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# Discrete Potentials on an Infinite Network

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**Some properties of discrete Green potentials and discrete harmonic functions on an infinite network are discussed in this paper from the point of view of the standard potential theory. Our results serve a further characterization of the infinite network by the classes of discrete harmonic functions.**

## Introduction

Discrete potential theory has been developed by Beurling and Deny [1], Blanc [2], Duffin [5; 6], Hundhausen [7], Saltzer [10] and many other mathematicians. Most of their results were concerned with the Laplace difference equation on a lattice domain in the $n$-dimensional Euclidean space. In the study of some classes of discrete harmonic functions on an infinite network which can be regarded as a generalization of the lattice domain, we often need a theory which is similar to the (continuous) potential theory as in [3] and [11] (see for instance [13]). In this paper, we present a discrete potential theory on an infinite network for the further study of some graph-theoretic properties of an infinite network. Most of our results have analogies in the standard books in potential theory. Our presentation is elementary by virtue of the discreteness, but our methods are essentially the same as in [3] and [11].

Some definitions and notation related to network theory are given in § 1. Several functional spaces related to discrete harmonic functions and known results will be stated there. Elementary properties of (discrete) superharmonic functions are resumed in § 2. We shall discuss the existence of the Green function of an infinite network in § 3 in relation to the classification of infinite networks in [13] and the existence of a non-constant positive superharmonic function. We obtain a new characterization of the Green function by the aid of the notion of flows. Discrete analogies of the Royden decomposition theorem and the Riesz decomposition theorem will be given in § 4 and § 5 respectively. We shall study some fundamental properties of Green potentials in § 5 and give a classification of infinite networks by the classes of harmonic functions in § 6, which is a special case of [8].

## § 1. Prelimiaries

Let $X$ be a countable set of nodes, $Y$ be a countable set of arcs and $K$ be a function on $X \times Y$ satisfying the following conditions:

(1.1) The range of $K$ is $\{-1, 0, 1\}$.

(1.2) For each $y \in Y$, $e(y)=\{x \in X; K(x, y) \neq 0\}$ consists of exactly two nodes $x_1$, $x_2$ and $K(x_1, y)K(x_2, y)=-1$.

(1.3) For each $x \in X$, $Y(x)=\{y \in Y; K(x, y) \neq 0\}$ is a nonempty finite set.

(1.4) For any $x, x' \in X$, there are $x_1, \dots, x_n \in X$ and $y_1, \dots, y_{n+1} \in Y$ such that $e(y_j) =\{x_{j-1}, x_j\}$, $j=1, \dots, n+1$ with $x_0=x$ and $x_{n+1}=x'$.

Let $r$ be a strictly positive real function on $Y$. Then $N=\{X, Y, K, r\}$ is called an infinite network.

Let $X'$ and $Y'$ be subsets of $X$ and $Y$ respectively and $K'$ and $r'$ be the restrictions of $K$ and $r$ onto $X' \times Y'$ and $Y'$ respectively. Then $N'=\{X', Y', K', r'\}$ is called a subnetwork of $N$ if (1.2), (1.3) and (1.4) are fulfilled replacing $X$, $Y$, $K$ by $X'$, $Y'$, $K'$ respectively. In order to emphasize the sets of nodes and arcs of $N'$, we often write $N'=<X', Y'>$. In case $X'$ (or $Y'$) is a finite set, $N'=<X', Y'>$ is called a finite subnetwork.

A sequence $\{N_n\}$ ($N_n=<X_n, Y_n>$) of finite subnetworks of $N$ is called an exhaustion of $N$ if

(1.5) $$X=\bigcup_{n=1}^{\infty} X_n \quad \text{and} \quad Y=\bigcup_{n=1}^{\infty} Y_n,$$

(1.6) $$Y(x) \subset Y_{n+1} \quad \text{for all} \quad x \in X_n.$$

For $x \in X$, let us put

$$X(x)=\{z \in X; K(z, y) \neq 0 \text{ for some } y \in Y(x)\}.$$

For a subset $A$ of $X$, denote by $\varepsilon_A$ the characteristic function of $A$, i.e., $\varepsilon_A(x)=1$ if $x \in A$ and $\varepsilon_A(x)=0$ if $x \in X-A$. In case $A=\{a\}$, we set $\varepsilon_a=\varepsilon_A$.

Let $L(X)$ and $L(Y)$ be the sets of all real functions on $X$ and $Y$ respectively. For $u \in L(X)$, we define $Eu \in L(Y)$ by

$$Eu(y)=\sum_{x \in X} K(x, y)u(x)=u(x_1)-u(x_2),$$

where $K(x_1, y)=1$ and $K(x_2, y)=-1$. For $u, v \in L(X)$, the support $Su$ of $u$, the Dirichlet integral $D(u)$ of $u$ and the inner product $(u, v)$ of $u$ and $v$ are defined by

$$Su=\{x \in X; u(x) \neq 0\},$$

$$D(u)=\sum_{y \in Y} r(y)^{-1}[Eu(y)]^2,$$

$$(u, v)=\sum_{y \in Y} r(y)^{-1}[Eu(y)][Ev(y)].$$

The Laplacian $\Delta u \in L(X)$ of $u$ is defined by

$$\Delta u(x) = -\sum_{y \in Y} r(y)^{-1} K(x, y)[Eu(y)].$$

A function $u \in L(X)$ is said to be harmonic (resp. superharmonic) on a set $A$ if $\Delta u(x) = 0$ (resp. $\Delta u(x) \leq 0$) for all $x \in A$.

