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UNIFORMITIES FOR FUNCTION SPACES AND CONTINUITY CONDITIONS

By

Takashi KARUBE

Using uniform structures, Dieudonné [3] systematized a number of results on topologies for homeomorphism groups which had been published till 1947. Since then, as to properties on continuity, only sufficient conditions have been given under special uniformities.

Let X be a set, Y be a uniform space endowed with a uniform structure $\mathfrak{U}$, $\mathfrak{S}$ be a family of subsets of X, and $\mathfrak{F}$ be the family of all mappings of X into Y. For each set $A \in \mathfrak{S}$ and each entourage $U \in \mathfrak{U}$, let $W(A, U)$ denote the set of all pairs (u, v) of mappings of X into Y such that $(u(x), v(x)) \in U$ for all $x \in A$. Then $\{W(A, U) \mid A \in \mathfrak{S}, U \in \mathfrak{U}\}$ form a fundamental system of entourages of a uniformity $\mathfrak{W}$ on $\mathfrak{F}$ under the proper conditions on $\mathfrak{S}$ (Theorem 1).

The purpose of this paper is to find the most general conditions possible, expressed by the properties of $\mathfrak{S}$ and $\mathfrak{U}$, that satisfy the following basal conditions on continuity with respect to the uniformity $\mathfrak{W}$: i) the mapping $(u, x) \rightarrow u(x)$ of $\mathfrak{C} \times X$ into Y is continuous, where X is a topological space and $\mathfrak{C}$ is a family of continuous mappings of X into Y (Theorems 2 and 3), ii) the mapping $(u, v) \rightarrow uv$ of $\mathfrak{C} \times \mathfrak{C}$ into $\mathfrak{C}$ is continuous, where X and Y are the same uniform space and $\mathfrak{C}\mathfrak{C} \subset \mathfrak{C}$ (Propositions 4 and 5 ; Theorems 4 and 5). These are the basal conditions often required to be satisfied for semi-groups of continuous transformations of a uniform space.

From our results, it is conjectured that if a uniformity $\mathfrak{W}$ on $\mathfrak{F}$ satisfies these basal conditions for the family of all continuous mappings of a space into itself which has several properties similar to those of euclidean spaces, then $\mathfrak{W}$ must be the uniformity of compact convergence. In fact it is affirmative (cf. Karube [5]).

For topological terms and notations we follow the usage of N. Bourbaki [2].

§ 1. Uniformizability conditions.

Theorem 1. *Let X be a set ; let Y be a set endowed with a uniform structure*

$\mathfrak{U}$ *which is not the coarsest ; let* $\mathfrak{S}$ *be a family of subsets of* X *; and let* $\mathfrak{F}$ *be the family of all mappings of* X *into* Y*. For each* $A\in\mathfrak{S}$ *and each* $U\in\mathfrak{U}$, *let* $W(A, U)$ *denote the set of all pairs* (u, v) *of mappings of* X *into* Y *such that* $(u(x), v(x))\in U$ *for all* $x\in A$*. Then, as* A *runs through* $\mathfrak{S}$ *and* U *runs through* $\mathfrak{U}$, *the sets* $W(A, U)$ *form a fundamental system of entourages of a uniformity on* $\mathfrak{F}$ *if and only if*

1) $\mathfrak{S}$ *is non-empty, and*

2) *for any two sets* $A_1, A_2\in\mathfrak{S}$ *there exists a set* $A_3\in\mathfrak{S}$ *such that* $A_3\supset A_1\cup A_2$.

In the case where $\mathfrak{U}$ *is the coarsest uniformity, the condition* 2) *is superfluous, and the corresponding uniformity on* $\mathfrak{F}$ *is the coarsest one.*

Proof. Put $\mathfrak{B}=\{W(A, U)\mid A\in\mathfrak{S}, U\in\mathfrak{U}\}$. It is easy to see that the above two conditions 1) and 2) are sufficient for $\mathfrak{B}$ to be a fundamental system of entourages of a uniformity on $\mathfrak{F}$. We shall show the necessity. In fact

a) the condition 1) is equivalent to the fact that every $W(A, U)\in\mathfrak{B}$ contains the diagonal set in $\mathfrak{F}\times\mathfrak{F}$, and

b) if $\mathfrak{U}$ is not the coarsest uniformity, then the condition 2) is equivalent to the following

2)′ for any two sets $W(A_1, U_1)$, $W(A_2, U_2)\in\mathfrak{B}$ there exists a set $W(A_3, U_3)\in\mathfrak{B}$ such that $W(A_3, U_3)\subset W(A_1, U_1)\cap W(A_2, U_2)$.

We shall only prove that 2)′ implies 2). Since $\mathfrak{U}$ is not the coarsest uniformity, there exists an entourage $U\in\mathfrak{U}$ different from $Y\times Y$. For the entourage U and any two sets $A_1, A_2\in\mathfrak{S}$, there exist an entonrage $U'\in\mathfrak{U}$ and a set $A_3\in\mathfrak{S}$ such that $W(A_3, U')\subset W(A_1\cup A_2, U)$. If $A_1\cup A_2$ is not contained in A_3, take a point $a\in A_1\cup A_2-A_3$ and two points p and q of Y such that $(p, q)\notin U$. There exist two functions $u, v\in\mathfrak{F}$ such that u and v coincide on A_3 and $u(a)=p$, $v(a)=q$. Then $(u, v)\notin W(A_1\cup A_2, U)$ while $(u, v)\in W(A_3, U')$. The contradiction shows that $A_3\supset A_1\cup A_2$.

Definition 1. Each family of subsets of X which satisfies conditions 1) and 2) in Theorem 1 is called a *u-family*.

The following Proposition 1 shows that our uniformity on $\mathfrak{F}$ is essentially the same as the uniformity of $\mathfrak{S}$-convergence on $\mathfrak{F}$ (cf. N. Bourbaki [2]). The former is rather more direct than the latter for us to define uniformities on function spaces.

