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Orthodox Semigroups on which Green's Relations are Compatible*)

Kunitaka SHOJI and Miyuki YAMADA
Department of Mathematics, Shimane University, Matsue, Japan
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This paper is a continuation of the previous paper [11]. Krishna Iyengar [2] has shown that a regular semigroup is *D*-compatible if and only if it is a semilattice of bisimple semigroups. In this paper, the structure of bisimple orthodox semigroups, especially that of *H*-compatible bisimple orthodox semigroups, is clarified. Further, we investigate the structure of orthodox semigroups *S* on which some of the Green's relations $\mathscr{H}_S$, $\mathscr{L}_S$, $\mathscr{R}_S$ and $\mathscr{D}_S$ are compatible.

A semigroup S is said to be H [L, R, D]-*compatible* if the Green's H [L, R, D]-relation $\mathscr{H}_S$ [$\mathscr{L}_S$, $\mathscr{R}_S$, $\mathscr{D}_S$] on S is a congruence.

In the previous paper [11], one of the authors has clarified the structure of H [L, R]-compatible orthodox semigroups. On the other hand, it has been shown by Krishna Iyengar [2] that a regular semigroup is D-compatible if and only if it is a semilattice of bisimple semigroups. Accordingly, it is obvious that an orthodox semigroup is D-compatible if and only if it is a semilattice of bisimple orthodox semigroups. In the first half of this paper, the structure of bisimple orthodox semigroups, especially that of H-compatible bisimple orthodox semigroups, will be clarified. By using the results obtained in the first half, we shall next investigate the structure of orthodox semigroups S on which some of the Green's relations $\mathscr{H}_S$, $\mathscr{L}_S$, $\mathscr{R}_S$ and $\mathscr{D}_S$ are compatible. Throughout the whole paper, the set [the band] of idempotents of a regular [an orthodox] semigroup S will be denoted by E_S.

§1. *H*-compatible bisimple orthodox semigroups

If f is a regular semigroup A onto a regular semigroup B, then the collection $\{ef^{-1} : e \in E_B\}$ of subsemigroups ef^{-1} ($e \in E_B$) of A is called the *kernel* of f and is denoted by Ker f.

Let T be an inversive semigroup (that is, an orthogroup (orthodox union of groups)), and Γ an inverse semigroup. If a regular semigroup S contains T and if there exists a surjective homomorphism $\xi : S \to \Gamma$ such that

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(C1) $\cup \operatorname{Ker} \xi \equiv \cup \{\lambda\xi^{-1} : \lambda \in E_\Gamma\} = T$ and

(C2) the structure decomposition (see [7], [11]) of T is given as $T \sim \Sigma\{\lambda\xi^{-1} : \lambda \in E_\Gamma\}$, then S is called a *regular extension* of T by Γ (see [11]).

The following results have been given by the previous paper [8] and [11]:
A. An orthodox semigroup is a regular extension of a band by an inverse semigroup, and vice-versa.
B. An H-compatible orthodox semigroup is a regular extension of a strictly inversive semigroup (that is, an orthodox band of groups; see [7], [11]) by an H-degenerated inverse semigroup, and vice-versa.

Now, let S be a regular extension of a strictly inversive semigroup T by an inverse semigroup Γ. By the definition of a regular extension, it follows that $S \supset T$ and there exists a surjective homomorphism $\xi: S \to \Gamma$ such that $\cup \operatorname{Ker} \xi \equiv \cup \{\lambda\xi^{-1} : \lambda \in E_\Gamma\} = T$ and the structure decomposition of T is given as $T \sim \Sigma\{\lambda\xi^{-1} : \lambda \in E_\Gamma\}$. (That is, T is a semilattice E_Γ of the rectangular groups $\lambda\xi^{-1}$.)

