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On Homogeneous Systems II

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The relations between the properties of analytic homogeneous systems on manifolds and those of their tangent Lie triple algebras are investigated in this paper, by using the results obtained in the preceding paper [2]. It is shown that an isomorphism class of Lie triple algebras corresponds to each isomorphism class of simply connected homogeneous systems (Theorem 2.2). After constructing a universal covering system of a homogeneous system (Theorem 3.1), we show the decomposition theorem of simply connected homogeneous systems (Theorem 4.2). Some particular cases are treated in which the homogeneous system is reduced to that of a Lie group, or reduced to a symmetric space (Theorems 1.1 and 1.3).

Preliminary Remarks

Let (G, η) be a geodesic C^∞-homogeneous system on a connected C^∞-manifold G. In [2] we have introduced the canonical connection of (G, η) and we have seen that both of the torsion and curvature of the connection are parallel on G. Hence, by choosing for each point of G a normal coordinate system, we can regard G as an analytic manifold with the analytic linear connection (see, e.g., Ch. VI, Theorem 7.7 in [3]). If we consider a normal neighborhood U of a fixed point e in G, the point $\eta(e, x, y)$ is the end point of the geodesic arc obtained by the parallel displacement of the geodesic arc joining e to y along the geodesic arc from e to x in U, for any $x, y \in U$. Therefore, the binary operation $\mu^{(e)}(x, y) := \eta(e, x, y)$ at e (cf. [2]) is a local analytic mapping of $U \times U$ into G. Moreover, since any affine transformation of G is analytic, the differentiable automorphisms of η are analytic. Especially every displacement $\eta(a, b)$, $a, b \in G$, of η is an analytic diffeomorphism of G. If (G, η) is a homogeneous system of a Lie group, that is, the binary system $\mu^{(e)}$ is a Lie group, it is well known that $\mu^{(e)}$ is analytic with respect to normal coordinate systems of the canonical connection (canonical coordinate system of the 1st kind [4]) and so η is analytic. The proof of this fact is based on local analyticity of $\mu^{(e)}$ and analyticity of the left translations and automorphisms of the Lie group.

We leave the following as an open problem: Is any geodesic C^∞-homogeneous system analytic? In the rest of this paper we assume that all homogeneous systems are geodesic and analytic homogeneous systems defined on connected and second countable analytic manifolds.

§ 1. Flat Homogeneous Systems and Symmetric Homogeneous Systems

In this section we observe some special situations for homogeneous system (G, η) in which the tangent Lie triple algebra $\mathfrak{G}$ of η at a fixed point e has the trivial bilinear operation, or the trivial trilinear operation.

A *Lie triple algebra* (*general Lie triple system* [6]) is by definition a vector space $\mathfrak{G}$ over an arbitrary field with an anti-commutative binary operation denoted by XY and a trilinear operation denoted by $[X, Y, Z]$ for $X, Y, Z \in \mathfrak{G}$ satisfying the relations; $[X, X, Y]=0$, $\mathfrak{S}([X, Y, Z]+(XY)Z)=0$, $\mathfrak{S}D(XY, Z)=0$, $D(X, Y)(UV)=(D(X, Y)U)V+U(D(X, Y)V)$ and $[D(X, Y), D(U, V)]=D(D(X, Y)U, V)+D(U, D(X, Y)V)$ for $X, Y, Z, U, V \in \mathfrak{G}$, where $\mathfrak{S}$ means the cyclic sum with respect to X, Y and and Z, and $D(X, Y)$ denotes the endomorphism $Z \to [X, Y, Z]$ of $\mathfrak{G}$ called an *inner derivation* of the Lie triple algebra. A Lie triple subalgebra (i.e. a subsystem with respect to both operations) $\mathfrak{H}$ of a Lie triple algebra $\mathfrak{G}$ is said to be *invariant* if $\mathfrak{H}$ is invariant under all inner derivations of $\mathfrak{G}$, that is, $[\mathfrak{G}, \mathfrak{G}, \mathfrak{H}] \subset \mathfrak{H}$. $\mathfrak{H}$ is called an *ideal* of $\mathfrak{G}$ if $\mathfrak{G}\mathfrak{H} \subset \mathfrak{H}$ and $[\mathfrak{H}, \mathfrak{G}, \mathfrak{G}] \subset \mathfrak{H}$. Any ideal of $\mathfrak{G}$ is an invariant L. t. subalgebra. In the preceding paper [2] we have shown that the tangent space $\mathfrak{G}=T_e(G)$ at a fixed base point e of geodesic homogeneous system (G, η) forms a Lie triple algebra, called the *tangent L.t.a.* at e, under the operations $XY=S_e(X, Y)$ and $[X, Y, Z]=R_e(X, Y)Z$, where S and R are respectively the torsion and curvature of the canonical connection of (G, η). From the axioms of Lie triple algebra it follows that if the trilinear operation vanishes identically the Lie triple algebra is a Lie algebra under the bilinear operation, and that if the bilinear operation vanishes it is a Lie triple system under the trilinear operation. The Lie triple algebra is said to be *reduced to Lie algebra* in the former case, and to be *reduced to Lie triple system* in the latter case.

