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On *H*-Compatible Quasi-orthodox Semigroups

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This is a continuation of the previous papers [3], [4] and [5]. The structure of general quasi-orthodox semigroups has been studied in [3], and construction theorems for upwards directed quasi-orthodox semigroups and split quasi-orthodox semigroups have been given in [5]. On the other hand, the structure of H-compatible orthodox semigroups has been clarified in [4]. In this paper, we shall show that a regular extension S of a completely regular semigroup $M\sim\Sigma\{M_\lambda: \lambda\in\Lambda\}$ by an inverse semigroup $\Gamma(\Lambda)$ is H-compatible if and only if M is H-compatible and $\Gamma(\Lambda)$ is H-degenerated. By using this result, it will be also shown that a natural regular semigroup S in the sense of Warne [2] is H-compatible if and only if the union of maximal subgroups of S is an H-compatible subsemigroup of S. Further, a necessary and sufficient condition for a regular semigroup to be H-compatible is also given.

§1. Introduction

A semigroup S is said to be *quasi-orthodox* if S is regular and if there exist an inverse semigroup $\Gamma(\Lambda)$ (where Λ denotes the semilattice of idempotents) and surjective homomorphism $f: S\to\Gamma(\Lambda)$ such that λf^{-1} is a completely simple subsemigroup of S for each $\lambda\in\Lambda$. Let G be a completely regular semigroup, and $G\sim\Sigma\{S_\lambda: \lambda\in\Lambda\}$ the structure decomposition of G (hence, $G\sim\Sigma\{S_\lambda: \lambda\in\Lambda\}$ means that G is a semilattice Λ of completely simple semigroups S_λ).[1] Let S be a regular semigroup.

If

(1) $S\supset G\supset E(S)$ (where $E(S)$ denotes the set of idempotents of S),

and (2) there exist an inverse semigroup $\Gamma(\Lambda)$ and a surjective homomorphism

$$f: S\to\Gamma(\Lambda) \text{ such that } \lambda f^{-1}=S_\lambda \text{ for each } \lambda\in\Lambda,$$

then S is called *a regular extension of* $G\sim\Sigma\{S_\lambda: \lambda\in\Lambda\}$ *by* $\Gamma(\Lambda)$.

In the previous paper [5], it has been shown that a regular semigroup S is quasi-orthodox if and only if S is a regular extension of a completely regular semigroup by an inverse semigroup. Hereafter, "a completely regular semigroup $G\sim\Sigma\{S_\lambda: \lambda\in\Lambda\}$" means "$G$ is a completely regular semigroup and has $G\sim\Sigma\{S_\lambda: \lambda\in\Lambda\}$ as its structure decomposition". Further, "an inverse semigroup $\Gamma(\Lambda)$" means "Γ is an inverse

1) A completely regular semigroup M (that is, a semigroup M which is a union of groups) is uniquely decomposed into a semilattice Λ of completely simple subsemigroups $\{M_\lambda: \approx\Lambda\}$. This decomposition is called the structure decomposition of M, and denoted by $M\sim\Sigma\{M_\lambda: \lambda\in\Lambda\}$.

semigroup and has Λ as its basic semilattice (that is, the semilattice of idempotents)". Throughout this paper, when S is a semigroup, $E(S)$ will denote the set of all idempotents of S. We use the notations and terminology used in [5], unless otherwise stated.

§2. A structure theorem

Let S be a quasi-orthodox semigroup, and $E(S)$ the set of idempotents of S. Let $\mathscr{H}_S$ be Green's H-relation on S, and H_x the $\mathscr{H}_S$-class containing x $(\in S)$. Let S be H-compatible (that is, let $\mathscr{H}_S$ be a congruence on S). Then, $S/\mathscr{H}_S=\{H_x: x\in S\}$, and $H_xH_y\subset H_{xy}$ is satisfied for $x, y\in S$. It is well-known that H_e is the maximal subgroup (of S) containing e for each $e\in E(S)$. Now since S is a quasi-orthodox semigroup, there exist an inverse semigroup $\Gamma(\Lambda)$ and a surjective homomorphism $\phi: S\to\Gamma(\Lambda)$ such that
(C. 1) for each $\lambda\in\Lambda$, $\lambda\phi^{-1}=S_\lambda$ is a completely simple subsemigroup of S.
Hence, of course $\cup$ Ker $\phi=\cup\{S_\lambda: \lambda\in\Lambda\}$ is a semilattice Λ of the completely simple subsemigroups $\{S_\lambda: \lambda\in\Lambda\}$. Accordingly, $G=\cup$ Ker ϕ is a completely regular subsemigroup (of S) containing $E(S)$, and its structure decomposition is $G\sim\Sigma\{S_\lambda: \lambda\in\Lambda\}$. Since each S_λ is a completely simple semigroup, G is of course a union of groups. Hence, $G\subset\cup\{H_e: e\in E(S)\}$. Let G_e be the maximal subgroup of G containing e. Then, $G_e=H_e\cap G$. For $e, f\in E(S)$ and for $x\in G_e$ and $y\in G_f$, we have $xy\in G\cap H_eH_f\subset G\cap H_{ef}$. There exists G_h, where $h\in E(S)$, such that $xy\in G_h$. Hence, $G_h\subset H_{ef}$. That is, $H_{ef}=H_h$. Since $x'y'\mathscr{H}_S\, xy$ for $x'\in H_e$ and $y'\in H_f$, it follows that $x'y'\in H_h$. This implies that $M=\cup\{H_e: e\in E(S)\}$ is a subsemigroup of S. Accordingly, S is a natural regular semigroup in the sense of Warne [2].[2)]