Proposition 1. *Let* X, Y, $\mathfrak{U}$ *and* $\mathfrak{F}$ *be the same as those in Theorem* 1. *Let* $\mathfrak{S}$ *be a u-family of subsets of* X *; let* $\mathfrak{S}^*$ *be the family of all sets that are finite unions of sets belonging to* $\mathfrak{S}$ *; let* $\mathfrak{S}^{**}$ *be the family of all subsets of sets belonging to* $\mathfrak{S}^*$. *As* A *runs through* $\mathfrak{S}$, $\mathfrak{S}^*$ *and* $\mathfrak{S}^{**}$ *respectively and* U *runs through* $\mathfrak{U}$, *the sets* $W(A, U)$ *form a fundamental system of entourages* *of a uniformity on $\mathfrak{F}$. Let $\mathfrak{W}$, $\mathfrak{W}^*$ and $\mathfrak{W}^{**}$ denote the uniformity respectively. Moreover let $\widetilde{\mathfrak{W}}$, $\widetilde{\mathfrak{W}}^*$ and $\widetilde{\mathfrak{W}}^{**}$ denote the uniformity of $\mathfrak{S}$-, $\mathfrak{S}^*$- and $\mathfrak{S}^{**}$-convergence on $\mathfrak{F}$ respectively, and for each $A \in \mathfrak{S}$ let $\mathfrak{W}_A$ denote the uniformity of $\{A\}$-convergence on $\mathfrak{F}$, where $\{A\}$ is the family consisting of the set A only. Then*

$$\mathfrak{W} = \mathfrak{W}^* = \mathfrak{W}^{**} = \widetilde{\mathfrak{W}} = \widetilde{\mathfrak{W}}^* = \widetilde{\mathfrak{W}}^{**} = \bigcup_{A \in \mathfrak{S}} \mathfrak{W}_A.$$

Proof. It is well-known that $\widetilde{\mathfrak{W}} = \widetilde{\mathfrak{W}}^* = \widetilde{\mathfrak{W}}^{**}$, and it is easy to show that 1) $\bigcup_{A \in \mathfrak{S}} \mathfrak{W}_A$ is a uniformity on $\mathfrak{F}$, 2) $\bigcup_{A \in \mathfrak{S}} \mathfrak{W}_A$ is the least upper bound of $\{\mathfrak{W}_A \mid A \in \mathfrak{S}\}$ in the family of all uniform structures ordered by the relation "inclusion", consequently it coincides with $\widetilde{\mathfrak{W}}$, 3) $\mathfrak{W} = \widetilde{\mathfrak{W}}$, and 4) $\mathfrak{W}^* = \widetilde{\mathfrak{W}}^*$ and $\mathfrak{W}^{**} = \widetilde{\mathfrak{W}}^{**}$ as well.

As simple cases where $\mathfrak{W}$ determines $\mathfrak{S}$ and $\mathfrak{U}$, we consider the case where $\mathfrak{W}$ is the coarsest or the finest uniformity in the following Proposition 2 and Proposition 3, where the notations keep the same meanings as in Proposition 1.

Proposition 2. *The following conditions are equivalent :*

1) *$\mathfrak{W}$ is the coarsest uniformity on $\mathfrak{F}$,*

2) *the topology $\mathfrak{T}$ on $\mathfrak{F}$ indnced by $\mathfrak{W}$ is the coarsest one,*

3) *$\mathfrak{S}$ consists of the empty set only or $\mathfrak{U}$ is the coarsest uniformity on Y.*

Proof. It is evident that 1) implies 2), and 3) implies 1). To show that 2) implies 3), suppose that $\mathfrak{S}$ has a non-empty set A. If there exists an entourage $U \in \mathfrak{U}$ different from $Y \times Y$, we can take two points p, q of Y such that $(p, q) \notin U$, then for any fixed point a of A we can define two functions $u, v \in \mathfrak{F}$ with value p, q at a respectively and the same value otherwise. Now by 2), we have $W(A, U)(u) = \mathfrak{F}$ and in particular $(u(a), v(a)) = (p, q) \in U$, which is a contradiction.

Proposition 3. *If the space Y contains more than one point, then the following conditions are equivalent :*

1) *$\mathfrak{W}$ is the finest uniformity,*

2) *the topology $\mathfrak{T}$ on $\mathfrak{F}$ induced by $\mathfrak{W}$ is the finest one,*

3) *$X \in \mathfrak{S}$ and $\mathfrak{U}$ is the finest uniformity on Y.*

If the space Y consists of only one point, both $\mathfrak{W}$ and $\mathfrak{U}$ consist of only one point, and of course they are both the finest and the coarsest uniformity.

Proof. It is evident that 1) implies 2), and 3) implies 1). We shall show that 2) implies 3). By 2), for any fixed $u_0 \in \mathfrak{F}$ there exist a set $A \in \mathfrak{S}$ and an entourage $U \in \mathfrak{U}$ such that

if $(u_0(x), u(x)) \in U$ $(u \in \mathfrak{F})$ for all $x \in A$, then $u = u_0$. (1)

The set A must coincide with the space X. In fact if a point a of X does not belong to A, we can choose a function $u \in \mathfrak{F}$ whose value at a is not $u_0(a)$ but coincides with that of u_0 otherwise. This is a contradiction.

The entourage U must coincide with the diagonal set in $Y\times Y$. In fact if p and q are distinct points such that $(p, q)\in U$, then for the constant maps u_0 and u with value p and q respectively, the relation (1) does not hold.

It is natural that the uniformity $\mathfrak{W}$ depends on both $\mathfrak{S}$ and $\mathfrak{U}$, and in general any uniformity on $\mathfrak{F}$ can not always be defined by giving only $\mathfrak{S}$ adequate properties (cf. Proposition 3), while most uniformities on $\mathfrak{F}$ in the literature have been defined so.

§ 2. Simultaneous continuity.