We shall be concerned with the following classes of functions on $X$:

$$L_0(X) = \{u \in L(X);\ Su \text{ is a finite set}\},$$

$$\boldsymbol{B}(X) = \{u \in L(X);\ u \text{ is bounded on } X\},$$

$$\boldsymbol{D}(N) = \{u \in L(X);\ D(u) < \infty\},$$

$$\boldsymbol{H}(N) = \{u \in L(X);\ u \text{ is harmonic on } X\},$$

$$\boldsymbol{SH}(N) = \{u \in L(X);\ u \text{ is superharmonic on } X\},$$

$$\boldsymbol{HB}(N) = \boldsymbol{H}(N) \cap \boldsymbol{B}(N), \quad \boldsymbol{HD}(N) = \boldsymbol{H}(N) \cap \boldsymbol{D}(N),$$

$$\boldsymbol{HBD}(N) = \boldsymbol{H}(N) \cap \boldsymbol{B}(N) \cap \boldsymbol{D}(N).$$

For a subset $C$ of $L(X)$, denote by $C^+$ the subset of $C$ which consists of non-negative functions. We shall consider the subsets $L_0^+(X)$, $L^+(X)$, $\boldsymbol{H}^+(N)$, $\boldsymbol{SH}^+(N)$ etc. For a subset $C$ of $L(X)$, denote by $O_C$ the collection of those infinite networks $N$ for which $C$ consists only of constant functions.

Let $x_0 \in X$. For every $u, v \in \boldsymbol{D}(N)$, we define an inner product $((u, v))$ and a norm $\|u\|$ by

$$((u, v)) = (u, v) + u(x_0)v(x_0),$$

$$\|u\| = [((u, u))]^{1/2} = [D(u) + u(x_0)^2]^{1/2}.$$

We have by [12; Lemma 1]

LEMMA 1.1. *For every finite subset $F$ of $X$, there exists a constant $M(F)$ such that* $\sum_{x \in F} |u(x)| \leq M(F)\|u\|$ *for all* $u \in \boldsymbol{D}(N)$.

By this fact and by a standard argument, we can prove

THEOREM 1.1. $\boldsymbol{D}(N)$ *and* $\boldsymbol{HD}(N)$ *are Hilbert spaces with respect to the inner product* $((u, v))$.

Denote by $\boldsymbol{D}_0(N)$ the closure of $L_0(X)$ in $\boldsymbol{D}(N)$ with respect to the norm $\|u\|$, i.e., $u \in \boldsymbol{D}_0(N)$ if and only if there is a sequence $\{f_n\}$ in $L_0(X)$ such that $\|u - f_n\| \to 0$ as $n \to \infty$. Note that $\boldsymbol{D}_0(N)$ does not depend on the choice of $x_0$.

We have by [12; Lemma 3]

LEMMA 1.2. *For any* $u \in \boldsymbol{D}(N)$ *and* $f \in L_0(X)$,

$$(u, f) = -\sum_{x \in X} f(x)[\Delta u(x)].$$

From this lemma and the definition of $\boldsymbol{D}_0(N)$, we obtain

LEMMA 1.3. $(v, h)=0$ *for every* $v \in \boldsymbol{D}_0(N)$ *and* $h \in \boldsymbol{HD}(N)$.

We shall call a function $T$ on the real line $R$ into itself a normal contraction of $R$ if $T0=0$ and $|Ts_1 - Ts_2| \leq |s_1 - s_2|$ for any $s_1, s_2 \in R$. Define $Tu \in L(X)$ for $u \in L(X)$ by $(Tu)(x) = Tu(x)$.

We have by [12; Lemma 2]

LEMMA 1.4. *Let* $T$ *be a normal contraction of* $R$ *and* $u \in \boldsymbol{D}(N)$. *Then* $Tu \in \boldsymbol{D}(N)$, $D(Tu) \leq D(u)$ *and* $\|Tu\| \leq \|u\|$.

REMARK 1.1. In case $N$ is the lattice domain of $R^n$ $(n \geq 3)$, $\boldsymbol{D}_0(N)$ contains the set of all 0-chains which are regular at infinity in Saltzer's sense (cf. [10]).

## §2. Superharmonic functions

We shall study some properties of discrete superharmonic functions which are very analogous to the continuous case. In order to rewrite the Laplacian in a more familiar form as in [7], let us put $W_a = X(a) - \{a\}$ for $a \in X$,

$$t(x, a) = \sum_{y \in Y} r(y)^{-1} |K(x, y)K(a, y)| \text{ for } x \neq a,$$

$$t(a) = \sum_{y \in Y} r(y)^{-1} |K(a, y)|.$$

Then $t(x, a) = t(a, x)$, $\sum_{x \in W_a} t(x, a) = t(a)$ and

$$\Delta u(a) = -t(a)u(a) + \sum_{x \in W_a} t(x, a)u(x). \tag{2.1}$$

We shall prove the following minimum principle:

LEMMA 2.1. *Let* $X'$ *be a finite subset of* $X$. *If* $u$ *is superharmonic on* $X'$ *and* $u \geq 0$ *on* $X - X'$, *then* $u \geq 0$ *on* $X$.