LEMMA 1. *In an H-compatible quasi-orthodox semigroup S,*

(1) *the union* $M=\cup\{H_e: e\in E(S)\}$ *of maximal subgroups* $\{H_e: e\in E(S)\}$ *of S is a subsemigoup. Hence, S is a natural regular semigroup and M is an H-compatible completely regular semigroup,*

(2) *the completely regular semigroup* $M\sim\Sigma\{M_\lambda: \lambda\in\Lambda\}$ *is a kernel normal system of S, that is, there exist an inverse semigroup* $\Omega(\Lambda)$ *and a surjective homomorphism* $\psi: S\to\Omega(\Lambda)$ *such that* $\lambda\psi^{-1}=M_\lambda$ *for* $\lambda\in\Lambda$ *(in this case,* $\Omega(\Lambda)$ *is necessarily H-degenerated), and*

(3) *accordingly, S is a regular extension of the H-compatible completely regular semigroup* $M\sim\Sigma\{M_\lambda: \lambda\in\Lambda\}$ *by the H-degenerated inverse semigroup* $\Omega(\Lambda)$.

PROOF. (1): This is already seen above. The first half of (2) has been shown in Warne [2], while the fact that $\Omega(\Lambda)$ is H-degenerated follows from [5]. The part

2) See also R. J. Warne [Natural regular semigroups, Colloquia Math. Soc. Janos Bolyai **20**, Algebraic Theory of Semigroups, Szeged (Hungary), 1976, 685–720].

(3) is obvious from (1) and (2).

Conversely, the following result has been shown by [4]:

LEMMA 2. *If S is a regular extension of an H-compatible completely regular semigroup $M \sim \Sigma\{M_\lambda: \lambda \in \Lambda\}$ by an H-degenerated inverse semigroup $\Gamma(\Lambda)$, that is, if S is a regular semigroup satisfying*

(1) *$S \supset M$, and*

(2) *there exists a surjective homomorphism $\phi: S \to \Gamma(\Lambda)$ such that $\lambda\phi^{-1} = M_\lambda (\lambda \in \Lambda)$,*

then

(1) *S is a quasi-orthodox semigroup, and*

(2) *M is the union of maximal subgroups of S.*

From Lemma 1, an H-compatible quasi-orthodox semigroup is a regular extension of an H-compatible completely regular semigroup by an H-degenerated inverse semigroup. In this paper, we shall show that the converse of this result is also true; that is, we shall prove that:

A regular extension of an H-compatible completely regular semigroup $K \sim \Sigma\{K_\lambda: \lambda \in \Lambda\}$ by an H-degenerated inverse semigroup $\Gamma(\Lambda)$ is H-compatible.

Let S be a regular extension of an H-compatible completely regular semigroup $M \sim \Sigma\{M_\lambda: \lambda \in \Lambda\}$ by an H-degenerated inverse semigroup $\Gamma(\Lambda)$. As was shown in Lemma 2, S is a quasi-orthodox semigroup and M is the union of maximal subgroups of S. Further, there exists a surjective homomorphism $\phi: S \to \Gamma(\Lambda)$ such that $\lambda\phi^{-1} = M_\lambda$ for $\lambda \in \Lambda$. For each $\gamma \in \Gamma(\Lambda)$, put $\gamma\phi^{-1} = S_\gamma$. Hence, $S_\lambda = M_\lambda$ for $\lambda \in \Lambda$. Let u_γ be a representative of S_γ for each $\gamma \in \Gamma(\Lambda)$ such that u_λ is an idempotent for $\lambda \in \Lambda$. For each $\lambda \in \Lambda$, let L_λ and R_λ be the L-class and R-class of M_λ containing u_λ respectively. Put $E(L_\lambda) = L_\lambda^*$ and $E(R_\lambda) = R_\lambda^*$. Every L_λ and R_λ are a left group and a right group respectively, while L_λ^* and R_λ^* are a left zero semigroup and a right zero semigroup respectively. Further, $L_\lambda^* \cap R_\lambda = \{u_\lambda\}$ and $L_\lambda \cap R_\lambda^* = \{u_\lambda\}$. As is seen in [3] (and [5]), $L[\Lambda] = \Sigma\{L_\lambda: \lambda \in \Lambda\}$ (Σ means disjoint sum) and $R[\Lambda] = \Sigma\{R_\lambda: \lambda \in \Lambda\}$ are a lower partial chain Λ of the left groups $\{L_\lambda: \lambda \in \Lambda\}$ and an upper partial chain Λ of the right groups $\{R_\lambda: \lambda \in \Lambda\}$ with respect to the multiplication in M (hence in S)[3]; $L[\Lambda] = LP\{L_\lambda: \lambda \in \Lambda\}$ and $R[\Lambda] = UP\{R_\lambda: \lambda \in \Lambda\}$.