THEOREM 1.1. *The following conditions for a geodesic homogeneous system (G, η) are mutually equivalent*:

(i) *(G, η) is the homogeneous system of a Lie group, i.e., G is a Lie group and $\eta(x, y, z)=yx^{-1}z$;*

(ii) *the tangent L.t.a. $\mathfrak{G}$ of (G, η) at a base point e is reduced to Lie algebra;*

(iii) *the canonical connection of (G, η) is flat.*

In this case, the binary system $xy=\mu^{(e)}(x, y)$ on G is a Lie group with its Lie algebra isomorphic to $\mathfrak{G}$.

A homogeneous system satisfying the conditions of this theorem will be called *flat*.

PROOF. From the definition $D(X, Y)=R_e(X, Y)$ of inner derivations of $\mathfrak{G}$ and from the fact that $\nabla R=0$, it follows directly that (ii) is equivalent to (iii). We have seen in [2] that the condition (i) implies (ii) and (iii), and that the bracket operation of the Lie algebra of $(G, \mu^{(e)})$ is $[X, Y]=XY$. Suppose that the canonical connection is flat. Since η is geodesic, the differential $dL_{x,y}=d\eta(xy, e)d\eta(x, xy)d\eta(e, x)$ of the left inner mapping $L_{x,y}$ is an element of the restricted holonomy group at e given by the parallel displacement along the geodesic triangle from e passing through x and xy and ending to e, if x and y belong to some normal neighborhood U of e. Hence $dL_{x,y}=1_{\mathfrak{G}}$, the identity map of $\mathfrak{G}$, for $x, y \in U$. The map of $G\times G\times G$ onto G given by $(x, y, z)\to L_{x,y}z$ is an analytic map and hence so is the map ϕ: $G\times G\to GL(\mathfrak{G})$; $\phi(x, y)=dL_{x,y}$, being a constant map on $U\times U$ with the constant value $1_{\mathfrak{G}}$. Therefore $dL_{x,y}=1_{\mathfrak{G}}$ holds for all $x, y\in G$. Since every left inner map is an affine transformation of the canonical connection, it follows that $L_{x,y}=1_G$ for $x, y\in G$. Thus we see that $\mu^{(e)}$ forms a Lie group on G. *q. e. d.*

We apply this result to the nullity subspace of the curvature of the canonical connection which is given by $\mathfrak{N}=\{X\in\mathfrak{G}: D(X, Y)=0 \text{ for all } Y\in\mathfrak{G}\}$ in the tangent Lie triple algebra $\mathfrak{G}$ at e.

THEOREM 1.2. *Let $\mathfrak{N}$ be the nullity subspace of the canonical connection of a homogeneous system (G, η) at e. Then $\mathfrak{N}$ is an invariant L. t. subalgebra of the tangent L. t. a. $\mathfrak{G}$ of (G, η) at e and it is reduced to Lie algebra. If (G, η) is regular, there exists a unique invariant flat subsystem (N, η_N) of (G, η) containing e at which $\mathfrak{N}$ is its tangent L. t. a..*