Notation 1. Let X be a topological space, Y be a uniform space endowed with a uniform structure $\mathfrak{U}$, $\mathfrak{S}$ be a u-family of subsets of X, and $\bar{\mathfrak{S}}$ be the family $\{\bar{A}\mid A\in\mathfrak{S}\}$. Let $\mathfrak{F}(X ; Y)$ be the family of all mappings of X into Y, $\mathfrak{C}(X ; Y)$ be the family of all continuous mappings of X into Y, and $\mathfrak{F}$ (resp. $\mathfrak{C}$) be any non-empty subfamily of $\mathfrak{F}(X ; Y)$ (resp. $\mathfrak{C}(X ; Y)$). Let $\mathfrak{W}$ (resp. $\bar{\mathfrak{W}}$) be the uniformity on $\mathfrak{F}(X ; Y)$ defined by $\mathfrak{S}$ (resp. $\bar{\mathfrak{S}}$) as in Proposition 1. These notations will keep these meanings throughout this section.

Definition 2. A uniformity on $\mathfrak{F}(X ; Y)$ is called a *uniformity which gives simultaneous continuity for* $\mathfrak{C}$ in brief an *s. c.-uniformity for* $\mathfrak{C}$, if for each $u_0\in\mathfrak{C}$ and each $x_0\in X$ the following condition holds : for any entourage U of Y there exist an entourage W of $\mathfrak{F}(X ; Y)$ and a neighborhood V of x_0 in X such that

$$\text{the relation “}u\in W(u_0)\text{ and }x\in V\text{” implies the relation “}u(x)\in U(u_0)(x_0))\text{”.} \tag{1}$$

A uniformity on $\mathfrak{F}(X ; Y)$ is called an *admissible uniformity which gives simultaneous continuity for* $\mathfrak{C}$ in brief an *a. s. c.-uniformity for* $\mathfrak{C}$, if the similar condition that "$u\in W(u_0)$" is replaced by "$u\in W(u_0)\cap\mathfrak{C}$" in (1), holds.

It is evident that an s. c.-uniformity for $\mathfrak{C}$ implies an a. s. c.-uniformity for $\mathfrak{C}$.

Theorem 2. *Let X, Y, $\mathfrak{U}$, $\mathfrak{S}$, $\mathfrak{W}$ and $\mathfrak{C}$ be the same as those in Notation 1. Let $\mathfrak{W}_c$ be the uniformity of compact convergence on $\mathfrak{F}(X ; Y)$. If $\mathfrak{U}$ is not the coarsest uniformity on Y, the following conditions are equivalent* :

1) *every point of X is interior to at least one set of $\mathfrak{S}$,*

2) *$\mathfrak{W}$ is an s. c.-uniformity for $\mathfrak{C}$.*

Moreover if X is locally compact, each of 1) *and* 2) *is equivalent to the following* :

3) *$\mathfrak{W}$ is finer than $\mathfrak{W}_c$.*

Proof. 1) implies 2) : (Dieudonné [3] had shown without the proof that 1) implies that $\mathfrak{W}$ is an a. s. c.-uniformity for $\mathfrak{C}(X ; Y)$ in our terminology). For the sake of completeness we give a proof. For each $x_0\in X$, choose a set $A\in\mathfrak{S}$ and a neighborhood V of x_0 in X such that $V\subset A$. For any entourage

$U \in \mathfrak{U}$, choose an entourage $U_1 \in \mathfrak{U}$ such that $U_1^2 \subset U$. For each $u_0 \in \mathfrak{C}$, there exists a neighborhood V_1 of x_0 in X such that $u_0(V_1) \subset U_1(u_0(x_0))$. If $x \in V \cap V_1$ and $u \in W(A, U_1)(u_0)$, then $u(x) \in U(u_0(x_0))$.

2) implies 1) : Since $\mathfrak{U}$ is not the coarsest uniformity, there exist an entourage $U' \neq Y \times Y$ and a symmetric entourage U_0 such that $U_0^2 \subset U'$. If $U_0(y_0) = Y$ for some $y_0 \in Y$, then $Y \times Y \subset U'$, which is a contradiction. Therefore $U_0(y) \neq Y$ for any $y \in Y$.

Suppose that there exists a point x_0 which is not interior to any set belonging to $\mathfrak{S}$. Take a mapping $u_0 \in \mathfrak{C}$ and a point p such that $p \notin U_0(u_0(x_0))$. By 2), there exist a neighborhood V of x_0 in X, a set $A_1 \in \mathfrak{S}$ and an entourage $U_1 \in \mathfrak{U}$ such that "$x \in V$ and $u \in W(A_1, U_1)(u_0)$" implies "$u(x) \in U_0(u_0(x_0))$". Take a point x_1 of $V - A_1$ and let u_1 be a mapping belonging to $\mathfrak{F}(X; Y)$ whose value is p at x_1 and the same as u_0 on A_1. Then $u_1 \in W(A_1, U_1)(u_0)$ but $u_1(x_1) \notin U_0(u_0(x_0))$. This is a contradiction.

The remaining part of our proof is similar to the proof of Arens' theorem ([1], p. 482) for the compact-open topology.

1) implies 3) : Let K be any compact subset of X and U be any entourage of Y. For each $x \in K$ take a set $A(x) \in \mathfrak{S}$ which has the point x in its interior. From the covering $\{A(x) \mid x \in K\}$ of K we can choose a finite subcovering, say $\{A(x_i) \mid x_i \in K, i=1, 2, \cdots, n\}$. Take a set $A \in \mathfrak{S}$ such that $A(x_1) \cup A(x_2) \cup \cdots \cup A(x_n) \subset A$, then $W(K, U) \supset W(A, U)$.