Since M is H-compatible, it follows that

(C. 2) for $\tau > \lambda$, $a, b \in L_\tau$ and $c, d \in L_\lambda$ [$a, b \in R_\tau$ and $c, d \in R_\lambda$], $a \mathscr{H}_{L_\tau} b$ and $c \mathscr{H}_{L_\lambda} d$

3) Let Λ be a semilattice, and S_λ a semigroup for each $\lambda \in \Lambda$. Let $T[\Lambda] = \Sigma\{S_\lambda: \lambda \in \Lambda\}$ (disjoint sum). If a binary operation $\circ$ is defined in $T[\Lambda]$ such that (1) if $a \in S_\lambda$, $b \in S_\tau$ and $\lambda \leqq \tau$, then $a \circ b$ [$b \circ a$] is defined and $a \circ b \in S_\lambda$ [$b \circ a \in S_\lambda$]; (2) $a \circ b = ab$ for $a, b \in S_\lambda$, $\lambda \in \Lambda$; and (3) if $a \in S_\lambda$, $b \in S_\tau$, $c \in S_\delta$ and $\lambda \leqq \tau \leqq \delta$, then $a \circ (b \circ c) = (a \circ b) \circ c$ [$c \circ (b \circ a) = (c \circ b) \circ a$], then the resulting system $T[\Lambda]$ is called an upper [lower] partial chain Λ of $\{S_\lambda: \in \Lambda\}$.

imply $ac\,\mathscr{H}_{L_\lambda}\,bd$ [$a\,\mathscr{H}_{R_\tau}\,b$ and $c\,\mathscr{H}_{R_\lambda}\,d$ imply $ca\,\mathscr{H}_{R_\lambda}\,db$].

In general, a lower partial chain of left groups [an upper partial chain of right groups] satisfying (C. 2) is said to be *H-compatible.*

Now, it follows from [3] that every element a of S_γ ($\gamma\in\Gamma(\Lambda)$) can be uniquely written in the form $a=xu_\gamma e$, $x\in L_{\gamma\gamma^{-1}}$, $e\in R^*_{\gamma^{-1}\gamma}$ [$a=fu_\gamma y$, $f\in L^*_{\gamma\gamma^{-1}}$, $y\in R_{\gamma^{-1}\gamma}$]. Therefore, $S=\{xu_\gamma e: x\in L_{\gamma\gamma^{-1}}, e\in R^*_{\gamma^{-1}\gamma}, \gamma\in\Gamma(\Lambda)\}$ [$S=\{fu_\gamma y: f\in L^*_{\gamma\gamma^{-1}}, y\in R_{\gamma^{-1}\gamma}, \gamma\in\Gamma(\Lambda)\}$].

First, we have

LEMMA 3. *For* $a=xu_\gamma e$, $x\in L_{\gamma\gamma^{-1}}$, $e\in R^*_{\gamma^{-1}\gamma}$ [$a=eu_\gamma x$, $e\in L^*_{\gamma\gamma^{-1}}$, $x\in R_{\gamma^{-1}\gamma}$] *and* $b=yu_\tau f$, $y\in L_{\tau\tau^{-1}}$, $f\in R^*_{\tau^{-1}\tau}$ [$b=fu_\tau y$, $f\in L^*_{\tau\tau^{-1}}$, $y\in R_{\tau^{-1}\tau}$],

(C. 3) *$a\,\mathscr{H}_S\,b$ if and only if $\gamma=\tau$, $e=f$ and $x\,\mathscr{H}_{L_{\gamma\gamma^{-1}}}\,y$*[4]

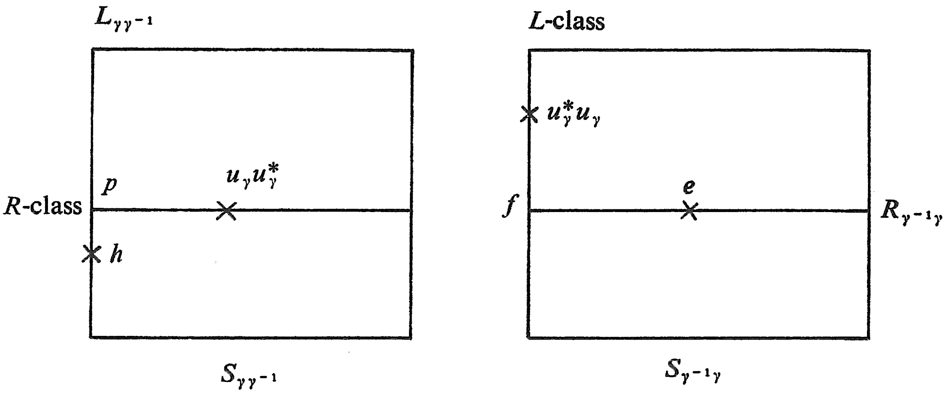
[$a\,\mathscr{H}_S\,b$ if and only if $\gamma=\tau$, $e=f$ and $x\,\mathscr{H}_{R_{\gamma^{-1}\gamma}}\,y$]

PROOF. We shall show that for $a=xu_\gamma e$, $x\in L_{\gamma\gamma^{-1}}$, $e\in R^*_{\gamma^{-1}\gamma}$ and $b=yu_\tau f$, $y\in L_{\gamma\gamma^{-1}}$, $f\in R^*_{\tau^{-1}\tau}$, $a\,\mathscr{H}_S\,b$ if and only if $\gamma=\tau$, $e=f$ and $x\,\mathscr{H}_{L_{\gamma\gamma^{-1}}}\,y$.