PROOF. From the formula $\mathfrak{S}D(XY, Z)=0$ for $X, Y, Z\in\mathfrak{G}$, we get $D(XY, Z)=0$ for any Z of $\mathfrak{G}$ if X and Y belong to $\mathfrak{N}$. Hence $\mathfrak{N}$ is a subalgebra of the bilinear multiplication XY in $\mathfrak{G}$. Another formula $[D(U, V), D(X, Y)]=D(D(U, V)X, Y)+D(X, D(U, V)Y)$ implies that $D(U, V)X$ is contained in $\mathfrak{N}$ for $X\in\mathfrak{N}$ and $U, V\in\mathfrak{G}$. Thus $\mathfrak{N}$ is an invariant L. t. subalgebra of $\mathfrak{G}$. Since $D(X, Y)=0$ for $X, Y\in\mathfrak{N}$, it follows that $\mathfrak{N}$ is reduced to Lie algebra. If (G, η) is regular then Theorem 5 in [2] implies that there exists a unique connected invariant subsystem (N, η_N) of (G, η) whose tangent L.t.a. at e is equal to $\mathfrak{N}$. This homogeneous system is geodesic and analytic and so it is flat by Theorem 1.1. *q. e. d.*

REMARK. In general, the integrability of nullity distributions of linearly connected manifolds has been discussed by S.-L. Tan in [5].

A homogeneous system (G, η) is said to be *symmetric* if, for a fixed point e, the transformation J_e of G sending x to $x^{-1}=\eta(x, e, e)$, the inverse element of x with respect to $\mu^{(e)}$, is an automorphism of η (cf. § 4 in [1]). In this case, $J_a(x)=\eta(x, a, a)$ is also an automorphism of η for each $a\in G$. Suppose that (G, η) is symmetric. Then the reductive homogeneous space expression $G=A/K$ of $A=G\times K_e$ (semi-direct product) by $K=\{e\}\times K_e$ at e (cf. [2]) is a symmetric homogeneous space under the involutive automorphism $\sigma=J_e\times 1_G$ of the Lie group A. The canonical connection is then locally symmetric, i.e., the torsion S vanishes identically, and so the tangent L.t.a. $\mathfrak{G}$ at e is reduced to Lie triple system.

Conversely, suppose that the tangent L.t.a. $\mathfrak{G}$ of a homogeneous system (G, η) at e is reduced to Lie triple system. Then $S=0$ and so the canonical connection is locally symmetric. The reversion J of geodesics through e across e is a local affine transformation. Since (G, η) is assumed to be geodesic and analytic, there exists a normal neighborhood U of e on which J is an analytic transformation satisfying $J=J_e$ on U and $J\circ\eta(x, y, z)=\eta(J(x), J(y), J(z))$ for x, y and z in U. From the assumption that G is connected it follows that two analytic mappings of $G\times G\times G$ into G given by $\phi(x, y, z)=J_e\circ\eta(x, y, z)$ and $\psi(x, y, z)=\eta(J_e(x), J_e(y), J_e(z))$ are coincident with each other. This shows that (G, η) is symmetric. Thus we have the following;

THEOREM 1.3. *For a homogeneous system (G, η) the following conditions are mutually equivalent:*

(i) *(G, η) is symmetric;*

(ii) *the tangent L.t.a. of (G, η) at a point e is reduced to Lie triple system;*

(iii) *the canonical connection is locally symmetric.*

In this case, the reductive homogeneous space expression $G=A/K$ at e is reduced to a symmetric homogeneous space.

§2. Simply Connected Homogeneous Systems

In this section we consider a problem to extend a homomorphism of tangent L.t. algebras of two homogeneous systems to a global homomorphism of homogeneous systems.

PROPOSITION 2.1. *Let $\mathfrak{G}$ and $\mathfrak{G}'$ be the tangent Lie triple algebras of two homogeneous systems (G, η) and (G', η') at respective base points e and e'. Assume that G is simply connected. Then, for each injective homomorphism F: $\mathfrak{G}\to\mathfrak{G}'$ of Lie triple algebras such that the image $F(\mathfrak{G})$ is invariant under the left inner mapping group of G' at e', there exists a unique analytic homomorphism f: $G\to G'$ of homogeneous systems satisfying $f(e)=e'$ and $df|_e=F$.*

PPOOF. We set $\mathfrak{H}=F(\mathfrak{G})$ and define an analytic distribution $\mathfrak{S}$ on G' by $\mathfrak{S}_{x'}=d\eta'(e', x')\mathfrak{H}$ for each x' of G'. Since (G', η') is assumed to be geodesic, approximating any curve in G' by piecewise geodesic curves we can show that the distribution $\mathfrak{S}$ is parallel with respect to the canonical connection. The same discussion as in §4 of [2] is available here and we get a unique connected analytic subsystem (H, η'_H) of (G', η') through e' whose tangent L.t.a. at e' is $\mathfrak{H}$. The submanifold H is autoparallel and the canonical connection of η'_H is equal to the induced connection on H from the canonical connection of (G', η'). Now F: $\mathfrak{G}\to\mathfrak{H}$ is a L.t.a. isomorphism and so it maps the torsion and curvature at e to those at e', respectively. Since G is simply connected there exists a unique analytic affine mapping f: $G\to H$ such that $f(e)=e'$

