Mem. Gra. Sci. Eng. Shimane Univ.
Series B: Mathematics
51 (2018), pp. 1–5

GEOMETRY OF GEODESIC SPHERES IN A COMPLEX PROJECTIVE SPACE IN TERMS OF THEIR GEODESICS

SADAHIRO MAEDA

Communicated by Toshihiro Nakanishi

(Received: March 9, 2017)

Abstract. This paper is the survey of joint works with K. Ogiue ([7]) and B.H. Kim, I.B. Kim ([5]). Geodesic spheres $G(r)$ are fundamental examples of (real) hypersurfaces in a Riemannian manifold. In this paper, as an ambient space we take an n–dimensional complex projective space $\mathbb{C}P^n(c)$, $n \geqq 2$ of constant holomorphic sectional curvature $c(>0)$. By observing geodesics on $G(r)$ in $\mathbb{C}P^n(c)$ we characterize all $G(r)\,(0 < r < \pi/\sqrt{c}\,)$ (Theorems 1 and 2) and some $G(r)$ which are called Berger spheres (Theorem 3).

1. Introduction

In differential geometry, there are two geometries. One is intrinsic geometry and the other is extrinsic geometry (submanifold geometry). Among real hypersurfaces M^{2n-1} isometrically immersed into $\mathbb{C}P^n(c)\,, n \geqq 2$, we pay particular attention to geodesic spheres $G(r)$ of radius $r\,(0 < r < \pi/\sqrt{c}\,)$. The class of geodesic spheres $G(r)$ in $\mathbb{C}P^n(c)$ is an abundant class which gives fruitful results in both of intrinsic geometry and extrinsic geometry.

In intrinsic geometry, we recall a well-known result due to W. Klingenberg : Let M be an even dimensional compact simply connected Riemannian manifold having the sectional curvature K with $0 < K \leqq L$ on M, where L is a constant. Then the length ℓ of every closed geodesic on M satisfies $\ell \geqq 2\pi/\sqrt{L}$ (see [4]). However the odd dimensional version of the above result does *not* hold (cf. [11]). Indeed, let $G(r)$ be a $(2n-1)$-dimensional geodesic sphere of radius $r\,(0 < r < \pi/\sqrt{c}\,)$ with $\tan^2(\sqrt{c}\,r/2) > 2$ in $\mathbb{C}P^n(c)$. Then in this case there exists a closed geodesic on $G(r)$ whose length is shorter than $2\pi/\sqrt{L}$, where L is the maximal sectional curvature of $G(r)$. The sectional curvature K of every geodesic sphere $G(r)$ of radius $r\,(0 < r < \pi/\sqrt{c})$ satisfies sharp inequalities $0 < (c/4)\cot^2(\sqrt{c}\,r/2) \leqq K \leqq c + (c/4)\cot^2(\sqrt{c}\,r/2)(= L)$ at its each point. In this paper, geodesic spheres of

2010 *Mathematics Subject Classification.* 53B25, 53C40.

Key words and phrases. geodesic spheres, complex projective space, geodesics, circles, Berger spheres, Sasakian space forms.

radius $r\,(0 < r < \pi/\sqrt{c}\,)$ with $\tan^2(\sqrt{c}\,r/2) > 2$ in $\mathbb{C}P^n(c)$, $n \geqq 2$ are called *Berger spheres*.

In extrinsic geometry, we can say that all $G(r)$ are the simplest examples of real hypersurfaces in $\mathbb{C}P^n(c)$. It is known that $\mathbb{C}P^n(c)$ does not have a totally umbilic real hypersurface and T. Cecil and P. Ryan ([3]) show that every geodesic sphere is the only example of real hypersurfaces in $\mathbb{C}P^n(c)\,(n \geqq 3)$ with at most two distinct principal curvatures at its each point. This means that every $G(r)$ is the *simplest* real hypersurface of $\mathbb{C}P^n(c)$.

The purpose of this paper is to characterize geodesic spheres in $\mathbb{C}P^n(c)$ in extrinsic geometry (see Theorems 1, 2 and 3) and clarify geometric properties of such hypersrfaces from the viewpoint of intrinsic geometry (see the last section).

This paper is related to a talk of the author in 2017 (October 6 $\sim$ 8) International Conference on Differential Geometry and Applications, Mykonos Island, Greece.

We adopt usual notations and terminologies in the theory of real hypersurfaces in $\mathbb{C}P^n(c)$ (for examples, see [8, 9]).

2. circles in Riemannian geometry

The key word in our paper is a circle. We recall the definition of circles in Riemannian geometry.