PROOF. Suppose that $\min\{u(x); x \in X'\} = u(a) < 0$ for some $a \in X'$. Since $\Delta u(a) \leq 0$ and $u(x) \geq u(a)$ for all $x \in X$, we have by (2.1)

$$t(a)u(a) \geq \sum_{x \in W_a} t(x, a)u(x) \geq \sum_{x \in W_a} t(x, a)u(a) = t(a)u(a),$$

so that $u(x) = u(a)$ for all $x \in X(a)$. If $(X - X') \cap X(a) \neq \emptyset$, then we arrive at a contradiction. In case $(X-X')\cap X(a)=\emptyset$, we see by the same reasoning as above that $u(x)=u(a)$ for all $x\in X_1=\cup\{X(b);\ b\in X(a)\}$. By repeating this procedure a finite number of times, we obtain $u(x)=u(a)$ for some $x\in X-X'$. This is a contradiction. Thus $u\geq 0$ on $X$.

Similarly we can prove

LEMMA 2.2. *Every non-constant superharmonic function does not attain its minimum on* $X$.

If $u_1$ and $u_2$ are superharmonic on a set $A$ and if $\alpha$ is a positive number, then $\alpha u_1$, $u_1+u_2$ and $\min(u_1, u_2)$ are superharmonic on $A$.

We can easily prove

LEMMA 2.3. *Let* $\{u_n\}$ *be a sequence in* $L(X)$ *which converges pointwise to* $u\in L(X)$, *i.e.,* $u_n(x)\to u(x)$ *as* $n\to\infty$ *for every* $x\in X$. *Then* $\{\Delta u_n\}$ *converges pointwise to* $\Delta u$.

LEMMA 2.4. *Let* $\{X_n\}$ *be an increasing sequence of subsets of* $X$ *whose union is equal to* $X$ *and let* $\{u_n\}$ *be an increasing sequence in* $L(X)$ *which converges pointwise to* $u$. *If* $u_n$ *is superharmonic on* $X_n$, *then either* $u\equiv\infty$ *or* $u\in \boldsymbol{SH}(N)$.

## §3. Green function of an infinite network

An infinite network $N$ is said to be of parabolic type (of order 2) in [13] if there exists a nonempty finite subset $A$ of $X$ for which the value of the following extremum problem vanishes:

$$d(A,\infty)=\inf\{D(u);\ u\in L_0(X),\ u=1 \text{ on } A\}. \tag{3.1}$$

This definition does not depend on $A$. Let $O_G$ be the collection of infinite networks which are of parabolic type. We say that $N$ is of hyperbolic type if it is not of parabolic type.

We have by [13; Theorem 3.2]

LEMMA 3.1. *An infinite network* $N$ *is of parabolic type if and only if any one of the following conditions is fulfilled*:

$$1\in \boldsymbol{D}_0(N). \tag{3.2}$$

$$\boldsymbol{D}_0(N)=\boldsymbol{D}(N), \tag{3.3}$$

COROLLARY. *Assume that* $N$ *is of hyperbolic type. If* $u\in\boldsymbol{D}_0(N)$ *and* $D(u)=0$, *then* $u=0$.

Let $A$ be a nonempty finite subset of $X$ and let $\{N_n\}$ ($N_n=\langle X_n, Y_n\rangle$) be an exhaustion of $N$ such that $A \subset X_1$. Consider the following extremum problem:

(3.4) Find $d(A, X-X_n)=\inf\{D(u);\ u \in L(X;\ A,\ X-X_n)\}$,

where $L(X;\ A,\ X-X_n)=\{u \in L(X);\ u=1 \text{ on } A,\ u=0 \text{ on } X-X_n\}$. We see by Lemma 1.4 and Dirichlet principle [12; Theorem 2] that there exists a unique optimal solution $u_n=u_n^A$ of problem (3.4) and that $u_n$ has the following properties:

(3.5) $u_n=1$ on $A$ and $0 \leq u_n \leq 1$ on $X$.

(3.6) $d(A, X-X_n)=D(u_n)=-\sum_{x \in A} \Delta u_n(x)=(u_n, v)$

for all $v \in L(X;\ A,\ X-X_n)$.

(3.7) $u_n$ is harmonic on $X_n-A$.

From the relation $D(u_{n+k}-u_n)=D(u_n)-D(u_{n+k})$, it follows that $\{u_n\}$ is a Cauchy sequence in $\boldsymbol{D}_0(N)$. There exists $u^A \in \boldsymbol{D}_0(N)$ such that $\|u^A-u_n\| \to 0$ as $n \to \infty$. Notice that $d(A, X-X_n) \to d(A, \infty)$ as $n \to \infty$ by [9; Theorems 2.1 and 2.2] and [13; Lemma 4.5]. Thus we see that $u^A$ has the following properties:

(3.8) $u^A=1$ on $A$ and $0 \leq u^A \leq 1$ on $X$.

(3.9) $d(A, \infty)=D(u^A)=-\sum_{x \in A} \Delta u^A(x)$.

(3.10) $u^A$ is harmonic on $X-A$.

Now we shall prove

THEOREM 3.1. *$N \in O_G$ if and only if $N \in O_{SH^+}$.*

PROOF. First we assume that $N \in O_G$. Then $D(u^A)=0$, so that $u^A=1$ on $X$. Suppose that there exists a non-constant positive superharmonic function $v$ on $X$ and put $t=\min\{v(x);\ x \in A\}$. Then we have $t>0$ and $u_n^A \leq v/t$ on $X$ by Lemma 2.1, so that $1 \leq v/t$ on $X$. Thus $v$ attains its minimum value $t$ at a point of $A$. This is a contradiction by Lemma 2.2. Thus $N \in O_{SH^+}$. Next we assume that $N$ is of hyperbolic type. Taking $A=\{a\}$ in the above observation and noting that $d(A, \infty)>0$, we see by (3.8), (3.9) and (3.10) that $u^A$ is a non-constant positive superharmonic function on $X$. Thus $N$ does not belong to $O_{SH^+}$.