The "only if" part: Since $a\,\mathscr{H}_S\,b$, we have $\gamma\,\mathscr{H}_{\Gamma(\Lambda)}\,\tau$. Since $\Gamma(\Lambda)$ is H-degenerated, $\gamma\,\mathscr{H}_{\Gamma(\Lambda)}\,\tau$ implies $\gamma=\tau$. Now, of course $a\,\mathscr{R}_S\,b$ ($\mathscr{R}_S$ denotes Green's R-relation on S). Hence, there exists $t\in S$ such that at$=b$. Let $S_\alpha\in t$. Then, $\gamma\alpha=\gamma$, whence $\gamma^{-1}\gamma\alpha=\gamma^{-1}\gamma$. Since $u_\gamma et\in S_\gamma$, $u_\gamma et=zu_\gamma h$ for some $z\in L_{\gamma\gamma^{-1}}$ and $h\in R^*_{\gamma^{-1}\gamma}$. Thus, $xu_\gamma et=yu_\gamma f$ implies $xzu_\gamma h=yu_\gamma f$. Since y, $xz\in L_{\gamma\gamma^{-1}}$ and $h, f\in R^*_{\gamma^{-1}\gamma}$, it follows that $xz=y$ and $h=f$. Similarly, $bs=a$, $s\in S$ implies that there exists $v\in L_{\gamma\gamma^{-1}}$ such that $yv=x$. Hence, $x\,\mathscr{R}_{L_{\gamma\gamma^{-1}}}\,y$. Consequently, we have $x\,\mathscr{H}_{L_{\gamma\gamma^{-1}}}\,y$.

The "if" part: Assume that $\gamma=\tau$, $e=f$ and $x\,\mathscr{H}_{L_{\gamma\gamma^{-1}}}\,y$. Hence, $a=xu_\gamma e$ and $b=yu_\gamma e$. Since $x\,\mathscr{H}_{L_{\gamma\gamma^{-1}}}\,v$, there exist z, $v\in L_{\gamma\gamma^{-1}}$ such that $zx=y$ and $vy=x$. Therefore, $za=b$ and $vb=a$. Hence, $a\,\mathscr{L}_S\,b$ ($\mathscr{L}_S$ denotes Green's L-relation on S). Since $x\,\mathscr{H}_{L_{\gamma\gamma^{-1}}}\,y$, there exists an H-class H_h (where h is an idempotent) of $L_{\gamma\gamma^{-1}}$ such that $H_h\ni x, y$. Therefore, there exists $s\in H_h$ such that $x=ys$. Let u^*_γ be an inverse of u_γ, and p an idempotent contained in the intersection of the R-class containing $u_\gamma u^*_\gamma$ and $L_{\gamma\gamma^{-1}}$ (see the diagram below). Further, let f be an idempotent contained in the intersection of the L-class containing $u^*_\gamma u_\gamma$ and $R_{\gamma^{-1}\gamma}$. Then, $a=ysu_\gamma e=ypsu_\gamma e$ $=yu_\gamma u^*_\gamma psu_\gamma e=yu_\gamma(u^*_\gamma psu_\gamma e)=yu_\gamma(u^*_\gamma u_\gamma)f(u^*_\gamma psu_\gamma e)=yu_\gamma(u^*_\gamma u_\gamma)ef(u^*_\gamma psu_\gamma e)$ $=yu_\gamma e(fu^*_\gamma psu_\gamma e)=bc$, where $c=fu^*_\gamma psu_\gamma e$. Hence, $a=bc$. Similarly, there exists d such that $ad=b$. Hence, $a\,\mathscr{R}_S\,b$, and accordingly $a\,\mathscr{H}_S\,b$.

4) If M is semigroup, $\mathscr{H}_M$ denotes Green's H-relation on M.



The part [ ] can be also proved in the same way.

LEMMA 4. *The semigroup S is an H-compatible quasi-orthodox semigroup; that is,*

$$a\mathscr{H}_S b \text{ implies } ac\mathscr{H}_S bc \text{ and } ca\mathscr{H}_S cb \text{ for } c\in S.$$

PROOF. First, we prove that $a\mathscr{H}_S b$ implies $ac\mathscr{H}_S bc$ for $c\in S$. Since $a\mathscr{H}_S b$, by Lemma 3 we can assume that $a=xu_\gamma e$, $b=yu_\gamma e$, $x, y\in L_{\gamma\gamma^{-1}}$ and $e\in R^*_{\gamma^{-1}\gamma}$. Let $c=zu_\tau f$, where $z\in L_{\tau\tau^{-1}}$ and $f\in R^*_{\tau^{-1}\tau}$, be any element of S. Now, $ac=xu_\gamma ezu_\tau f$ and $bc=yu_\gamma ezu_\tau f$. Since $u_\gamma ezu_\tau\in S_{\gamma\tau}$, $u_\gamma ezu_\tau$ can be written in the form $u_\gamma ezu_\tau=vu_{\gamma\tau}h$, where $v\in L_{\gamma\tau(\gamma\tau)^{-1}}$ and $h\in R^*_{(\gamma\tau)^{-1}\gamma\tau}$. Hence, $ac=(xv)u_{\gamma\tau}(hf)$ and $bc=(yv)u_{\gamma\tau}(hf)$. Since $u_{\gamma\tau}(hf)\in S_{\gamma\tau}$, there exist $t\in L_{\gamma\tau(\gamma\tau)^{-1}}$ and $k\in R^*_{(\gamma\tau)^{-1}\gamma\tau}$ such that $u_{\gamma\tau}(hf)=tu_{\gamma\tau}k$. Hence, $ac=xvtu_{\gamma\tau}k$, $bc=yvtu_{\gamma\tau}k$, $xvt\in L_{\gamma\tau(\gamma\tau)^{-1}}$, $yvt\in L_{\gamma\tau(\gamma\tau)^{-1}}$ and $k\in R^*_{(\gamma\tau)^{-1}\gamma\tau}$. Since $\gamma\gamma^{-1}\geqq\gamma\tau(\gamma\tau)^{-1}$ and since $x\mathscr{H}_{L_{\gamma\gamma^{-1}}}y$ follows from Lemma 3, it follows from (C. 2) that $xvt\mathscr{H}_{L_{\gamma\tau(\gamma\tau)}}yvt$. Hence again by Lemma 3, we have $ac\mathscr{H}_S bc$. Similarly, we can prove that $a\mathscr{H}_S b$ implies $ca\mathscr{H}_S cb$ for $c\in S$. Thus, S is H-compatible.