3) implies 2) : (The local compactness of X is used only in this case.) Let u_0 be any mapping of $\mathfrak{C}$ and x_0 be any point of X. For any entourage $U \in \mathfrak{U}$, there is an entourage $U_1 \in \mathfrak{U}$ such that $U_1^2 \subset U$. Take a compact neighborhood V of x_0 in X such that $u_0(V) \subset U_1(u_0(x_0))$. By 3), there exist a set $A \in \mathfrak{S}$ and an entourage $U_2 \in \mathfrak{U}$ such that $W(V, U_1) \supset W(A, U_2)$. Then for $u \in W(A, U_2)(u_0)$ and $x \in V$, we have $u(x) \in U(u_0(x_0))$.

Remarks. In the case where $\mathfrak{U}$ is the coarsest uniformity, both $\mathfrak{W}$ and $\mathfrak{W}_c$ are the coarsest uniformity, the conditions 2) and 3) hold trivially, and the condition 1) is independent of them.

Lemma 1. *Uniformities $\mathfrak{W}$ and $\overline{\mathfrak{W}}$ coincide on $\mathfrak{C} \times \mathfrak{C}$.*

Proof. The similar proof as in Bourbaki [2], Proposition 6, p. 280, is valid.

Theorem 3. *If X is a uniformizable topological space and Y a uniform space which contains a non-degenerate arc, then the following conditions are equivalent:*

1) *every point of X is interior to at least one set of $\overline{\mathfrak{S}}$,*

2) *$\overline{\mathfrak{W}}$ is an s. c.-uniformity for $\mathfrak{C}(X; Y)$,*

3) *$\overline{\mathfrak{W}}$ is an a. s. c.-uniformity for $\mathfrak{C}(X; Y)$,*

4) *$\mathfrak{W}$ is an a. s. c.-uniformity for $\mathfrak{C}(X; Y)$.*

Proof. By Theorem 2, 1) and 2) are equivalent. It is evident that 2) implies 3). By Lemma 1, 3) and 4) are equivalent. We shall prove that 3) implies 1).

For any two distinct points p and q of a non-degenerate arc in Y, there is an entourage $U \in \mathfrak{U}$ such that $q \notin U(p)$. Let u_0 be the constant mapping of X into $\{p\}$. Suppose that a point a of X exists that is not interior to any set of $\bar{\mathfrak{S}}$. Since $u_0 \in \mathfrak{C}(X; Y)$, there exist a neighborhood V of a in X, a set $A \in \mathfrak{S}$ and an entourage $U_1 \in \mathfrak{U}$ such that

the relations "$x \in V$ and $u \in W(\bar{A}, U_1)(u_0) \cap \mathfrak{C}(X; Y)$" imply the relation "$u(x) \in U(p)$".

Take a point b of $V - \bar{A}$. Since X is uniformizable (cf. [2], p. 144), there exists a real-valued continuous function f defined on X such that

$f(\bar{A}) = 0$, $f(b) = 1$ and $0 \leq f(x) \leq 1$ for every $x \in X$.

Moreover there exists a continuous mapping g of the closed interval $[0, 1]$ into Y such that

$g(0) = p$ and $g(1) = q$.

Let u be the composition $g \circ f$. Then

$b \in V$ and $u \in W(\bar{A}, U_1)(u_0) \cap \mathfrak{C}(X; Y)$,

but $u(b) \notin U(p)$, which is a contradiction.

Corollary. *If $\mathfrak{S}$ consists of closed subsets of X, then $\mathfrak{W}$ is an a.s.c.-uniformity for $\mathfrak{C}(X; Y)$ if and only if it is an s.c.-uniformity for $\mathfrak{C}(X; Y)$.*

Remarks. When we replace $\mathfrak{C}(X; Y)$ by $\mathfrak{C}$, Theorem 3 and its corollary don't hold in general. In fact, the uniformity $\mathfrak{W}_p$ of pointwise convergence is an a. s. c.-uniformity for an equicontinuous family $\mathfrak{C}_e$ of mappings of X into Y (cf. Bourbaki [2], p. 286, Corollary 4), and all finite subsets of X are closed if X is a T_1-space. On the other hand, $\mathfrak{W}_p$ is not an s. c.-uniformity for $\mathfrak{C}_e$ by Theorem 2 if X is not a discrete space and $\mathfrak{U}$ is not the coarsest uniformity.

§ 3. Sufficient conditions for the continuity of *uv*.

Notation 2. Let X be a uniform space endowed with a uniform structure $\mathfrak{U}$, $\mathfrak{S}$ be a u-family of subsets of X, and $\bar{\mathfrak{S}}$ be the family $\{\bar{A} \mid A \in \mathfrak{S}\}$. Let $\mathfrak{F}(X)$ be the family of all mappings of X into itself, $\mathfrak{C}(X)$ be the family of all continuous mappings of X into itself, and $\mathfrak{F}$ (resp. $\mathfrak{C}$) be any non-empty subfamily of $\mathfrak{F}(X)$ (resp. $\mathfrak{C}(X)$). Let $\mathfrak{W}$ (resp. $\bar{\mathfrak{W}}$) be the uniformity on $\mathfrak{F}(X)$ defined by $\mathfrak{S}$ (resp. $\bar{\mathfrak{S}}$) as in Proposition 1. For any two mappings u, v belonging to $\mathfrak{F}(X)$, uv will always be the composite mapping $x \rightarrow u(v(x))$ $(x \in X)$. These notations will keep the meanings hereafter throughout the paper.

Definition 3. Let $\mathfrak{F}$ be a non-empty subfamily of $\mathfrak{F}(X)$. A uniformity $\mathfrak{W}$ on $\mathfrak{F}(X)$ is called a *p.-uniformity for* $\mathfrak{F}$, if the mapping $(u, v) \rightarrow uv$ is continuous at every point $(u_0, v_0) \in \mathfrak{F} \times \mathfrak{F}$ with respect to the topologies on $\mathfrak{F}(X) \times \mathfrak{F}(X)$ that are induced by the uniformity $\mathfrak{W}$. Let $\mathfrak{F}'$ be another non-empty subfamily of $\mathfrak{F}(X)$ which is closed under the function composition. A uniformity $\mathfrak{W}$ on $\mathfrak{F}(X)$ is called an *a. p.-uniformity for* $\mathfrak{F}'$, if the mapping $(u, v) \rightarrow uv$ is continuous at every point $(u_0, v_0) \in \mathfrak{F}' \times \mathfrak{F}'$ with respect to the relative topologies on $\mathfrak{F}' \times \mathfrak{F}'$ and $\mathfrak{F}'$ that are induced by the uniformity $\mathfrak{W}$.