DEFINITION 3.1. We say that a function $g_a \in L(X)$ is the Green function of $N$ with pole at $a \in X$ if

(3.11) $g_a \in \boldsymbol{D}_0^+(N)=\boldsymbol{D}_0(N) \cap L^+(X)$,

(3.12) $\Delta g_a(x)=-\varepsilon_a(x)$ on $X$.

We show the uniqueness of the Green function if it exists. Let $g_a$ and $g'_a$ be Green functions of $N$ with pole at $a$ and put $v=g_a-g'_a$. Then $N$ is of hyperbolic type by Theorem 3.1 and $v$ belongs to both $\boldsymbol{HD}(N)$ and $\boldsymbol{D}_0(N)$, so that $D(v)=(v, v)=0$ by Lemma 1.3. Thus $v=0$ by the Corollary of Lemma 3.1, hence $g_a=g'_a$.

If $N$ is of hyperbolic type, then we see that $u^A/D(u^A)$ satisfies (3.11) and (3.12) with $A=\{a\}$ by our construction. Thus $g_a=u^A/D(u^A)$ with $A=\{a\}$. We have proved

THEOREM 3.2. *The Green function $g_a$ of $N$ with pole at $a$ exists if and only if $N$ is of hyperbolic type.*

We can easily prove

THEOREM 3.3. *$(g_a, v)=v(a)$ for every $v\in\boldsymbol{D}_0(N)$.*

COROLLARY. *$g_a(b)=g_b(a)$ for all $a, b\in X$.*

REMARK 3.1. For $a\in X_n$, let us put

$$g_a^{(n)}=u_n/D(u_n) \quad \text{with} \quad u_n=u_n^A \text{ and } A=\{a\}.$$

Then we have

$$\Delta g_a^{(n)}(x)=-\varepsilon_a(x) \quad \text{on} \quad X_n, \tag{3.13}$$

$$g_a^{(n)}(x)=0 \quad \text{on} \quad X-X_n. \tag{3.14}$$

A function which satisfies (3.13) and (3.14) is determined uniquely by Lemma 2.1. We call $g_a^{(n)}$ the Green function of $N_n$ with pole at $a$. By Lemma 2.1, $g_a^{(n)}\le g_a^{(n+1)}$ on $X$. If $N$ is of hyperbolic type, then $\{g_a^{(n)}\}$ converges pointwise to $g_a$.

Now we give a new characterization of the Green function $g_a$ by using the concept of flows from $\{a\}$ to the ideal boundary $\infty$ of $N$. For $w\in L(Y)$, let us put

$$I(w; x)=\sum_{y\in Y}K(x, y)w(y),$$

$$H(w)=\sum_{y\in Y}r(y)w(y)^2.$$

We say that $w\in L(Y)$ is a flow from $\{a\}$ to $\infty$ if $I(w; x)=0$ for all $x\in X, x\neq a$. Denote by $F(\{a\}, \infty)$ the set of all flows from $\{a\}$ to $\infty$. Consider the following extremum problem:

$$\text{Find} \quad d^*(\{a\}, \infty)=\inf\{H(w);\ w\in F(\{a\}, \infty),\ I(w; a)=1\}. \tag{3.15}$$

If $N$ is of hyperbolic type, then we can prove that problem (3.15) has a unique optimal solution and $d^*(\{a\}, \infty)=d(\{a\}, \infty)^{-1}$ (cf. [9; Theorem 5.1]). Let us put $w_a(y)=r(y)^{-1}Eg_a(y)$ for $y\in Y$. Then $I(w_a; x)=-\Delta g_a(x)=\varepsilon_a(x)$ and $H(w_a)=D(g_a)=d(\{a\}, \infty)^{-1}=d^*(\{a\}, \infty)$. Thus $w_a$ is the optimal solution of problem (3.15). Let $P=$

$(C_X(P), C_Y(P), p)$ be a path from $\{a\}$ to $\{x\}$ (cf. [12]). Then we have

$$\sum_{y \in C_Y(P)} r(y)p(y)w_a(y) = g_a(x) - g_a(a) = g_a(x) - d^*(\{a\}, \infty).$$

Thus we have

THEOREM 3.4. *Assume that $N$ is of hyperbolic type and let $w_a$ be the optimal solution of problem* (3.15) *and let $P$ be a path from $\{a\}$ to $\{x\}$ $(x \neq a)$. Then*

$$g_a(a) = d^*(\{a\}, \infty), \quad g_a(x) = \sum_{y \in C_Y(P)} r(y)p(y)w_a(y) + d^*(\{a\}, \infty),$$