and $df|_e=F$. (See, e.g., Theorems 6.1 and 7.2 in [3].) There exists a neighborhood U of e on which the restriction $f|U$ of f is an affine isomorphism. The inclusion map of (H, η') into (G', η') is an analytic homomorphism and so it is sufficient to show that f is a homomorphism of (G, η) into (H, η'). If we choose U to be a normal neighborhood of e, then $f|U$ sends the parallel displacement of vectors along each geodesic arc in U from e to x to the parallel displacement along its image. Hence we have the equality $df|_x \circ d\eta(e, x)|_e = d\eta'(e', x')|_{e'} \circ df|_e$ of linear isomorphisms of $T_e(G)$ onto $T_{x'}(H)$, where $e'=f(e)$ and $x'=f(x)$ for $x \in U$. Then we see that the two affine transformations $f \circ \eta(e, x)$ and $\eta'(e', x') \circ f$ coincide on G for $x \in U$. For each z of G two analytic mappings ϕ_z and ψ_z of G into H given by $\phi_z(x)=f \circ \eta(e, x, z)$ and $\psi_z(x) = \eta'(f(e), f(x), f(z))$ which coincide on U must coincide on G. Thus f is an analytic homomorphism of (G, η) into (H, η'). *q.e.d.*

From this proposition we have immediately the following two theorems.

THEOREM 2.2. *If G and G' are simply connected and if F: $\mathfrak{G} \rightarrow \mathfrak{G}'$ is an isomorphism of the tangent Lie triple algebras of homogeneous systems (G, η) and (G', η') at respective base points e and e', then there exists a unique isomorphism f of (G, η) onto (G', η') satisfying $f(e)=e'$ and $df|_e=F$.*

THEOREM 2.3. *If G is simply connected, then for each automorphism F of the tangent L.t.a. at e of a homogeneous system (G, η), there corresponds a unique automorphism of (G, η) leaving e fixed and satisfying $df|_e=F$.*

Now, denote by $A(\eta)$ the group of all analytic automorphisms of (G, η) and by $Aff(\eta)$ the affine transformation group of the canonical connection of η. For a fixed base point e of G, let $A_e(\eta)$ be the isotropy subgroup of $A(\eta)$ at e, $Aut(\mu^{(e)})$ the automorphism group of the binary system $\mu^{(e)}$ and $Aut(\mathfrak{G})$ the automorphism group of the tangent L. t. a. $\mathfrak{G}$ at e, respectively.

PROPOSITION 2.4. $Aut(\mu^{(e)}) = A_e(\eta)$.

PROOF. If $\phi \in A_e(\eta)$, from the definition $\mu^{(e)}(x, y)=\eta(e, x, y)$ of $\mu^{(e)}$, it is clear that ϕ is an automorphism of $\mu^{(e)}$. Conversely, suppose that ϕ is an automorphism of $\mu^{(e)}$. The unit element e of $\mu^{(e)}$ is left fixed by ϕ, and the equation $\eta(e, \phi(x)) \circ \phi = \phi \circ \eta(e, x)$ holds for every element x. Composing the diffeomorphism $\eta(e, \phi(a))$ to both sides of this equation on the left we get

$$\eta(\phi(a), \phi \circ \eta(e, a, x)) \circ \phi = \phi \circ \eta(a, \eta(e, a, x))$$

for $a \in G$. Since $\eta(e, a)$ is a diffeomorphism of G we see that the equation $\eta(\phi(a), \phi(b), \phi(c)) = \phi(\eta(a, b, c))$ holds for $a, b, c \in G$. *q.e.d.*

THEOREM 2.5. *If the underlying manifold G of a homogeneous system (G, η) is simply connected, the following isomorphisms are valid for any point e of G:*

(i) *$Aut(\mu^{(e)})\cong Aut(\mathfrak{G})$, where $\mathfrak{G}$ is the tangent L.t.a. at e;*

(ii) *$Aff(\eta)\cong G\times Aut(\mu^{(e)})$ (semi-direct product at e), where the group multiplication of the semi-direct product is defined by $(x, \alpha)(y, \beta):=(\mu^{(e)}(x, \alpha(y)), L^{(e)}_{x,\alpha(y)}\circ\alpha\circ\beta)$ for $(x, \alpha), (y, \beta)\in G\times Aut(\mu^{(e)})$, $L^{(e)}{}_{x,y}$ denoting a left inner mapping at e, i.e., $L^{(e)}_{x,y}:=\eta(\mu^{(e)}(x, y), e)\circ\eta(e, x)\circ\eta(e, y)$* (cf. § 3 in [1]).