Let $\gamma = \gamma(s)$ be a smooth real curve parametrized by its arclength s on a Riemannian manifold M with Riemmanian connection ∇. If the curve γ satisfies the following ordinary differential equations with some constant $k(\geqq 0)$:

$$\nabla_{\dot{\gamma}}\dot{\gamma} = kY_s \quad \text{and} \quad \nabla_{\dot{\gamma}}Y_s = -k\dot{\gamma},$$

where $\nabla_{\dot{\gamma}}$ is the covariant differentiation along γ with respect to the Riemmanian connection ∇ of M and Y_s is the so-called unit principal normal vector of γ, we call a circle of curvature k on M. We may regard a geodesic of null curvature. By virtue of the existence and uniqueness theorem on solutions to ordinary differential equations we can see that at any point p of a Riemannian manifold M for each positive constant k and every pair $\{X, Y\}$ of orthonormal vectors $X, Y(\in T_pM)$ there exists locally the unique circle γ of curvature k on M with initial condition that $\gamma(0) = p, \dot{\gamma}(0) = X$ and $Y_0 = Y$.

3. Main results

Since $\mathbb{C}P^n(c)$ does not admit totally umbilic real hypersurfaces, there does not exist real hypersurfaces *all* of whose geodesics are mapped to circles in this ambient space (See [10] and Proposition 2 in [6]). So it is natural to investigate real hypersurfaces *some* of whose geodesics are mapped to circles in $\mathbb{C}P^n(c)$.

The following three theorems are established from this viewpoint. Theorem 3.3 is an immediate consequence of Theorem 3.2.

Theorem 3.1 ([7]). *Let M^{2n-1} be a real hypersurface of $\mathbb{C}P^n(c)$, $n \geqq 2$ through an isometric immersion. Then M is locally congruent to a geodesic sphere $G(r)\,(0 < r < \pi/\sqrt{c}\,)$ with respect to the full isometry group $SU(n+1)$ of the ambient space $\mathbb{C}P^n(c)$ if and only if there exist such orthonormal vectors $v_1, v_2, \dots, v_{2n-2}$*

orthogonal to the characteristic vector ξ_p *at each point* p *of* M *that all geodesics of* M *through* p *in the direction* $v_i + v_j\,(1 \leqq i \leqq j \leqq 2n-2)$ *are circles of positive curvature in* $\mathbb{C}P^n(c)$.

Theorem 3.2 ([7]). *Let* M^{2n-1} *be a real hypersurface of* $\mathbb{C}P^n(c)$, $n \geqq 2$. *Then* M *is locally congruent to either a geodesic sphere* $G(r)\,(0 < r < \pi/\sqrt{c}\,)$ *with* $n \geqq 2$ *or a tube of radius* $\pi/(2\sqrt{c}\,)$ *around a totally geodesic* $\mathbb{C}P^\ell(c)\,(1 \leqq \ell \leqq n-2)$ *with* $n \geqq 3$ *if and only if there exist such orthonormal vectors* $v_1, v_2, \ldots, v_{2n-2}$ *orthogonal to the characteristic vector* ξ_p *at each point* p *of* M *that all geodesics of* M *through* p *in the direction* $v_i\,(1 \leqq i \leqq 2n-2)$ *are circles of the same positive curvature* $k(p)$ *in* $\mathbb{C}P^n(c)$. *Here, the function* $k = k(p)$ *on* M *is automatically constant. When* M *is locally congruent to* $G(r)$, k *is expressed as*: $k = (\sqrt{c}/2)\cot(\sqrt{c}\,r/2)$, *and when* M *is locally congruent to a tube of radius* $\pi/(2\sqrt{c}\,)$ *around a totally geodesic* $\mathbb{C}P^\ell(c)\,(1 \leqq \ell \leqq n-2)$, k *is written as:* $k = \sqrt{c}/2$.

Theorem 3.3 ([5]). *Let* M^{2n-1} *be a real hypersurface of* $\mathbb{C}P^n(c)$, $n \geqq 2$. *Then* M *is locally congruent to a Berger sphere, namely a geodesic sphere* $G(r)$ *of radius* r *with* $\tan^2(\sqrt{c}\,r/2) > 2$ *if and only if at each point* p *of* M *there exist such orthonormal vectors* $v_1, v_2, \ldots, v_{2n-2}$ *orthogonal to the characteristic vector* ξ_p *at each point* p *of* M *that all geodesics of* M *through* p *in the direction* $v_i\,(1 \leqq i \leqq 2n-2)$ *are circles of the same positive curvature* $k(p)$ *with* $k(p) < \sqrt{c}/(2\sqrt{2}\,)$ *in* $\mathbb{C}P^n(c)$.