It is evident that a p.-uniformity for $\mathfrak{F}'$ implies an a. p.-uniformity for $\mathfrak{F}$.

Lemma 2 (Dieudonné [3]). *Let u_0 and v_0 be two fixed mappings belonging to $\mathfrak{F}(X)$. If, for an arbitrary set $A \in \mathfrak{S}$, there exist a set $B \in \mathfrak{S}$ and an entourage $U \in \mathfrak{U}$ such that :*

1) *$U(v_0(A)) \subset B$, and* 2) *u_0 is uniformly continuous on B,*

then the mapping $(u, v) \rightarrow uv$ of $\mathfrak{F}(X) \times \mathfrak{F}(X)$ into $\mathfrak{F}(X)$ in continuous at (u_0, v_0) with respect to the topologies induced by the uniformity $\mathfrak{W}$.

From this Lemma the following Proposition 4 follows directly.

Proposition 4. *Let $\mathfrak{F}$ be any non-empty subfamily of $\mathfrak{F}(X)$. If*

1) *for each mapping $u \in \mathfrak{F}$ and each set $A \in \mathfrak{S}$, there exists a set $B \in \mathfrak{S}$ such that $u(A) \subset B$,*

2) *for each set $A \in \mathfrak{S}$, there exist an entourage $U \in \mathfrak{U}$ and a set $C \in \mathfrak{S}$ such that $U(A) \subset C$, and*

3) *each mapping $u \in \mathfrak{F}$ is uniformly continuous on every set $A \in \mathfrak{S}$,*

then the uniformity $\mathfrak{W}$ is a p.-uniformity for $\mathfrak{F}$.

Remarks. i) If the identity mapping of X is contained in $\mathfrak{F}$, the unified condition of 1) and 2) in Proposition 4 is equivalent to the condition 1) for any $v_0 \in \mathfrak{F}$ in Lemma 2.

ii) If the condition (closely related to the fact that $\mathfrak{W}$ is Hausdorff —— Bourbaki [2], p. 318)

4) X is covered by $\mathfrak{S}$,

is combined with the conditions 1), 2), and 3) in Proposition 4, it is easily seen that

a) the condition 2) implies that $\overline{\mathfrak{W}}$ coincides with $\mathfrak{W}$,

b) the conditions 2) and 4) imply that every point of X is interior to at least one set of $\mathfrak{S}$, and

c) the conditions 2), 3), and 4) imply that $\mathfrak{F} \subset \mathfrak{C}(X)$.

Proposition 5. *Let $\mathfrak{C}$ be any non-empty subfamily of $\mathfrak{C}(X)$ which is closed under the function composition. If*

1)′ *for each mapping $u \in \mathfrak{C}$ and each set $A \in \mathfrak{S}$, there exists a set $B \in \mathfrak{S}$ such that $u(A) \subset \bar{B}$,*

2)′ *for each set $A \in \mathfrak{S}$, there exist an entourage $U \in \mathfrak{U}$ and a set $C \in \mathfrak{S}$ such that $U(A) \subset \bar{C}$, and*

3)$'$ *every mapping* $u\in\mathfrak{C}$ *is uniformly continuous on* $\bar{A}$ *for every set* $A\in\mathfrak{S}$, *then the uniformity* $\mathfrak{W}$ *is an a.p.-uniformity for* $\mathfrak{C}$.

Proof. By conditions 1)$'$, 2)$'$, and 3)$'$, the conditions 1), 2), and 3) in Proposition 4 are valid for $\bar{\mathfrak{S}}$ and $\mathfrak{C}$ in place of $\mathfrak{S}$ and $\mathfrak{F}$ respectively. So $\mathfrak{W}$ is an a. p.-uniformity for $\mathfrak{C}$ by Proposition 4 and Lemma 1.

Remarks. It is easy to see that "the conditions 1), 2), and 3) for $\mathfrak{F}=\mathfrak{C}$ in Proposition 4 "imply" the conditions 1)$'$, 2)$'$, and 3)$'$".

§ 4. Necessary conditions for the continuity of uv —— p.-uniformity.

In this section we shall show that the conditions in Proposition 4 are also necessary conditions in a natural sense.

Lemma 3. *Let* $\mathfrak{F}$ *be any subfamily of* $\mathfrak{F}(X)$ *which contains the identity mapping of* X, *and* $\mathfrak{U}$ *be a non-coarsest uniformity on* X. *If* $\mathfrak{W}$ *is a p.-uniformity for* $\mathfrak{F}$, *then the conditions* 1), 2), *and* 3) *in Proposition* 4 *hold.*

Proof. Since $\mathfrak{U}$ is a non-coarsest uniformity, there exists an entourage $U_1\in\mathfrak{U}$ such that $U_1^2\neq X\times X$ and $U_1^{-1}=U_1$. Then

$U_1(x)\neq X$ for any $x\in X$. (1)

Let u_1 be any fixed mapping belonging to $\mathfrak{F}$, and v_1 be the identity mapping of X. Since $\mathfrak{W}$ is a p.-uniformity for $\mathfrak{F}$, for u_0, $v_0\in\mathfrak{F}$, $A_0\in\mathfrak{S}$, and $U_0\in\mathfrak{U}$ that are given arbitrarily, there exist B_1, $B_2\in\mathfrak{S}$ and V_1, $V_2\in\mathfrak{U}$ such that

if u, $v\in\mathfrak{F}(X)$, $(u_0, u)\in W(B_1, V_1)$ and $(v_0, v)\in W(B_2, V_2)$, then (u_0v_0, uv) $\in W(A_0, U_0)$. (2)

We shall consider three cases for u_0, v_0, A_0, and U_0.