From the result above, we have the following theorem:

THEOREM 5. *An H-compatible quasi-orthodox semigroup S is a regular extension of an H-compatible completely regular semigroup $M\sim\Sigma\{M_\lambda: \lambda\in\Lambda\}$ by an H-degenerated inverse semigroup $\Gamma(\Lambda)$. Conversely, a regular extension of an H-compatible completely regular semigroup $M\sim\Sigma\{M_\lambda: \lambda\in\Lambda\}$ by an H-degenerated inverse semigroup $\Gamma(\Lambda)$ is an H-compatible quasi-orthodox semigroup.*

PROOF. Obvious from Lemmas 1–4.

§3. H-compatible regular semigroups

Next, we shall show the following theorem which will give another proof for the latter half of Therorem 5:

THEOREM 6. *A regular semigroup S is H-compatible if and only if S satisfies the following* (C. 4):

(C. 4) *$e, f \in E(S)$, $e \geqq f$ and $xx^* = x^*x = e$ (where x^* is an inverse of x) imply $xf = fx$.*

To prove Theorem 6, we firstly show the following lemma:

LEMMA 7. *If a regular semigroup S satisfies the following* (C. 5), *then $\mathscr{H}_S$ is left* [*right*] *compatible*:

(C. 5) *If g and h are contained in a subgroup of S, then $ag \mathscr{H}_S ah$ [$ga \mathscr{H}_S ha$]* for *$a \in S$.*

PROOF. Suppose that a, b are elements of S such that $a \mathscr{H}_S b$. By [1], p. 49, there exist an inverse a^* of a and an inverse b^* of b such that $a^*a = b^*b$ and $aa^* = bb^*$. Now, $(ab^*)(ba^*) = (ba^*)(ab^*) = bb^*$ and $(b^*a)(a^*b) = (a^*b)(b^*a) = b^*b$. Hence, ab^* $ba^* \in H_{bb^*}$ (the H-class (of S) containing bb^*) and a^*b, $b^*a \in H_{b^*b}$. For any $x \in S$, $xa = xaa^*a = xbb^*a$. Hence, it follows from (C. 5) that $xa = xbb^*a \mathscr{H}_S xbb^*b = xb$. Thus, $\mathscr{H}_S$ is left compatible.

Proof for Theorem 6: Let S be an H-compatible regular semigroup. Since $\mathscr{H}_S$ is a congruence on S, if $e, f \in E(S)$, $e \geqq f$ and $xx^* = x^*x = e$ (where x^* is an inverse of x) then $x \mathscr{H}_S e$, $fx \mathscr{H}_S fe = f$ and $xf \mathscr{H}_S ef = f$. Hence, $fx = fxf = xf$. Conversely, suppose that a regular semigroup S satisfies (C. 4). First, it is proved that $\mathscr{H}_S$ is left compatible as follows: Let e be an element of $E(S)$, and G any subgroup (of S) containing e. For any $x \in S$ and $g \in G$, we have $xg \mathscr{R}_S xe$ (where $\mathscr{R}_S$ denotes Green's R-relation on S) since $\mathscr{R}_S$ is left compatible. Let $(xg)^*$ and $(xe)^*$ be inverses of xg and xe respectively. Also, let g^{-1} be the group-inverse of g in G. Then, we have $e(xe)^*xe = (e(xe)^*xe)^2$ and $e(xg)^*xg = (e(xg)^*xg)^2$. Further, $e(xe)^*xe \leqq e$ and $e(xg)^*xg \leqq e$. By the condition (C. 4), it follows that $ge(xe)^*xe = e(xe)^*xeg = e(xe)^*xg$ and $g^{-1}e(xg)^*xg = e(xg)^*xgg^{-1}$ $= e(xg)^*xe$. Hence, $Sxe = Sxe(xe)^*xe = Sxg^{-1}ge(xe)^*xe = Sxg^{-1}e(xe)^*xg$, that is, Sxe $\subset Sxg$. On the other hand, $Sxg = Sxg(xg)^*xg = Sxg^2g^{-1}e(xg)^*xg = Sxg^2e(xg)^*xe$, that is, $Sxg \subset Sxe$. Thus, $Sxe = Sxg$, which implies $xe \mathscr{H}_S xg$ (where $\mathscr{L}_S$ denotes Green's L-relation on S). Hence, $xe \mathscr{H}_S xg$. By Lemma 7, this implies that $\mathscr{H}_S$ is left compatible. Similarly, $\mathscr{H}_S$ is also right compatible. Consequently, $\mathscr{H}_S$ is a congruence; that is, S is H-compatible.