PROOF. (i) is an immediate consequence of Theorem 2.3 and Proposition 2.4 shown above. To show (ii) we consider a mapping Φ of $G\times A_e(\eta)$ onto $A(\eta)$ given by $\Phi(x, \alpha)=\eta(e, x)\circ\alpha$, which is easily seen to be a group isomorphism. Since $A_e(\eta)=Aut(\mu^{(e)})$ as proved above, it is sufficient to show that $A(\eta)=Aff(\eta)$. In fact we see $A(\eta)\subset Aff(\eta)$ from the definition of the canonical connection (cf. Proposition 2 in [2]). Conversely, if f is an affine transformation of G, then it sends the torsion and curvature at e of G to those at $e'=f(e)$, that is, $df|_e$ is a L.t.a. isomorphism of the tangent L.t.a. $\mathfrak{G}$ at e to $\mathfrak{G}'$ at e'. Theorem 2.2 and the uniqueness of affine transformation for its differential at any point imply that f is an automorphism of η.

q. e. d.

§ 3. Universal Covering Homogeneous Systems

Let e be a fixed base point of a homogeneous system (G, η). As a simply connected covering manifold $\tilde{G}$ of G we consider the space of end point fixed homotopy classes $[\tilde{x}]$ of paths $\tilde{x}(t)$, $0\leq t\leq 1$, in G from the base point $e=\tilde{x}(0)$ to each point $x=\tilde{x}(1)$ of G. The space $\tilde{G}$ is supposed to be endowed with the topology and the analytic structure in a natural manner so that the covering map $p: \tilde{G}\to G$, $p([\tilde{x}])=\tilde{x}(1)$, is an analytic local diffeomorphism at each point of $\tilde{G}$.

A homogeneous system $\tilde{\eta}$ on $\tilde{G}$ is induced from η as follows. For any homotopy classes $[\tilde{x}], [\tilde{y}], [\tilde{z}]\in\tilde{G}$ of paths $\tilde{x}$, $\tilde{y}$, and $\tilde{z}$ joining e to x, y and z in G, respectively, put

$$\tilde{\eta}([\tilde{x}], [\tilde{y}], [\tilde{z}]):=[\tilde{y}\cdot(\eta(x, y)(\tilde{x}^{-1}\cdot\tilde{z}))],$$

where $\tilde{x}\cdot\tilde{y}$ denotes the product path of $\tilde{x}$ followed by $\tilde{y}$ and $\tilde{x}^{-1}$ denotes the inverse path of $\tilde{x}$. In fact, since the displacement $\eta(x, y)$ is a diffeomorphism sending x to y, the path in the bracket of the right hand side of the equation above is well defined and the definition of $\tilde{\eta}([\tilde{x}], [\tilde{y}], [\tilde{z}])$ depends only on the homotopy classes $[\tilde{x}]$, $[\tilde{y}]$ and $[\tilde{z}]$. By definition of the analytic structure of $\tilde{G}$ we see that the map $\tilde{\eta}: \tilde{G}\times\tilde{G}\times\tilde{G}\to\tilde{G}$ is analytic. Moreover, using directly the definition of $\tilde{\eta}$ given above, we can show that $(G, \tilde{\eta})$ is a homogeneous system, that is, the following equalities are valid: (i) $\tilde{\eta}([\tilde{x}], [\tilde{x}], [\tilde{y}])=\tilde{\eta}([\tilde{x}], [\tilde{y}], [\tilde{x}])=[\tilde{y}]$; (ii) $\tilde{\eta}([\tilde{x}], [\tilde{y}], \tilde{\eta}([\tilde{y}], [\tilde{x}], [\tilde{z}]))=[\tilde{z}]$; (iii) $\tilde{\eta}([\tilde{u}], [\tilde{v}], \tilde{\eta}([\tilde{x}], [\tilde{y}], [\tilde{z}]))=\tilde{\eta}(\tilde{\eta}([\tilde{u}], [\tilde{v}], [\tilde{x}]), \tilde{\eta}([\tilde{u}], [\tilde{v}], [\tilde{y}]), \tilde{\eta}([\tilde{u}], [\tilde{v}], [\tilde{z}]))$. The terminal point of the path $\tilde{y}\cdot(\eta(x, y)(\tilde{x}^{-1}\cdot\tilde{z}))$ is the point $\eta(x, y, z)$ and hence the covering map $p: (\tilde{G}, \tilde{\eta})\to(G, \eta)$ is a homomorphism of homogeneous systems and induces a local isomorphism around each point of $\tilde{G}$.