Case a). Suppose that there exist a mapping $v_2\in\mathfrak{F}$ and a set $A_1\in\mathfrak{S}$ such that

$v_2(A_1)\not\subset A$ for any set $A\in\mathfrak{S}$. (3)

For $u_0=u_1$, $v_0=v_2$, $A_0=A_1$, and $U_0=U_1$, choose B_1, $B_2\in\mathfrak{S}$ and V_1, $V_2\in\mathfrak{U}$ such that the relation (2) holds. Take a Point $p\in v_2(A_1)-B_1$ (cf. (3)), a point $q\in v_2^{-1}(p)\cap A_1$, and a point $r\notin U_1(u_1(p))$ (cf. (1)). There exist two mappings u, $v\in\mathfrak{F}(X)$ such that

$u(x)=u_1(x)$ for $x\in B_1$, $u(p)=r$; $v(x)=v_2(x)$ for $x\in B_2\cup\{q\}$.

Then $(u_1(v_2(q)), u(v(q)))\notin U_1$, but the premise of (2) is satisfied. This is a contradiction.

Case b). Suppose that there exists a set $A_2\in\mathfrak{S}$ such that

$U(A_2)\not\subset B$ for any $U\in\mathfrak{U}$ and any $B\in\mathfrak{S}$. (4)

For $u_0=u_1$, $v_0=v_1$, $A_0=A_2$, and $U_0=U_1$, choose B_1, $B_2\in\mathfrak{S}$ and V_1, $V_2\in\mathfrak{U}$ such that the relation (2) holds. Take a point $p\in V_2(A_2)-B_1$ (cf.(4)), a point $q\in A_2$ such that $(q, p)\in V_2$, and a point $r\notin U_1(u_1(q))$ (cf. (1)). There exist two mappings u, $v\in\mathfrak{F}(X)$ such that

$u(x)=u_1(x)$ for $x\in B_1$, $u(p)=r$; $v(x)=v_1(x)$ for $x\in B_2-\{q\}$, $v(q)=p$. Then $(u_1(v_1(q)), u(v(q)))\notin U_1$, which is a contradiction.

Case c). Suppose that a mapping $u_2\in\mathfrak{F}$ is not uniformly continuous on a set $A_3\in\mathfrak{S}$, i. e. there exists an entourage $U_2\in\mathfrak{U}$ such that for each entourage $U_\alpha\in\mathfrak{U}$ there are two points x_α, $y_\alpha\in A_3$ as follows :

$$(x_\alpha, y_\alpha)\in U_\alpha \text{ and } (u_2(x_\alpha), u_2(y_\alpha))\notin U_2. \tag{5}$$

For $u_0=u_2$, $v_0=v_1$, $A_0=A_3$, and $U_0=U_2$, choose B_1, $B_2\in\mathfrak{S}$ and V_1, $V_2\in\mathfrak{U}$ such that the relation (2) holds and $B_1\supset A_3$. For $U_\alpha=V_2$, let x^*, y^* be the corresponding x_α, y_α in (5). There exist two mappings u, $v\in\mathfrak{F}(X)$ such that

$u(x)=u_2(x)$ for $x\in B_1$; $v(x)=v_1(x)=x$ for $x\in B_2-\{x^*\}$, $v(x^*)=y^*$. Then $(u_2(v_1(x^*)), u(v(x^*)))\notin U_2$ which is a contradiction.

From Proposition 4 and Lemma 3, we have the following theorem.

Theorem 4. *Let X be a uniform space endowed with a non-coarsest uniform structure $\mathfrak{U}$, and $\mathfrak{F}(X)$, $\mathfrak{S}$, and $\mathfrak{W}$ be the same as those in Notation 2. Let $\mathfrak{F}$ be any subfamily of $\mathfrak{F}(X)$ which contains the identity mapping of X. Then $\mathfrak{W}$ is a p.-uniformity for $\mathfrak{F}$ if and only if the following three conditions hold :*

1) *for each mapping $u\in\mathfrak{F}$ and each set $A\in\mathfrak{S}$, there exists a set $B\in\mathfrak{S}$ such that $u(A)\subset B$,*

2) *for each set $A\in\mathfrak{S}$, there exist an entourage $U\in\mathfrak{U}$ and a set $C\in\mathfrak{S}$ such that $U(A)\subset C$, and*

3) *each mapping $u\in\mathfrak{F}$ is uniformly continuous on every set $A\in\mathfrak{S}$.*

§ 5. Necessary conditions for the continuity of uv—a. p.-uniformity.

In this section we shall show that the three conditions in Proposition 5 are also necessary conditions in a sense.

Lemma 4. *Let X be a uniform space that contains a non-degenerate arc, and $\mathfrak{C}(X)$ be the family of all continuous mappings of X into itself. If $\mathfrak{W}$ is an a. p.-uniformity for $\mathfrak{C}(X)$, then, for each mapping $u\in\mathfrak{C}(X)$ and each set $A\in\mathfrak{S}$, there exists a set $B\in\mathfrak{S}$ such that $u(A)\subset\bar{B}$.*

Proof. Take two distinct points p and q on a non-degenerate arc in X, and an entourage $U_1\in\mathfrak{U}$ such that $(p, q)\notin U_1$. Let u_1 be the constant mapping of X into p. Now suppose that there exist a mapping $v_1\in\mathfrak{C}(X)$ and a set $A_1\in\mathfrak{S}$ such that $v_1(A_1)\not\subset\bar{A}$ for any set $A\in\mathfrak{S}$. Since $\mathfrak{W}$ is an a. p.-uniformity for $\mathfrak{C}(X)$, for u_1, v_1, A_1, and U_1, there exist A_2, $A_3\in\mathfrak{S}$ and U_2, $U_3\in\mathfrak{U}$ such that

if $u\in W(A_2, U_2)(u_1)\cap\mathfrak{C}(X)$ and $v\in W(A_3, U_3)(v_1)\cap\mathfrak{C}(X)$, then $uv\in W(A_1, U_1)(u_1v_1)\cap\mathfrak{C}(X)$.