Now, let S be a regular extension of an H-compatible completely regular semigroup $M \sim \Sigma\{M_\lambda : \lambda \in \Lambda\}$ by an H-degenerated inverse semigroup $\Gamma(\Lambda)$. Since M is an H-compatible completely regular semigroup, S satisfies the condition (C. 4). Hence, S is H-compatible. Since S is a quasi-orthodox semigroup by Lemma 2, S is an H-compatible quasiorthodox semigroup. Thus, another proof for the latter half of Theorem 5 was given by using Theorem 6.

Finally, we present the following theorem which gives characterizations for H-

compatible quasi-orthodox semigroups:

THEOREM 8. *For a regular semigroup S, the following* (1)–(4) *are equivalent*:

(1) *S is an H-compatible quasi-orthodox semigroup.*

(2) *S is a natural regular semigroup (in the sense of Warne* [2]), *and the union of maximal subgroups of S is an H-compatible subsemigroup.*

(3) *The union of maximal subgroups of S is a band of groups.*

(4) *S is a quasi-orthodox semigroup and satisfies the condition* (C. 4).

PROOF. Obvious.

§4. Construction

A construction for regular extensions of a (H-compatible) completely regular semigroup $M \sim \Sigma\{M_\lambda: \lambda \in \Lambda\}$ by an (H-degenerated) inverse semigroup $\Gamma(\Lambda)$ has been given in [3] as follows: Let u_λ be an idempotent of M_λ for each $\lambda \in \Lambda$, and L_λ and R_λ the L-class and the R-class (of M_λ) containing u_λ. Then, $L[\Lambda]=\Sigma\{L_\lambda: \lambda \in \Lambda\}$ is a lower partial chain Λ of $\{L_\lambda: \lambda \in \Lambda\}$ and $R[\Lambda]=\Sigma\{R_\lambda: \lambda \in \Lambda\}$ is an upper partial chain Λ of $\{R_\lambda: \lambda \in \Lambda\}$ (with respect to the multiplication in M). Now, put $E(R_\lambda)=R_\lambda^*$. The system $\Sigma\{R_\lambda^*: \lambda \in \Lambda\}$ is not necessarily an upper partial chain Λ of $\{R_\lambda^*: \lambda \in \Lambda\}$. Put $R^*[\Lambda]=\Sigma\{R_\lambda^*: \lambda \in \Lambda\}$. For $\gamma, \tau \in \Gamma(\Lambda)$, let $f_{<\gamma,\tau>}$: $R^*_{\gamma^{-1}\gamma} \times L_{\tau\tau^{-1}} \times R^*_{\tau^{-1}\tau} \to L_{\gamma\tau(\gamma\tau)^{-1}}$ and $g_{<\gamma,\tau>}$: $R^*_{\gamma^{-1}\gamma} \times L_{\tau\tau^{-1}} \times R^*_{\tau^{-1}\tau} \to R^*_{(\gamma\tau)^{-1}\gamma\tau}$ be mappings such that $\Delta=\{f_{<\gamma,\tau>}: \gamma, \tau \in \Gamma(\Lambda)\} \cup \{g_{<\gamma,\tau>}: \gamma, \tau \in \Gamma(\Lambda)\}$ satisfies the following (I)–(III):

(I) If $g \in R^*_{\gamma^{-1}\gamma}$, $t \in L_{\tau\tau^{-1}}$, $h \in R^*_{\tau^{-1}\tau}$, $v \in L_{\delta\delta^{-1}}$ and $w \in R^*_{\delta^{-1}\delta}$, then

$$(q, t, h)f_{<\gamma,\tau>}((q, t, h)g_{<\gamma,\tau>}, v, w)f_{<\gamma\tau,\delta>}=(q, t(h, v, w)f_{<\tau,\delta>}, (h, v, w)g_{<\tau,\delta>})f_{<\gamma,\tau\delta>}$$

and $$((q, t, h)g_{<\gamma,\tau>}, v, w)g_{<\gamma\tau,\delta>}=(q, t(h, v, w)f_{<\tau,\delta>}, (h, v, w)g_{<\tau,\delta>})g_{<\gamma,\tau\delta>}.$$

(II) If $\gamma \in \Gamma(\Lambda)$, $p \in L_{\gamma\gamma^{-1}}$, $q \in R^*_{\gamma^{-1}\gamma}$, then there exist $k \in L_{\gamma^{-1}\gamma}$ and $n \in R^*_{\gamma\gamma^{-1}}$ such that

$$p(q, k, n)f_{<\gamma,\gamma^{-1}>}((q, k, n)g_{<\gamma,\gamma^{-1}>}, p, q)f_{<\gamma\gamma^{-1},\gamma>}=p \text{ and}$$

$$((q, k, n)g_{<\gamma,\gamma^{-1}>}, p, q)g_{<\gamma\gamma^{-1},\gamma>}=q.$$

(III) For $\lambda, \tau \in \Lambda$, $j \in R_\lambda^*$, $k \in L_\tau$ and $h \in R_\tau^*$,

$$u_\lambda jku_\tau h=(j, k, h)f_{<\lambda,\tau>}u_{\lambda\tau}(j, k, h)g_{<\lambda,\tau>}.$$