*where $p$ is the path index of $P$.*

As an application of this theorem, we show

EXAMPLE 3.1. Let us take $X = \{x_n;\ n = 0, 1, 2,\dots\}$ and $Y = \{y_n;\ n = 1, 2,\dots\}$ and define $K(x, y)$ by $K(x_{n-1}, y_n) = -1$, $K(x_n, y_n) = 1$ for $n = 1, 2,\dots$ and $K(x, y) = 0$ for any other pair. Let $r$ be a strictly positive function on $Y$. Then $N = \{X, Y, K, r\}$ is an infinite network. Notice that $N \in O_G$ if and only if $\sum_{n=1}^{\infty} r(y_n) = \infty$ (cf. [9; Proposition 2.1] and [13; Theorem 4.1]). Assume that $\sum_{n=1}^{\infty} r(y_n) < \infty$ and put $c_k = \sum_{n=k}^{\infty} r(y_n)$. If $w \in F(\{x_m\}, \infty)$ and $I(w; x_m) = 1$, then $w(y_k) = 0$ for $k \leq m$ and $w(y_k) = -1$ for $k \geq m+1$. Thus $d^*(\{x_m\}, \infty) = c_{m+1}$ and $g_{x_m}(x_k) = c_{m+1}$ if $0 \leq k \leq m$ and $g_{x_m}(x_k) = -\sum_{n=m+1}^{k} r(y_n) + c_{m+1} = c_{k+1}$ if $k \geq m+1$ by Theorem 3.4.

## §4. Royden decomposition theorem

In this section we always assume that $N$ is of hyperbolic type. First we shall prove the following discrete analogy of the well-known Royden decomposition theorem:

THEOREM 4.1. *Every $u \in \boldsymbol{D}(N)$ can be decomposed uniquely in the form: $u = v + h$, where $v \in \boldsymbol{D}_0(N)$ and $h \in \boldsymbol{HD}(N)$.*

PROOF. Denote by $\boldsymbol{D}'(N)$, $\boldsymbol{D}_0'(N)$ and $\boldsymbol{HD}'(N)$ the quotient spaces of $\boldsymbol{D}(N)$, $\boldsymbol{D}_0(N)$ and $\boldsymbol{HD}(N)$ respectively with respect to the equivalence relation $D(u-v) = 0$. Then $\boldsymbol{D}_0'(N) = \boldsymbol{D}_0(N)$ by the Corollary of Lemma 3.1. We see easily that $\boldsymbol{D}'(N)$ and $\boldsymbol{HD}'(N)$ are Hilbert spaces with respect to the inner product $(u, v)$. Since $\boldsymbol{D}_0'(N)$ and $\boldsymbol{HD}'(N)$ are orthogonal by Lemma 1.3, our assertion follows from the orthogonal decomposition theorem.

Denote by $\boldsymbol{P}_0(N)$ the subset of $\boldsymbol{D}_0(N)$ which consists of superharmonic functions on $X$, i.e., $\boldsymbol{P}_0(N) = \boldsymbol{D}_0(N) \cap \boldsymbol{SH}(N)$. In order to characterize $\boldsymbol{P}_0(N)$ like Beurling and Deny [1], we begin with

THEOREM 4.2. *Let $T$ be a normal contraction of $R$ and let $u \in \boldsymbol{D}_0(N)$. Then $Tu \in \boldsymbol{D}_0(N)$ and $D(Tu) \leq D(u)$.*

PROOF. In view of Lemma 1.4, we have only to prove that $Tu \in \boldsymbol{D}_0(N)$. There exists a sequence $\{f_n\}$ in $L_0(X)$ such that $\|u - f_n\| \to 0$ as $n \to \infty$. Then $Tf_n \in L_0(X)$ and $\|Tf_n\| \leq \|f_n\|$ by Lemma 1.4. Thus $\{\|Tf_n\|\}$ is bounded. We can find a weakly convergent subsequence of $\{Tf_n\}$ in $\boldsymbol{D}_0(N)$. Denote it again by $\{Tf_n\}$ and let $u' \in \boldsymbol{D}_0(N)$ be the limit. For any $x \in X$, $g_x \in \boldsymbol{D}_0(N)$ and

$$((Tf_n, g_x)) = Tf_n(x) + g_x(x_0)Tf_n(x_0),$$

$$((u', g_x)) = u'(x) + g_x(x_0)u'(x_0)$$

by Theorem 3.3, so that $\{Tf_n\}$ converges pointwise to $u'$. It follows from Lemma 1.1 that

$$|Tf_n(x) - Tu(x)| \leq |f_n(x) - u(x)| \leq M(\{x\})\|f_n - u\|,$$

so that $\{Tf_n\}$ converges pointwise to $Tu$. Therefore $Tu = u' \in \boldsymbol{D}_0(N)$ and $((Tf_n, v)) \to ((Tu, v))$ as $n \to \infty$ for every $v \in \boldsymbol{D}_0(N)$.

Taking $Ts = \max(s, 0)$ for $s \in R$ in the above proof, we can easily obtain

LEMMA 4.1. *For every $u \in \boldsymbol{D}_0^+(N)$, there exists a sequence $\{u_n\}$ in $L_0^+(X)$ such that $(u_n, v) \to (u, v)$ as $n \to \infty$ for all $v \in \boldsymbol{D}_0(N)$.*

We have

LEMMA 4.2. $\boldsymbol{P}_0(N) = \{u \in \boldsymbol{D}_0(N);\ D(u+v) \geq D(u)$ *for all* $v \in \boldsymbol{D}_0^+(N)\}$.