For each $a \in S$, put $a\xi = \bar{a}$. Then, the following result can be proved by slightly modifying the proof of Lemma 1 of [9]:

LEMMA 1. *$a \mathscr{D}_S b$ if and only if $\bar{a} \mathscr{D}_\Gamma \bar{b}$.*

PROOF. The "only if" part is obvious. The "if" part: Let a, b be elements of S such that $\bar{a} \mathscr{L}_\Gamma \bar{b}$. Let a^*, b^* be inverses of a, b, respectively. Since Γ is an inverse semigroup, $\overline{a^*a} = \overline{b^*b}$. Also since $(\overline{a^*a})\xi^{-1}$ is a rectangular group, $a^*a \mathscr{D}_S b^*b$. Hence, $a \mathscr{L}_S a^*a \mathscr{D}_S b^*b \mathscr{L}_S b$, that is, $a \mathscr{D}_S b$. Dually if $\bar{a} \mathscr{R}_\Gamma \bar{b}$ $(a, b \in S)$, then $a \mathscr{D}_S b$. Therefore, $\bar{a} \mathscr{D}_\Gamma \bar{b}$ implies $a \mathscr{D}_S b$.

By using Lemma 1 and the results A, B above, we can obtain the following theorem.

THEOREM 2. (1) *A bisimple orthodox semigroup is a regular extension of a band by a bisimple inverse semigroup, and vice-versa.* (2) *An H-compatible bisimple orthodox semigroup is a regular extension of a strictly inversive semigroup by an H-degenerated bisimple inverse semigroup, and vice-versa.*

REMARK. A method of constructing all possible regular extensions of T by Γ for a given strictly inversive semigroup T and a given inverse semigroup Γ has been given by [10]; in particular for the case where T is a band, see also [8]. The structure of bisimple inverse semigroups has been also clarified by Reilly [4] and Reilly and Clifford [5]. Hence, we can know the gross structure of bisimple orthodox semigroups from Theorem 2, (1). A somewhat different construction of bisimple orthodox semigroups has been also given in Clifford [1], by extending Reilly's construction (see [4]) of bisimple inverse semigroups to bisimple orthodox semigroups.

By Theorem 2, (2) and Remark above, the problem of describing all H-compatible bisimple orthodox semigroups is reduced to that of describing all H-degenerated bisimple inverse semigroups. Therefore, we shall investigate the construction of H-degenerated bisimple inverse semigroups from now on.

Let E be a uniform semilattice, that is, a semilattice satisfying the following condition (C3):

(C3) For any $e, f \in E$, eE is isomorphic to fE; $eE \cong fE$.

Put $E \times E = \Delta$, and take an isomorphism $\xi_{(e,f)}$ of eE onto fE for each $(e, f) \in \Delta$. Assume that $F_\Delta(E) = \{\xi_{(e,f)}: (e, f) \in \Delta\}$ satisfies the conditions (3), (4) of (3.1) of [11], that is, the conditions

(C4) (1) $\xi_{(e,f)}$ is the identity mapping on eE for each $e \in E$.

(2) for $(e, f), (h, t) \in \Delta$,

$$\xi_{((fh)\xi_{(f,e)},(fh)\xi_{(h,t)})} = \xi_{(e,f)}{}^*\xi_{(h,t)}|(fh)\xi_{(f,e)}E.$$

Then, it is easily seen from [11] that $F_\Delta(E)$ is an H-degenerated inverse subsemigroup of the symmetric inverse semigroup $\mathscr{I}_E(*)$ on E. Further, we have $\xi_{(e,f)}{}^*\xi_{(f,e)} = \xi_{(e,e)}$ and $\xi_{(f,e)}{}^*\xi_{(e,f)} = \xi_{(f,f)}$ for any $(e, f) \in \Delta$. Hence, any two idempotents $\xi_{(e,e)}$ and $\xi_{(f,f)}$ are contained in the same $\mathscr{D}_{F_\Delta(E)}$-class. This implies that $F_\Delta(E)$ is bisimple.

Remark. This result is closely related with Theorem 3.2 of Munn [3].

Now, we have the following main theorem.

Theorem 3. *Any H-degenerated bisimple inverse semigroup is isomorphic to some $F_\Delta(E)$ constructed as above.*