In this case, if we define multiplication in $S=\{(i, \gamma, j): \gamma \in \Gamma(\Lambda), i \in L_{\gamma\gamma^{-1}}, j \in R^*_{\gamma^{-1}\gamma}\}$ by

$$(i, \gamma, j)(p, \tau, q)=(i(j, p, q)f_{<\gamma,\tau>}, \gamma\tau, (j, p, q)g_{<\gamma,\tau>}),$$

then

(1) $\underline{M}=\{(i, \lambda, j): i\in L_\lambda,\ j\in R^*_\lambda,\ \lambda\in\Lambda\}$ is a completely regular semigroup and its structure decomposition is $\underline{M}\sim\Sigma\{\underline{M}_\lambda: \lambda\in\Lambda\}$, where $\underline{M}=\{(i, \lambda, j): i\in L_\lambda, j\in R^*_\lambda\}$; and $\phi: M\to\underline{M}$ defined by $(iu_\lambda j)\phi=(i, \lambda, j)$ (where $i\in L_\lambda,\ j\in R^*_\lambda,\ \lambda\in\Lambda$) is an isomorphism, and the restriction of ϕ to M_λ, say $\phi|M_\lambda$, is an isomorphism of M_λ onto $\underline{M}_\lambda$;

(2) further $\psi: S\to\Gamma(\Lambda)$ defined by $(i, \gamma, j)\psi=\gamma$ is a surjective homomorphism, and Ker $\psi=\{\underline{M}_\lambda: \lambda\in\Lambda\}$ and $\cup$ Ker $\psi=\underline{M}$ (hence, S is a regular extension of $\underline{M}\sim\Sigma\{\underline{M}_\lambda: \lambda\in\Lambda\}$ by $\Gamma(\Lambda)$).

Since $\underline{M}\overset{\phi}{\cong}M$ and $\underline{M}_\lambda\overset{\phi|M_\lambda}{\cong}M_\lambda$, S can be considered as a regular extension of $M\sim\Sigma\{M_\lambda: \lambda\in\Lambda\}$ by $\Gamma(\Lambda)$, up to isomorphism. Conversely, every regular extension of $M\sim\Sigma\{M_\lambda: \lambda\in\Lambda\}$ by $\Gamma(\Lambda)$ can be constructed in this way.

§5. Partial chains

The concept of a lower [upper] partial chain of semigroups seems to be important not only in this paper but also in the construction theory of various kinds of regular semigroups. In this section, we shall consider the construction of lower [upper] partial chains of right [left] reductive semigroups. Of course, it is well-known that a left [right] group is right [left] reductive.

If G is a right reductive semigroup, the set $\Lambda(G)$ of all left translations is a semigroup and has the set Λ_0 (G) of all inner left translations on G as its left ideal under the multiplication defined as follows: For any $\lambda_1,\ \lambda_2\in\Lambda(G)$, $(\lambda_1\lambda_2)x=\lambda_1(\lambda_2x)\,(x\in G)$. Of course, the mapping f: $G\to\Lambda_0(G)\subset\Lambda(G)$ defined by $af=\lambda_a$ (where λ_a denotes the inner left translation on G induced by a) gives an isomorphism since G is right reductive. Hence, G is embedded in $\Lambda(G)$ as its left ideal. Now, let $D(G)$ be a semigroup such that $D(G)\supset G$ and there exists an isomorphism ϕ: $D(G)\to\Lambda(G)$ satisfying $a\phi=\lambda_a$ for $a\in G$.

Hence, $z=x*y$ (in $D(G)$) if and only if $\lambda^x\lambda^y=\lambda^z$ (in $\Lambda(G)$), where $x\phi=\lambda^x$ (hence, $\lambda^a=\lambda_a$ for $a\in G$). That is, $D(G)$ is an isomorphic copy of $\Lambda(G)$ which contains G and in which every element a of G corresponds to λ_a. Now, we have

THEOREM 9. *Let Y be a semilattice, and S_γ a right reductive semigroup for each $\gamma\in Y$. For α, $\beta\in Y$ with $\alpha\geq\beta$, let $\phi_{\alpha,\beta}$: $S_\alpha\to D(S_\beta)$ be a homomorphism such that the set $\{\phi_{\alpha,\beta}: \alpha, \beta\in Y, \alpha\geq\beta\}$ satisfies the following conditions:*

(1) *$\phi_{\gamma,\gamma}$ is the identity mapping on S_γ for each $\gamma\in Y$,*

(2) *$(a\phi_{\alpha,\beta}*b)\phi_{\beta,\gamma}=(a\phi_{\alpha,\gamma})*(b\phi_{\beta,\gamma})$ for $\alpha\geq\beta\geq\gamma$, $a\in S_\alpha$ and $b\in S_\beta$.*

Then, $S=\Sigma\{S_\gamma: \gamma\in Y\}$ is a lower partial chain Y of $\{S_\gamma: \gamma\in Y\}$ under the partial binary operation $\circ$ defined by

$$a\circ b=(a\phi_{\alpha,\beta})*b \quad \textit{for} \quad a\in S_\alpha,\ \ b\in S_\beta\ (\alpha\geq\beta),$$

where $*$ denotes the multiplication in $D(S_\gamma)\,(\gamma\in Y)$. Further, every lower partial *chain* Y *of* $\{S_\gamma: \gamma \in Y\}$ *can be obtained in this way.*

PROOF. To show that the resulting system $S(\circ)$ constructed as above is a lower partial chain Y of $\{S_\gamma: \gamma \in Y\}$, we need only to prove that $(a\circ b)\circ c = a\circ(b\circ c)$ for $a \in S_\alpha$, $b \in S_\beta$ and $c \in S_\gamma$ with $\alpha \geq \beta \geq \gamma$.