PROOF. Let $u \in \boldsymbol{D}_0(N)$. Assume that $D(u+v) \geq D(u)$ for all $v \in \boldsymbol{D}_0^+(N)$. For each $x \in X$ and $t > 0$, we have $t\varepsilon_x \in \boldsymbol{D}_0^+(N)$ and

$$D(u) \leq D(u + t\varepsilon_x) = D(u) + 2t(u, \varepsilon_x) + t^2 D(\varepsilon_x),$$

so that $0 \leq (u, \varepsilon_x) = -\Delta u(x)$. Thus $u \in \boldsymbol{P}_0(N)$. Next we assume that $u \in \boldsymbol{P}_0(N)$ and let $v \in \boldsymbol{D}_0^+(N)$. We can find a sequence $\{v_n\}$ in $L_0^+(X)$ such that $(v_n, u) \to (v, u)$ as $n \to \infty$ by Lemma 4.1. It follows from Lemma 1.2 that

$$(v_n, u) = -\sum_{x \in X} v_n(x)\,[\Delta u(x)] \geq 0,$$

and hence $(v, u) \geq 0$. Thus $D(u+v) \geq D(u)$.

THEOREM 4.3. $\boldsymbol{P}_0(N) \subset \boldsymbol{SH}^+(N)$.

PROOF. Let $u \in \boldsymbol{P}_0(N)$. Then $|u| \in \boldsymbol{D}_0^+(N)$ and $D(|u|) \leq D(u)$ by Theorem 4.2. Since $|u| - u \in \boldsymbol{D}_0^+(N)$, we have by Lemma 4.2 $D(u) \leq D(u + (|u| - u)) = D(|u|)$, and hence

$D(|u|)=D(u)$. By using Lemma 4.2 again, we have

$$D(u)\leq D(u+(|u|-u)/2)\leq D(u+|u|)/4+D(u-|u|)/4$$
$$=[D(u)+D(|u|)]/2=D(u),$$

so that $D(u-|u|)=0$. Thus $u=|u|$ by the Corollary of Lemma 3.1, and hence $u\in L^+(X)$.

THEOREM 4.4. $\boldsymbol{HD}(N)=\{u_1-u_2;\ u_1, u_2\in \boldsymbol{HD}(N)\}$.

PROOF. Let $u\in \boldsymbol{HD}(N)$. Then $-u^+=\min(-u, 0)$ and $-u^-=\min(u, 0)$ belong to both $\boldsymbol{D}(N)$ and $\boldsymbol{SH}(N)$. By the Royden decomposition theorem, there exist $v_i\in \boldsymbol{D}_0(N)$ and $h_i\in \boldsymbol{HD}(N)$ such that $-u^+=v_1+h_1$ and $-u^-=v_2+h_2$. Then $v_i\in \boldsymbol{P}_0(N)$, so that $v_i\in L^+(X)$ by Theorem 4.3. Therefore $-h_i\in \boldsymbol{HD}^+(N)$. From the relation $u=u^+-u^-=(v_2-v_1)+(h_2-h_1)$ and the uniqueness of the Royden decomposition, we conclude that $v_1=v_2$ and $u=h_2-h_1$.

## §5. Green potentials

In this section we shall study the classes $\boldsymbol{SH}^+(N)$ and $\boldsymbol{P}_0(N)$. Thus we always assume that $N$ is of hyperbolic type. Denote by $g_a$ the Green function of $N$ with pole at $a\in X$. The results in this section are discrete analogies of the well-known results in potential theory (cf. [3] or [11]).

We take $L^+(X)$ for the set of all non-negative Radon measures in potential theory and define the Green potential $G\mu$ of $\mu\in L^+(X)$ and the mutual energy $G(\mu, \nu)$ of $\mu, \nu\in L^+(X)$ by

$$G\mu(x)=\sum_{b\in X} g_b(x)\mu(b),$$

$$G(\nu, \mu)=\sum_{x\in X}[G\mu(x)]\nu(x).$$

Clearly $G(\nu, \mu)=G(\mu, \nu)$ for all $\mu, \nu\in L^+(X)$. We call $G(\mu, \mu)$ the energy of $\mu$. Let us consider subsets of $L^+(X)$ which are related to Green potentials:

$$M(G)=\{\mu\in L^+(X);\ G\mu\in L(X)\},$$

$$E(G)=\{\mu\in L^+(X);\ G(\mu, \mu)<\infty\}.$$

Then $L_0^+(X)\subset E(G)\subset M(G)$.

We can easily prove

LEMMA 5.1. *Let $\{N_n\}$ ($N_n=<X_n, Y_n>$) be an exhaustion of $N$ and let $\mu\in M(G)$ and $\mu_n=\mu\varepsilon_{X_n}$. Then $G(\mu_n, \mu_n)\to G(\mu, \mu)$ as $n\to\infty$, $G\mu_n\leq G\mu_{n+1}$ on $X$ and $\{G\mu_n\}$ converges pointwise to $G\mu$.*

COROLLARY. $\{G\mu;\ \mu\in M(G)\}\subset \boldsymbol{SH}^{+}(N)$.

LEMMA 5.2. *$\Delta G\mu=-\mu$ for every $\mu\in M(G)$.*

PROOF. Let $\{N_n\}$ ($N_n=<X_n, Y_n>$) be an exhaustion of $N$ and let $\mu\in M(G)$ and $\mu_n=\mu\varepsilon_{X_n}$. Then

$$\Delta G\mu_n(x)=\Delta[\sum_{b\in X_n} g_b(x)\mu(b)]=\sum_{b\in X_n}[\Delta g_b(x)]\mu(b)=-\mu_n(x).$$

Our assertion follows from Lemmas 2.3 and 5.1.