Proof. Let S be an H-degenerated bisimple inverse semigroup and E its basic semilattice. Put $\Omega = \{(e, f): xx^* = e, x^*x = f$ for some $x \in S$, $e, f \in E\}$ (where x^* is an inverse of x in S). Then, by Munn [3] we have $\Omega = E \times E = \Delta$. Let $(e, f) \in \Delta$. There exists a unique $x \in S$ such that $xx^* = e$ and $x^*x = f$. Define $\xi_{(e,f)}: eE \to fE$ by $u\xi_{(e,f)} = x^*ux$, $u \in eE$. It is obvious from Munn [3] that $\xi_{(e,f)}$ is an isomorphism of eE onto fE. Put $F_\Delta(E) = \{\xi_{(e,f)}: (e, f) \in \Delta\}$. First, it is obvious that $F_\Delta(E)$ satisfies the condition (1) of (C4). Let $(e, f), (h, t) \in \Delta$. There exist x, y such that $xx^* = e$, $x^*x = f$, $yy^* = h$ and $y^*y = t$. Since $(fh)\xi_{(f,e)} = xfhx^* = xyy^*x^*$ and $(fh)\xi_{(h,t)} = y^*fhy = y^*x^*xy$, it follows that $\xi_{((fh)\xi_{(f,e)},(fh)\xi_{(h,t)})}$ is an isomorphism of xhx^*E onto y^*fyE, and $u\xi_{((fh)\xi_{(f,e)},(fh)\xi_{(h,t)})} = (xy)^*u(xy) = y^*x^*uxy = u\xi_{(e,f)}{}^*\xi_{(h,t)}$ for $u \in xhx^*E$. Thus, $\xi_{((fh)\xi_{(f,e)},(fh)\xi_{(h,t)})} = \xi_{(e,f)}{}^*\xi_{(h,t)} | (fh)\xi_{(f,e)}E$. Therefore, $F_\Delta(E)$ satisfies the condition (2) of (C4). Then $F_\Delta(E)$ is an H-degenerated inverse subsemigroup of $\mathscr{I}_E(*)$. Define $\phi: S \to F_\Delta(E)$ by $a\phi = \xi_{(aa^*,a^*a)}$. For $a, b \in S$, $(ab)\phi = \xi_{(abb^*a^*,b^*a^*ab)} = \xi_{((fh)\xi_{(f,e)},(fh)\xi_{(h,t)})}$ (where $aa^* = e$, $a^*a = f$, $bb^* = h$ and $b^*b = t$) $= \xi_{(e,f)}{}^*\xi_{(h,t)} = \xi_{(aa^*,a^*a)}{}^*\xi_{(bb^*,b^*b)} = (a\phi)^*(b\phi)$.

Then it follows from the above that ϕ is an isomorphism of S onto $F_{\Delta}(E)$.

§ 2. Relationship between Green's relations; and some remarks

By using Krishna Iyengar [2] and [7], [11], firstly we have the following theorem which shows the structure of orthodox semigroups S on which some of the Green's relations $\mathscr{H}_S$, $\mathscr{L}_S$, $\mathscr{R}_S$ and $\mathscr{D}_S$ are compatible.

THEOREM 4. *Let S be an orthodox semigroup.*

(1) *If S is $L[R]$-compatible, then S is D-compatible.*

(2) *If S is both L-compatible and R-compatible, then S is H-compatible.*

(3) *S is both H-compatible and $L[R]$-compatible if and only if S is a strictly inversive semigroup in which the set E_S of idempotents is a right [left] semiregular band (that is, a band satisfying the identity $xyzx = xyzxyxzx$ $[xyzx = xyxzxyzx]$).*

(4) *S is both H-compatible and D-compatible if and only if S is a semilattice of H-compatible bisimple orthodox semigroups and the union of maximal subgroups of S is a strictly inversive subsemigroup.*

PROOF. (1): Let S be an $L[R]$-compatible orthodox semigroup. Then, by [11] S is a semilattice of rectangular groups. Then, by Krishna Iyengar [2] it is D-compatible.

(2): This is obvious.

(3): Let S be a both H-compatible and L-compatible orthodox semigroup and E_S the set of idempotents of S. Then, by [11] S is an inversive semigroup (that is, an orthogroup). Since S is H-compatible, it is a strictly inversive semigroup. Next, note that E_S is L-compatible. Then it follows from [6] that E_S is a right semiregular band Conversely, let S be a strictly inversive semigroup in which the set E_S of idempotents is a right semiregular band. Then, by [7] S is a band of groups, hence it is H-compatible. Since $S/\mathscr{H}_S$ is isomorphic to E_S, it follows that $S/\mathscr{H}_S$ is a right semiregular band, that is, $S/\mathscr{H}_S$ is a right regular band of left zero semigroups. Then it is easily seen that S is a right regular band of left groups. Hence, by [11] it is L-compatible.