Now, $(a\circ b)\circ c = (a\phi_{\alpha,\beta}*b)\circ c = (a\phi_{\alpha,\beta}*b)\phi_{\beta,\gamma}*c = (a\phi_{\alpha,\gamma})*(b\phi_{\beta,\gamma})*c$ and $a\circ(b\circ c) = a\circ (b\phi_{\beta,\gamma}*c) = (a\phi_{\alpha,\gamma})*(b\phi_{\beta,\gamma})*c$. Hence, $a\circ(b\circ c) = a\circ(b\circ c)$. Conversely, let $S(\circ) = \Sigma\{S_\gamma: \gamma \in Y\}$ be a lower partial chain Y of $\{S_\gamma: \gamma \in Y\}$. For each $\gamma \in Y$, let $f_\gamma: D(S_\gamma) \to \Lambda(S_\gamma)$ be an isomorphism such that $af_\gamma = \lambda_a$ $(a \in S_\gamma)$. Put $xf_\gamma = \lambda^x$ for $x \in D(S_\gamma)$ (hence, $\lambda^a = \lambda_a$ for $a \in S_\gamma$). Then, $x*y = z$ in $D(S_\gamma)$ if and only if $\lambda^x\lambda^y = \lambda^z$ in $\Lambda(S_\gamma)$. For $\alpha, \beta \in Y$ with $\alpha \geq \beta$, let $\phi_{\alpha,\beta}: S_\alpha \to D(S_\beta)$ be the mapping defined as follows: $a\phi_{\alpha,\beta} = a_{\alpha,\beta}$, where $a_{\alpha,\beta} = \lambda^{a_{\alpha,\beta}} f_\beta^{-1}$ and $\lambda^{a_{\alpha,\beta}} t = a\circ t$ $(t \in S_\beta)$. Then,

(1) $\phi_{\alpha,\alpha}$ is the identity mapping on S_α: For $x \in S_\alpha$, $x\phi_{\alpha,\alpha} = x_{\alpha,\alpha}$. For any $a \in S_\alpha$, $\lambda^{x_{\alpha,\alpha}} a = x\circ a = xa$ (in S_α) $= \lambda_x a$. Hence, $\lambda^{x_{\alpha,\alpha}} = \lambda_x$. Since $\lambda_x f_\alpha^{-1} = x$, we have $x_{\alpha,\alpha} = x$. Thus, $x\phi_{\alpha,\alpha} = x$ for all $x \in S_\alpha$.

(2) $(a\phi_{\alpha,\beta}*b)\phi_{\beta,\gamma} = (a\phi_{\alpha,\gamma})*(b\phi_{\beta,\gamma})$ for $a \in S_\alpha$, $b \in S_\beta$ with $\alpha \geq \beta \geq \gamma$: Let $c \in S_\gamma$. Now, $a\phi_{\alpha,\beta}*b = a_{\alpha,\beta}*b$ and $(a\phi_{\alpha,\beta}*b)\phi_{\beta,\gamma}*c = (a_{\alpha,\beta}*b)_{\beta,\gamma}*c$, and $(a\phi_{\alpha,\gamma})*(b\phi_{\beta,\gamma})*c = a_{\alpha,\gamma}*b_{\beta,\gamma}*c$. On the other hand, $(a\circ b)\circ c = (\lambda^{a_{\alpha,\beta}} b)\circ c = \lambda^{(\lambda^{a_{\alpha,\beta}} b)_{\beta,\gamma}} c$, and $a\circ(b\circ c) = a\circ (\lambda^{b_{\beta,\gamma}} c) = \lambda^{a_{\alpha,\gamma}}\lambda^{b_{\beta,\gamma}}*c$. Hence $a\circ(b\circ c) = (a\circ b)\circ c$ implies $\lambda^{(\lambda^{a_{\alpha,\beta}} b)_{\beta,\gamma}} c = \lambda^{a_{\alpha,\gamma}}\lambda^{b_{\beta,\gamma}} c$. Thus, we have $\lambda^{(\lambda^{a_{\alpha,\beta}} b)_{\beta,\gamma}} = \lambda^{a_{\alpha,\gamma}}\lambda^{b_{\beta,\gamma}}$. Therefore, $(\lambda^{a_{\alpha,\beta}} b)_{\beta,\gamma} = a_{\alpha,\gamma}*b_{\beta,\gamma}$. For any $x \in S_\beta$, $\lambda^{a_{\beta,\beta}}\lambda_b x = \lambda^{a_{\alpha,\beta}}(bx)$ (bx is the product of b, x in S_β) $= (\lambda^{a_{\alpha,\beta}} b)x$. Hence, $\lambda^{a_{\alpha,\beta}}\lambda_b = \lambda^{a_{\alpha,\beta}}{}_b$, whence $a_{\alpha,\beta}*b = \lambda^{a_{\alpha,\beta}} b$. Therefore, $a_{\alpha,\gamma}*b_{\beta,\gamma} = (a_{\alpha,\beta}*b)_{\beta,\gamma}$. Accordingly, we have $(a\phi_{\alpha,\beta}*b)\phi_{\beta,\gamma} = (a\phi_{\alpha,\gamma})*(b\phi_{\beta,\gamma})$. Finally, the multiplication $\circ$ is given as follows: For $a \in S_\alpha$, $b \in S_\beta$ such that $\alpha \geq \beta$, $a\circ b = \lambda^{a_{\alpha,\beta}} b = a_{\alpha,\beta}*b = (a\phi_{\alpha,\beta})*b$.

Finally, it should be noted that every upper partial chain Y of left reductive semigroups $\{S_\gamma: \gamma \in Y\}$ can be dually constructed.

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