Now we shall prove a discrete analogy of the Riesz decomposition theorem:

THEOREM 5.1. *Every $u\in \boldsymbol{SH}^{+}(N)$ can be decomposed uniquely in the form: $u=G\mu+h$, where $\mu\in M(G)$ and $h\in \boldsymbol{H}^{+}(N)$. In this decomposition, $\mu=-\Delta u$ and $h$ is the greatest harmonic minorant of $u$, i.e., $h'\leq h$ on $X$ for all $h'\in \boldsymbol{H}(N)$ such that $h'\leq u$ on $X$.*

PROOF. Let $\{N_n\}$ ($N_n=<X_n, Y_n>$) be an exhaustion of $N$ and let $u\in \boldsymbol{SH}^{+}(N)$. Denote by $g_a^{(n)}$ the Green function of $N_n$ with pole at $a$ (cf. Remark 3.1). Let us put $\mu=-\Delta u$,

$$u_n(x)=\sum_{b\in X_n} g_b^{(n)}(x)\mu(b) \quad \text{and} \quad h_n=u-u_n.$$

Then $\Delta u_n=-\mu$ on $X_n$ and $u_n=0$ on $X-X_n$, so that $h_n$ is harmonic on $X_n$ and $h_n\geq 0$ on $X-X_n$. Thus $h_n\geq 0$ on $X$ by Lemma 2.1. Since $g_b^{(n)}\leq g_b^{(n+1)}$ on $X$, we have $h_n\geq h_{n+1}$ on $X$. Let $h(x)$ be the limit of $\{h_n(x)\}$ for each $x\in X$. Then $h\in \boldsymbol{H}^{+}(N)$ by Lemma 2.3. By Remark 3.1, we see that $\{u_n\}$ converges pointwise to $G\mu$. Thus we have $u=G\mu+h$. The uniqueness of the decomposition follows from Lemma 5.2. Finally we show that $h$ is the greatest harmonic minorant of $u$. Assume that $h'\in \boldsymbol{H}(N)$ and $h'\leq u$ on $X$. Since $h_n-h'$ is harmonic on $X_n$ and $h_n-h'=u-h'\geq 0$ on $X-X_n$, we see by Lemma 2.1 that $h_n\geq h'$ on $X$, and hence $h\geq h'$ on $X$.

In order to characterize $\boldsymbol{P}_0(N)$ as a set of Green potentials, we prepare some lemmas. We can easily prove

LEMMA 5.3. *If $\mu\in L_0^{+}(X)$, then $G\mu\in \boldsymbol{D}_0(N)$ and $(G\mu, v)=\sum_{b\in X} v(b)\mu(b)$ for all $v\in \boldsymbol{D}_0(N)$.*

COROLLARY. *$G(\mu, \nu)=(G\mu, G\nu)$ and $G(\mu, \mu)=D(G\mu)$ for every $\mu, \nu\in L_0^{+}(X)$.*

LEMMA 5.4. *If $\mu\in E(G)$, then $G\mu\in \boldsymbol{D}_0(N)$ and $D(G\mu)=G(\mu, \mu)$.*

PROOF. Let $\{N_n\}$ ($N_n=<X_n, Y_n>$) be an exhaustion of $N$ and put $\mu_n=\mu\varepsilon_{X_n}$ and $u_n=G\mu_n$. Then $u_n\in \boldsymbol{D}_0(N)$ by Lemma 5.3 and we have for any $p>0$

$$D(u_n)=G(\mu_n, \mu_n)\leq G(\mu_n, \mu_{n+p})\leq D(u_{n+p})\leq G(\mu, \mu)<\infty$$

by the Corollary of Lemma 5.3, so that

$$\|u_n-u_{n+p}\|^2=D(u_n-u_{n+p})+[u_n(x_0)-u_{n+p}(x_0)]^2$$
$$\leq D(u_{n+p})-D(u_n)+[u_n(x_0)-u_{n+p}(x_0)]^2.$$

Since $\{u_n(x_0)\}$ converges to $G\mu(x_0)$ by Lemma 5.1, we see that $\{u_n\}$ is a Cauchy sequence in $\boldsymbol{D}_0(N)$. There exists $v\in\boldsymbol{D}_0(N)$ such that $\|u_n-v\|\to 0$ as $n\to\infty$. Since $\{u_n\}$ converges pointwise to $v$, we conclude that $G\mu=v\in\boldsymbol{D}_0(N)$ by Lemma 5.1. It follows that $D(u_n)\to D(G\mu)$ as $n\to\infty$, so that $D(G\mu)=G(\mu, \mu)$ by Lemma 5.1.

We shall prove

THEOREM 5.2. $\boldsymbol{P}_0(N)=\{G\mu;\ \mu\in E(G)\}$.

PROOF. On account of Lemmas 5.2 and 5.4, it suffices to show that $\boldsymbol{P}_0(N)\subset\{G\mu;\ \mu\in E(G)\}$. Let $u\in\boldsymbol{P}_0(N)$. Then $u\in\boldsymbol{SH}^+(N)$ by Theorem 4.3. By the Riesz decomposition theorem, there exist $\mu\in M(G)$ and $h\in\boldsymbol{H}^+(N)$ such that $u=G\mu+h$. Consider an exhaustion $\{N_n\}$ ($N_n=<X_n, Y_n>$) of $N$ and put $\mu_n=\mu\varepsilon_{X_n}$ and $u_n=G\mu_n$. Then

$$D(u_n)=G(\mu_n, \mu_n)\leq G(\mu, \mu_n)\leq \sum_{x\in X} u(x)\mu_n(x)=(u, u_n)$$
$$\leq[D(u)]^{1/2}[D(u_n)]^{1/2}$$

by Lemma 5.3, so that $G(\mu_n, \mu_n)\leq D(u)<\infty$. We see by Lemma 5.1 that $G(\mu, \mu)\leq D(u)$, and hence $\mu\in E(G)$. It follows from the Royden decomposition theorem that $u=G\mu$.