Take a point $r\in v_1(A_1)-A_2$ and a point $s\in A_1$ such that $r=v_1(s)$. Since X is uniformizable, there is a continuous mapping f of X into the closed interval $[0, 1]$ such that $f(\bar{A}_2)=0$ and $f(r)=1$ (cf. Bourbaki [2], p. 144). Moreover there is a contiunous mapping g of the closed interval [0, 1] onto an arc $\widehat{pq}$ such that $g(0)=p$ and $g(1)=q$. Then for $u=g\circ f$ and $v=v_1$, we have $(u_1(v_1(s)), u(v(s)))=(p, q)\notin U_1$, which is a contradiction.

For the proof of the remaining two conditions in Proposition 5, we need an auxiliary concept "uniform deformability" as follows.

Definition 4. A uniform space X with a uniformity $\mathfrak{U}$ is *uniformly deformable* if for any entourage $U\in\mathfrak{U}$ there exists an entourage $U^*\in\mathfrak{U}$ as follows : for any two U^*-close points p and q, there exists a continuous mapping f of X into itself such that $f(p)=q$ and $(x, f(x))\in U$ for any $x\in X$.

L. R. Ford, jr. [4] defined the similar notion "strong local homogeneity" which is stronger than ours in those points that f must be a homeomorphism and fixes the complement of a neighborhood of x, while weaker than ours since the uniform scale of such neighborhoods is not required. There are several examples common to his and ours.

Examples. The following spaces i), ii),$\cdots$, v) are uniformly deformable uniform spaces, whereas the space vi) is a manifold that is not uniformly deformable : i) locally euclidean, uniformly locally connected, uniform spaces, ii) locally euclidean, compact, uniform spaces, iii) convex subsets of a normed space, iv) the set of all rational points in a euclidean space, v) discrete uniform spaces, and vi) the set of all points (x, y) in the euclidean plane such that $(x^2+1)y^2>x$. (The uniformities of iii), iv), and vi) are the usual ones.)

Now we prove a lemma for Theorem 5 in the next page.

Lemma 5. *Let X be a uniformly deformable space that contains a non-degenerate arc, and $\mathfrak{C}(X)$ be the family of all continuous mappings of X into itself. If $\mathfrak{W}$ is an a. p.-uniformity for $\mathfrak{C}(X)$, then conditions* 2)′ *and* 3)′ *in Proposition* 5 *hold for* $\mathfrak{C}(X)$.

Proof. Since $\mathfrak{W}$ and $\bar{\mathfrak{W}}$ coincide on $\mathfrak{C}(X)\times\mathfrak{C}(X)$ by Lamme 1, $\bar{\mathfrak{W}}$ is an a. p.-uniformity for $\mathfrak{C}(X)$. Hence, for u_0, $v_0\in\mathfrak{C}(X)$, $A_0\in\mathfrak{S}$, and $U_0\in\mathfrak{U}$ that are given arbitrarily, there exist A_1, $A_2\in\mathfrak{S}$ and U_1, $U_2\in\mathfrak{U}$ such that

$$\text{if } u\in W(\bar{A}_1, U_1)(u_0)\cap\mathfrak{C}(X) \text{ and } v\in W(\bar{A}_2, U_2)(v_0)\cap\mathfrak{C}(X), \text{ then } uv\in W(\bar{A}_0, U_0)(u_0v_0)\cap\mathfrak{C}(X). \tag{1}$$

Since X is uniformly deformable, we can choose an entourage $U_3\in\mathfrak{U}$ as follows : for any U_3-close two points x^* and y^*, there is a mapping $v\in\mathfrak{C}(X)$ such that

$$1)\ v(x^*)=y^*, \text{ and } 2)\ v(x) \text{ is } U_2\text{-close to } x \text{ for any } x\in X. \tag{2}$$

Let e be the identity mapping of X. We shall consider two cases for u_0, v_0, A_0, and U_0.

Case a). Let $\widehat{pq}$ be any non-degenerate arc in X, and choose an entourage $V\in\mathfrak{U}$ such that $(p, q)\notin V$. Let k be the constant mapping of X into p. Now suppose that there exists a set $B \in \mathfrak{S}$ such that

$U(B) \not\subset \bar{A}$ for any $U \in \mathfrak{U}$ and any $A \in \mathfrak{S}$. (3)

For $u_0 = k$, $v_0 = e$, $A_0 = B$, and $U_0 = V$, choose A_1, $A_2 \in \mathfrak{S}$ and $U_1, U_2 \in \mathfrak{U}$ such that the relation (1) holds. For U_2, choose an entourage $U_3 \in \mathfrak{U}$ such that (2) holds. Take a point $r \in U_3(\bar{B}) - \bar{A}_1$ (cf. (3)), and a point $s \in B$ such that $(s, r) \in U_3$. By (2), we can choose a mapping $v \in \mathfrak{C}(X)$ for $x^* = s$ and $y^* = r$. Since X is uniformizable, there exists a mapping $u \in \mathfrak{C}(X)$ such that $u(\bar{A}_1) = p$ and $u(r) = q$. Then $(k(e(s)), u(v(s))) = (p, q) \notin V$, which is a contradiction (cf. (1)).