In general, if a homogeneous system (G', η') is a *covering homogeneous system* of a geodesic homogeneous system (G, η), that is, if the covering map p': $G' \to G$ induces an analytic homomorphism of the homogeneous system (G', η') onto (G, η), the map p' induces a local isomorphism of homogeneous systems. Hence the canonical connection of (G', η') is the linear connection on G' induced from the canonical connection of (G, η), and so the covering homogeneous system (G', η') is also geodesic. Moreover, if $p'(e')=e$, $dp'|_{e'}$ is a L.t.a. isomorphism of the tangent L.t.a. $\mathfrak{G}'$ of (G', η') at e' to the tangent L.t.a. $\mathfrak{G}$ at e. If G' is simply connected, then from Theorem 2.2 we get a unique isomorphism ϕ: $(G', \eta') \to (\tilde{G}, \tilde{\eta})$ satisfying $d\phi|_{e'} = dp|_{\tilde{e}}^{-1} \circ dp'|_{e'}$ so that $p \circ \phi = p'$. Thus a simply connected covering homogeneous system may be called a *universal covering homogeneous system.*

Summing up the results above, we have

THEOREM 3.1. *Every homogeneous system (G, η) has a universal covering homogeneous system. If p: $(\tilde{G}, \tilde{\eta}) \to (G, \eta)$ and p': $(G', \eta') \to (G, \eta)$ are two universal covering systems of (G, η), then there exists an isomorphism ϕ: $(\tilde{G}, \tilde{\eta}) \to (G', \eta')$ satisfying $p = p' \circ \phi$.*

From Theorems 1.1 and 1.3 and from the fact that the differential of a covering map is an isomorphism of the tangent L.t.a.'s we have the followings:

PROPOSITION 3.2. *A covering homogeneous system (G', η') of (G, η) is flat if and only if (G, η) is flat. In this case the Lie group $(G', \mu^{(e')})$ is the covering group of $(G, \mu^{(e)})$ under the covering map p, if $p(e')=e$.*

PROPOSITION 3.3. *A covering homogeneous system (G', η') of (G, η) is symmetric if and only if (G, η) is symmetric.*

§4. Decomposition Theorem

Let (G, η) and (G', η') be homogeneous systems. The product homogeneous system $(G, \eta) \times (G', \eta')$ is the product manifold $M = G \times G'$ with the homogeneous system $\theta = \eta \times \eta'$. For a fixed base point (e, e') of M, $(G \times \{e'\}, \eta \times 1_{G'})$ is a subsystem of (M, θ) isomorphic to (G, η) under the natural projection. It is invariant under the left inner mapping group of (M, θ) at (e, e'). In fact, for $(x, x'), (y, y') \in M$, the left inner mapping $L^{(e,e')}_{(x,x'),(y,y')}$ at (e, e') sends any (z, e') of $G \times \{e'\}$ to $(L^{(e)}_{x,y}z, e')$. If we identify the subsystem $G \times \{e'\}$ (resp. $\{e\} \times G'$) with G (resp. G'), the tangent L.t.a. $\mathfrak{M}$ of (M, θ) at (e, e') is identified with the direct sum $\mathfrak{G} \oplus \mathfrak{G}'$ of the tangent L.t.a.'s of (G, η) and (G', η') at the respective base points e and e'. Then $\mathfrak{G}$ (resp. $\mathfrak{G}'$) is an invariant L.t. subalgebra of $\mathfrak{M}$. The canonical connection of (M, θ) is equivalent to the affine product of the canonical connections of (G, η) and (G', η') and hence the torsion and curvature of M are given as follows;