COROLLARY. $D(u)=-\sum u(x)[\Delta u(x)]$ *for every* $u\in\boldsymbol{P}_0(N)$.

## §6. The classes $\boldsymbol{HP}(N)$ and $\boldsymbol{HBD}(N)$

Let us put

$$\boldsymbol{HP}(N)=\{u\in\boldsymbol{H}(N);\ u=u_1-u_2 \text{ with } u_1, u_2\in\boldsymbol{H}^+(N)\}.$$

Then we have

THEOREM 6.1. $\boldsymbol{HB}(N)\subset\boldsymbol{HP}(N)$.

PROOF. Let $u\in\boldsymbol{HB}(N)$ and let $c$ be a constant such that $|u|\leq c$ on $X$. Then $u_1=(c+u)/2$ and $u_2=(c-u)/2$ belong to $\boldsymbol{H}^+(N)$ and $u=u_1-u_2$. Thus $u\in\boldsymbol{HP}(N)$.

COROLLARY. $O_{HP}\subset O_{HB}$.

THEOREM 6.2. *For every* $u \in \boldsymbol{HD}(N)$, *there exists a sequence* $\{h_n\}$ *in* $\boldsymbol{HBD}(N)$ *such that* $\|u-h_n\| \to 0$ *as* $n \to \infty$.

PROOF. In case $N$ is of parabolic type, $\boldsymbol{HD}(N)$ consists only of constant functions, so that our assertion is clear. We consider the case where $N$ is of hyperbolic type. Let $u \in \boldsymbol{HD}^+(N)$. Then $u_n = \min(u, n) \in \boldsymbol{D}(N) \cap \boldsymbol{SH}^+(N)$. By the Royden decomposition theorem, we can find $v_n \in \boldsymbol{D}_0(N)$ and $h_n \in \boldsymbol{HD}(N)$ such that $u_n = v_n + h_n$. Noting that $v_n \in \boldsymbol{P}_0(N)$, we see by the Riesz decomposition theorem and Theorem 5.2 that $v_n$ and $h_n$ belong to $L^+(X)$ and $h_n$ is the greatest harmonic minorant of $u_n$, so that $0 \le h_n \le h_{n+1} \le u_{n+1} \le u$ on $X$ and $h_n \in \boldsymbol{HBD}(N)$. By Lemma 1.3 we have $D(u-u_n) = D(v_n) + D(u-h_n)$. Since $D(u-u_n) \to 0$ as $n \to \infty$ by [13; Lemma 3.1], we have $D(v_n) \to 0$ and $D(u-h_n) \to 0$ as $n \to \infty$. By the relations

$$D(v_n - v_{n+p}) \le 2[D(v_n) + D(v_{n+p})],$$

$$0 \le v_{n+p}(x_0) \le v_n(x_0) \le u(x_0) < \infty,$$

we see that $\{v_n\}$ is a Cauchy sequence in $\boldsymbol{D}_0(N)$. There exists $v \in \boldsymbol{D}_0(N)$ such that $\|v_n - v\| \to 0$ as $n \to \infty$. We have $D(v) = 0$, and hence $v = 0$ by the Corollary of Lemma 3.1. Thus $\{h_n(x_0)\}$ converges to $u(x_0)$ and

$$\|u-h_n\|^2 = D(u-h_n) + [u(x_0) - h_n(x_0)]^2 \to 0$$

as $n \to \infty$. Now we consider the case where $u \in \boldsymbol{HD}(N)$ is of any sign. There exist $u', u'' \in \boldsymbol{HD}^+(N)$ such that $u = u' - u''$ by Theorem 4.4. By the above observation, we can find sequences $\{h'_n\}$ and $\{h''_n\}$ in $\boldsymbol{HBD}(N)$ such that $\|u' - h'_n\| \to 0$ and $\|u'' - h''_n\| \to 0$ as $n \to \infty$. Writing $h_n = h'_n - h''_n$, we see that $h_n \in \boldsymbol{HBD}(N)$ and $\|u-h_n\| \to 0$ as $n \to \infty$.

COROLLARY. $O_{HD} = O_{HBD}$.

From the obvious relations $\boldsymbol{H}^+(N) \subset \boldsymbol{SH}^+(N)$, $\boldsymbol{HP}(N) \subset \boldsymbol{H}(N)$ and $\boldsymbol{HBD}(N) \subset \boldsymbol{HB}(N)$, it follows that $O_{SH^+} \subset O_{HP}$, $O_H \subset O_{HP}$ and $O_{HB} \subset O_{HBD}$. Thus we have the following classification of infinite networks by the corollaries of Theorems 6.1 and 6.2 and Theorem 3.1:

THEOREM 6.3. $O_G = O_{SH^+} \subset \underset{\substack{\cup \\ O_H}}{O_{HP}} \subset O_{HB} \subset O_{HBD} = O_{HD}$.

The inclusion relations in this theorem are strict, i.e., there exist infinite networks which are contained in one of the class but not in the preceding one (cf. [8]).

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