Case b). Suppose that a mapping $f \in \mathfrak{C}(X)$ is not uniformly continuous on a set $\bar{B} \in \mathfrak{S}$, i. e., there exists an entourage $V \in \mathfrak{U}$ such that for each entourage $U_\alpha \in \mathfrak{U}$ there are two points x_α, $y_\alpha \in \bar{B}$ as follows : $(x_\alpha, y_\alpha) \in U_\alpha$ and $(f(x_\alpha), f(y_\alpha)) \notin V$. For $u_0 = f$, $v_0 = e$, $A_0 = B$, and $U_0 = V$, choose $A_1, A_2 \in \mathfrak{S}$ and U_1, $U_2 \in \mathfrak{U}$ such that the relation (1) holds. For U_2, choose an entourage $U_3 \in \mathfrak{U}$ such that (2) holds. Moreover we can choose two points x^* and y^* in $\bar{B}$ such that $(x^*, y^*) \in U_3$ and $(f(x^*), f(y^*)) \notin V$. For these U_3-close points x^* and y^*, there is a mapping $v \in \mathfrak{C}(X)$ such that (2) holds. Then $(f(e(x^*)), f(v(x^*))) = (f(x^*), f(y^*)) \notin V$, which is a contradiction (cf. (1)).

By Proposition 5, Lemma 4 and Lemma 5, we have

Theorem 5. *Let X be a uniformly deformable space endowed with a uniform structure $\mathfrak{U}$ which contains a non-degenerate arc, and $\mathfrak{C}(X)$ be the family of all continuous mapping of X into itself. Then $\mathfrak{W}$ is an a. p.-uniformity for $\mathfrak{C}(X)$ if and only if the following three conditions hold :*

1) *for each mapping $u \in \mathfrak{C}(X)$ and each set $A \in \mathfrak{S}$, there exists a set $B \in \mathfrak{S}$ such that $u(A) \subset \bar{B}$,*

2) *for each set $A \in \mathfrak{S}$, there exist an entourage $U \in \mathfrak{S}$ and a set $C \in \mathfrak{S}$ such that $U(A) \subset \bar{C}$, and*

3) *every mapping $u \in \mathfrak{C}(X)$ is uniformly continuous on $\bar{A}$ for any set $A \in \mathfrak{S}$.*

Corollary 1. *Let X, $\mathfrak{U}$, and $\mathfrak{C}(X)$ be the same as those in Theorem 5. The following two conditions* i) *and* ii) *are equivalent :*

i) *$\mathfrak{W}$ is an a. p.-uniformity for $\mathfrak{C}(X)$,*

ii) *$\bar{\mathfrak{W}}$ is a p.-uniformity for $\mathfrak{C}(X)$.*

Proof. The condition i) implies conditions 1), 2), and 3) in Theorem 5. Now in 1), $u(A) \subset \bar{B}$ implies $u(\bar{A}) \subset \bar{B}$ by the continuity of u, and in 2), if we take an entourage $V \in \mathfrak{U}$ such that $V^2 \subset U$, we have $V(\bar{A}) \subset \bar{C}$. Hence the condition ii) holds by Proposition 4. Conversely, the condition ii) implies that $\bar{\mathfrak{W}}$ is an a. p.-uniformity, and so we have i) by Lemma 1.

Corollary 2. *Let X, $\mathfrak{U}$ and $\mathfrak{C}(X)$ be the same as those in Theorem 5. If all sets in $\mathfrak{S}$ are closed, then $\mathfrak{W}$ is an a. p.-uniformity for $\mathfrak{C}(X)$ if and only if it is a p.-uniformity for $\mathfrak{C}(X)$.*

§ 6. Uniformities which satisfy our continuity conditions.

In conclusion, we shall summarize several results in this paper in a form. To what extent $\mathfrak{S}$ and $\mathfrak{U}$ are restricted by the following conditions imposed on $\mathfrak{W}$?——

1) $\mathfrak{W}$ is a uniformity that is defined directly by $\mathfrak{S}$ and $\mathfrak{U}$ only, and coincides with the usual one,

2) $\mathfrak{W}$ ensures the joint continuity and the continuity of the product operation for a semi-group $\mathfrak{C}$ of continuous transformations of a uniform space X.

In this point of view, it seems to us from our results that the following five conditions are to be satisfied :

1) for any two sets A, $B\in\mathfrak{S}$ there exists a set $C\in\mathfrak{S}$ such that $A\cup B\subset C$,

2) $\bar{\mathfrak{S}}$ covers X,

3) for each mapping $u\in\mathfrak{C}$ and each set $A\in\mathfrak{S}$, there exists a set $B\in\mathfrak{S}$ such that $u(A)\subset\bar{B}$,

4) for each set $A\in\mathfrak{S}$, there exist an entourage $U\in\mathfrak{U}$ and a set $C\in\mathfrak{S}$ such that $U(A)\subset\bar{C}$,

5) every mapping $u\in\mathfrak{C}$ is uniformly continuous on $\bar{A}$ for every set $A\in\mathfrak{S}$.

We have examined these conditions in various cases in each of which $\mathfrak{S}$ is a family of all subsets of a euclidean space that have a particular topological property. After the trial, we have conjectured that if X is a euclidean space and $\mathfrak{C}$ is $\mathfrak{C}(X)$, then $\mathfrak{W}$ must be the uniformity of compact convergence. In fact it is affirmative in more general cases where X is either any locally euclidean, uniformly locally connected, metric space or any convex subset of a normed space (cf. Karube [5]). The fact together with many results on the compact-open topology shows that the compact-open topology for $\mathfrak{C}$ is the most natural set-open topology (cf. e. g., Kelley [6], p. 230).

References

[1] Arens, R.: A topology for spaces of transformations. *Ann. Math.* **47**, 480–495 (1946).

[2] Bourbaki, N. : General Topology, Part 2, translated from French. Paris-Reading-Palo Alto-London-Don Mills : Hermann and Addison-Wesley 1966.

[3] Dieudonné, J. : On topological groups of homeomorphisms. *Amer. J. Math.* **70**, 659–680 (1948).

[4] Ford, L. R. jr.: Homeomorphism groups and coset spaces. *Trans. Amer. Math. Soc.* **77**, 490–497 (1954).

[5] Karube, T. : On a kind of uniqueness of set-entourage uniformities for function spaces. *Proc. Japan Acad.* **44**, 204–206 (1968).

[6] Kelley, J. L.: General Topology. Toronto-New York-London : D. van Nostrand 1955.