$$S^{(M)}_{(e,e')}(X+X', Y+Y')=S^{(G)}_{e}(X, Y)+S^{(G')}_{e'}(X', Y'),$$

$$R^{(M)}_{(e,e')}(X+X', Y+Y')(Z+Z')=R^{(G)}_{e}(X, Y)Z+R^{(G')}_{e'}(X', Y')Z',$$

where $X+X'$ denotes the element of $\mathfrak{M}$ decomposed into the components $X\in\mathfrak{G}$ and $X'\in\mathfrak{G}'$ of the direct sum $\mathfrak{G}\oplus\mathfrak{G}'$. These formulas show that the Lie triple algebra $\mathfrak{G}$(resp. $\mathfrak{G}'$) is an ideal of $\mathfrak{M}$, that is, $\mathfrak{M}\mathfrak{G}\subset\mathfrak{G}$ and $[\mathfrak{G}, \mathfrak{M}, \mathfrak{M}]\subset\mathfrak{G}$. From the fact that M is the affine product of G and G', it follows that the homogeneous system M is regular if both of G and G' are regular.

Conversely to the facts above, we have;

PROPOSITION 4.1. *Let (M, θ) be a simply connected regular homogeneous system. Suppose that the tangent Lie triple algebra $\mathfrak{M}$ of (M, θ) at a fixed base point e is decomposed into the direct sum of two ideals $\mathfrak{G}$ and $\mathfrak{G}'$ of $\mathfrak{M}$. Then there exist simply connected invariant subsystems (G, η) and (G', η') of (M, θ) with the tangent Lie triple algebras $\mathfrak{G}$ and $\mathfrak{G}'$, respectively, at e, such that (M, θ) is isomorphic to the product homogeneous system $(G, \eta)\times(G', \eta')$.*

PROOF. Since any ideal $\mathfrak{G}$ of the Lie triple algebra $\mathfrak{M}$ is an invariant L.t. subalgebra and since the homogeneous system (M, θ) is assumed to be regular, there exists a unique connected invariant subsystem (G, θ_G) of (M, θ) containing the base point e at which the tangent L.t.a. of (G, θ_G) is equal to $\mathfrak{G}$ (cf. Theorem 5 in [2]). In the same manner we get an invariant subsystem $(G', \theta_{G'})$ for the ideal $\mathfrak{G}'$ at e'. Denote $\eta=\theta_G$ and $\eta'=\theta_{G'}$, and let $(\tilde{G}, \tilde{\eta})$ (resp. $(\tilde{G}', \tilde{\eta})$) be a universal covering homogeneous system of (G, η) (resp. (G', η')). We have a natural Lie triple algebra isomorphism of the tangent L.t.a. $\tilde{\mathfrak{M}}$ of the product homogeneous system $(\tilde{M}, \tilde{\theta})=(\tilde{G}, \tilde{\eta})\times(\tilde{G}', \tilde{\eta})$ to the tangent L.t.a. $\mathfrak{M}$ of (M, θ). Since both of $\tilde{M}$ and M are simply connected, this L.t.a. isomorphism is uniquely extended to an isomorphism ϕ: $(\tilde{M}, \tilde{\theta})\rightarrow(M, \theta)$ of homogeneous systems with $\phi(e, e)=e$ (Theorem 2.2). Under this isomorphism ϕ, the invariant subsystem $(\tilde{G}, \tilde{\eta})$ (resp. $(\tilde{G}', \tilde{\eta}')$) corresponds to (G, η) (resp. (G', η')). Thus we see that G and G' are simply connected and we get the isomorphism of (M, θ) onto $(G, \eta)\times(G', \eta')$. *q.e.d.*

This proposition and the properties of product homogeneous systems mentioned at the first part of this section imply the following decomposition theorem:

THEOREM 4.2. *Let (G, η) be a simply connected regular homogeneous system with the tangent Lie triple algebra $\mathfrak{G}$ at a base point e. Then $\mathfrak{G}$ is decomposed into a direct sum $\mathfrak{G}_1\oplus\mathfrak{G}_2\oplus\cdots\oplus\mathfrak{G}_k$ of ideals $\mathfrak{G}_i(i=1, 2,\dots, k)$ of $\mathfrak{G}$ if and only if the homogeneous system (G, η) is isomorphic to the product homogeneous system $(G_1, \eta_1)\times(G_2, \eta_2)\times\cdots\times(G_k, \eta_k)$, where each (G_i, η_i) is an invariant subsystem of (G, η) whose tangent L.t.a. is $\mathfrak{G}_i$.*

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