(4): This follows from [2] and [11].

REMARKS. 1. An orthodox semigroup which is a semilattice of H-compatible orthodox semigroups is not necessarily H-compatible. For example, an inversive semigroup (that is, an orthogroup) S is a semilattice of rectangular groups (accordingly, a semilattice of H-compatible bisimple orthodox semigroups), but not necessarily H-compatible. S is H-compatible only when S is strictly inversive.

2. An H-compatible orthodox semigroup is not necessarily D-compatible. Let A, B be two sets such that $A \cap B = \square$ and $|A| = |B|$ (where $|X|$ means the cardinality of X). For X, $Y = A$ or B, let $H_{X,Y}$ be the set of all 1–1 mappings of X onto Y. Put

$H_{A,A} \cup H_{B,B} \cup H_{A,B} \cup H_{B,A} \cup \{0\}$ (where 0 is a symbol which is different from any element of $H_{X,Y}$, X, $Y=A$ or $B)=S$. For $\delta, \xi \in S$, define the product $\delta*\xi$ as follows:

$$\delta*\xi=\begin{cases} 0 & \text{if } (1)\ \delta\in H_{A,A},\ \xi\in H_{B,B};\ (2)\ \xi\in H_{A,A},\ \delta\in H_{B,B}; \\ & (3)\ \delta,\ \xi\in H_{A,B} \text{ or } \delta,\ \xi\in H_{B,A};\ \text{ or } (4)\ \delta=0 \text{ or } \xi=0, \\ \text{resultant composition}, & \text{otherwise.} \end{cases}$$

Then, in the resulting system $S(*)$, the $\mathscr{D}_S$-classes are $H_{A,A} \cup H_{B,B} \cup H_{A,B} \cup H_{B,A}$ and $\{0\}$. On the other hand, the $\mathscr{H}_S$-classes are $H_{A,A}$, $H_{A,B}$, $H_{B,A}$, $H_{B,B}$ and $\{0\}$. Now, we can easily seen that this semigroup $S(*)$ is H-compatible but not D-compatible.

3. The full transformation semigroup $\mathscr{T}_X$ on the set $X=\{a, b\}$ is an orthodox semigroup which is D-compatible but not H-compatible.

4. A band B is H-compatible but not necessarily L-compatible [R-compatible]. It has been shown by [6] that B is $L[R]$-compatible if and only if B is a right [left] semiregular band.

5. Consider $\mathscr{T}_X$ above. $\mathscr{T}_X$ consists of four transformations $\begin{pmatrix} a & b \\ a & b \end{pmatrix}$, $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$, $\begin{pmatrix} a & b \\ a & a \end{pmatrix}$ and $\begin{pmatrix} a & b \\ b & b \end{pmatrix}$; that is $\mathscr{T}_X=\left\{\begin{pmatrix} a & b \\ b & a \end{pmatrix}, \begin{pmatrix} a & b \\ b & a \end{pmatrix}, \begin{pmatrix} a & b \\ a & a \end{pmatrix}, \begin{pmatrix} a & b \\ b & b \end{pmatrix}\right\}$. The set $\left\{\begin{pmatrix} a & b \\ a & b \end{pmatrix}, \begin{pmatrix} a & b \\ b & a \end{pmatrix}\right\}=R_1$ is a subgroup of $\mathscr{T}_X$ and the set $\left\{\begin{pmatrix} a & b \\ a & a \end{pmatrix}, \begin{pmatrix} a & b \\ b & b \end{pmatrix}\right\}=R_0$ is a right zero semigroup. Further, $\mathscr{T}_X$ is a semilattice $\{0, 1\}$ of the $\mathscr{R}_{\mathscr{T}_X}$-classes R_0 and R_1. Hence, $\mathscr{T}_X$ is R-compatible but not H-compatible. Similarly, there exists an orthodox semigroup which is L-compatible but not H-compatible.

6. A bicyclic semigroup is both D-compatible and H-compatible but neither L-compatible nor R-compatible.

7. A right semiregular band B is both H-compatible and L-compatible but not necessarily R-compatible. In fact, B is R-compatible if and only if B is a regular band. Similarly, there exists an orthodox semigroup which is R-compatible but not L-compatible.

Problem. Determine the structure of H-compatible regular semigroups.

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