g}^{(1)}]\subset[\mathfrak{A}^{(1)},\mathfrak{A}^{(1)}]=C^1\mathfrak{A}^{(1)}$ and, in general, $\mathfrak{g}^{(i+1)}=[D(\mathfrak{g},\mathfrak{g}),\mathfrak{g}^{(i)}]\subset[\mathfrak{A}^{(1)},C^{i-1}$

$\mathfrak{A}^{(1)}]=C^i\mathfrak{A}^{(1)}$ for $i=1, 2,\ldots$. If $\mathfrak{A}$ is a solvable Lie algebra, then $\mathfrak{A}^{(1)}$ is nilpotent, that is, $C^i\mathfrak{A}^{(1)}=0$ for some i, and the relations above imply that $\mathfrak{g}$ is solvable. Thus we have the following theorem which shows that, in the case of Lie triple system, solvability of Lie triple algebras introduced in this section is reduced to solvability in the sense of W. G. Lister [2] (cf. Cor. 2.6 in [2]).

THEOREM 2. *Let $\mathfrak{g}$ be a Lie triple system. The standard enveloping Lie algebra $\mathfrak{A}=\mathfrak{g}\oplus D(\mathfrak{g}, \mathfrak{g})$ is solvable if and only if $\mathfrak{g}$ is solvable as a Lie triple algebra.*

§2. Semi-simple Lie triple algebras

PROPOSITION 4. *If two ideals $\mathfrak{h}$ and $\mathfrak{m}$ of a Lie triple algebra $\mathfrak{g}$ are solvable, so is $\mathfrak{h}+\mathfrak{m}$.*

PROOF. If $\mathfrak{h}^{(i)}=\mathfrak{m}^{(i)}=0$ for sufficiently large i, then the relation (6) in Lemma 3 implies $(\mathfrak{h}+\mathfrak{m})^{(i)}\subset\mathfrak{h}\cap\mathfrak{m}$ and (5) in the same lemma implies $(\mathfrak{h}+\mathfrak{m})^{(2i)}\subset\mathfrak{h}^{(i)}\cap\mathfrak{m}^{(i)}=0$, which proves the proposition. *q.e.d.*

The proposition above establishes the existence of a unique maximal solvable ideal of $\mathfrak{g}$ which is the *radical* of $\mathfrak{g}$. A Lie triple algebra is *semi-simple* if its radical is zero.

PROPOSITION 5. *Let $\mathfrak{r}$ be the radical of a Lie triple algebra $\mathfrak{g}$. The quotient Lie triple algebra $\mathfrak{g}/\mathfrak{r}$ is semi-simple.*

PROOF. Let f: $\mathfrak{g}\rightarrow\mathfrak{g}/\mathfrak{r}$ be the natural homomorphism of Lie triple algebras. If $\bar{\mathfrak{h}}$ is an ideal of $\mathfrak{g}/\mathfrak{r}$, then $\mathfrak{h}=f^{-1}(\bar{\mathfrak{h}})$ is an ideal of $\mathfrak{g}$, and $f^{-1}(\bar{\mathfrak{h}}^{(i)})=\mathfrak{h}^{(i)}+\mathfrak{r}$ $(i=1, 2,\ldots)$. If $\bar{\mathfrak{h}}$ is solvable, then $\bar{\mathfrak{h}}^{(i)}=0$ for some i and $\mathfrak{h}^{(i)}\subset\mathfrak{r}$. Since $\mathfrak{r}$ is solvable, $\mathfrak{h}^{(j)}=0$ for some j, that is, $\mathfrak{h}$ is contained in $\mathfrak{r}$. Hence $\bar{\mathfrak{h}}=f(\mathfrak{h})=0$ so that the radical of $\mathfrak{g}/\mathfrak{r}$ is zero. *q.e.d.*

THEOREM 3. *Let $\mathfrak{g}$ be a Lie triple algebra and $\mathfrak{A}=\mathfrak{g}\oplus D(\mathfrak{g}, \mathfrak{g})$ its standard enveloping Lie algebra. If $\mathfrak{A}$ is a semi-simple Lie algebra, then $\mathfrak{g}$ is semi-simple.*

PROOF. Let $\mathfrak{r}$ be the radical of $\mathfrak{g}$. From Proposition 2 it follows that the Lie subalgebra $\mathfrak{B}=\mathfrak{r}\oplus D(\mathfrak{g}, \mathfrak{r})$ is a solvable ideal of $\mathfrak{A}$. If $\mathfrak{A}$ is semi-simple, then $\mathfrak{B}=0$ and so $\mathfrak{r}$ must be zero. *q.e.d.*

Let $\mathfrak{n}=\{X\in\mathfrak{g}\mid D(X, \mathfrak{g})=0\}$ be the *nullity subspace* of a Lie triple algebra $\mathfrak{g}$ (cf. [1]). If $\mathfrak{h}$ and $\mathfrak{m}$ are two ideals of $\mathfrak{g}$ contained in $\mathfrak{n}$, then it is evident that the ideal $\mathfrak{h}+\mathfrak{m}$ is also contained in $\mathfrak{n}$. Thus there exists a unique maximal ideal $\mathfrak{n}_0$ contained in $\mathfrak{n}$, which will be called the *nullity ideal* of $\mathfrak{g}$.

THEOREM 4. *Let $\mathfrak{g}$ be a Lie triple algebra. If the inner derivation algebra*

$D(\mathfrak{g}, \mathfrak{g})$ is semi-simple and if the nullity ideal of $\mathfrak{g}$ is zero, then $\mathfrak{g}$ is semi-simple.

PROOF. Let $\mathfrak{r}$ be the radical of $\mathfrak{g}$. Proposition 3 implies that $D(\mathfrak{g}, \mathfrak{r})$ is a solvable ideal of $D(\mathfrak{g}, \mathfrak{g})$. Since $D(\mathfrak{g}, \mathfrak{g})$ is assumed to be semi-simple, $\mathfrak{r}$ must be contained in the nullity ideal of $\mathfrak{g}$. Hence $\mathfrak{r}=0$. *q.e.d.*

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