4. Several comments on Theorems 1, 2 and 3

(1) In Theorems 3.1, 3.2 and 3.3, all geodesics on $G(r)$ are mapped to circles lying on a totally real totally geodesic $\mathbb{R}P^n(c/4)$ of constant sectional curvature $c/4$ in the ambient space $\mathbb{C}P^n(c)$. Note that there exist many geodesics on $G(r)$ which do not lie on $\mathbb{R}P^n(c/4)$ (cf. Proposition 2.1 in [2]).

(2) In Theorem 3.2, to distinguish geodesic spheres $G(r)\,(0 < r < \pi/\sqrt{c}\,)$ and a tube T of radius $\pi/(2\sqrt{c}\,)$ around a totally geodesic $\mathbb{C}P^\ell(c)\,(1 \leqq \ell \leqq n-2)$, we recall the notion of extrinsic geodesics. We denote by M^n a Riemannian submanifold of a Riemannian manifold $\widetilde{M}^{n+p}$ through an isometric immersion f. A smooth real curve $\gamma = \gamma(s)$ on the submanifold M^n is an *extrinsic geodesic* if the curve $f \circ \gamma$ is a geodesic in the ambient space $\widetilde{M}^{n+p}$. In this case, as a matter of course the curve γ is also a geodesic on the submanifold M^n.

The following fact implies that we can distinguish $G(r)$ and a tube T, i.e., a tube of radius $\pi/(2\sqrt{c}\,)$ around a totally geodesic $\mathbb{C}P^\ell(c)\,(1 \leqq \ell \leqq n-2)$, by counting the number of congruence classes of extrinsic geodesics (see [9]).

Fact. (i) *A geodesic sphere* $G(r)\,(0 < r < \pi/(2\sqrt{c}\,))$ *has no extrinsic geodesics.*
(ii) *A geodesic sphere* $G(r)\,(\pi/(2\sqrt{c}\,) \leqq r < \pi/\sqrt{c}\,)$ *has just one extrinsic geodesic with respect to the full isometry group* $\mathrm{I}(G(r))$ *of* $G(r)$.
(iii) *The tube* T *has uncountably infinite many extrinsic geodesics with respect to the full isometry group* $\mathrm{I}(T)$ *of* T.

(3) We comment on the length of an integral curve of the characteristic vector field ξ on a geodesic sphere $G(r)$ in $\mathbb{C}P^n(c)$. We note that every integral curve

γ_ξ of the characteristic vector field ξ is a closed geodesic on $G(r)$ with length $2\pi\sin(\sqrt{c}\,r)/\sqrt{c}$. In fact, the curve γ_ξ satisfies $\nabla_\xi\xi = 0$, $\widetilde{\nabla}_\xi\xi = \sqrt{c}\,\cot(\sqrt{c}\,r)\mathcal{N}$ and $\widetilde{\nabla}_\xi\mathcal{N} = -\sqrt{c}\,\cot(\sqrt{c}\,r)\xi$ with $\dot{\gamma}_\xi = \xi$, where ∇ and $\widetilde{\nabla}$ are the Riemannian connections of $G(r)$ and $\mathbb{C}P^n(c)$, respectively. These mean that the curve γ_ξ can be regarded as a small circle of positive curvature $\sqrt{c}\,|\cot(\sqrt{c}\,r)|$ on $S^2(c)(=\mathbb{C}P^1(c))$. Hence, the length ℓ of γ_ξ is represented as:

$$\ell = 2\pi/\sqrt{c\cot^2(\sqrt{c}\,r)+c} = 2\pi\sin(\sqrt{c}\,r)/\sqrt{c}.$$

We next consider an inequality:

$$2\pi\sin(\sqrt{c}\,r)/\sqrt{c} < 2\pi/\sqrt{c+(c/4)\cot^2(\sqrt{c}\,r/2)},$$

where $c+(c/4)\cot^2(\sqrt{c}\,r/2)$ is the maximal sectional curvature of $G(r)$. Solving this inequality, we get $\tan^2(\sqrt{c}\,r/2) > 2$.

(4) We review the length spectrum of every geodesic sphere $G(r)$ $(0<r<\pi/\sqrt{c}\,)$ (for details, see [2]). Every geodesic sphere $G(r)$ $(0<r<\pi/\sqrt{c}\,)$ admits countably infinite many congruence classes of closed geodesics with respect to the full isometry group $\mathrm{I}(G(r))$ of $G(r)$. All integral curves of the characteristic vector field ξ are congruent each other with respect to $\mathrm{I}(G(r))$ and the shortest closed geodesics (with common length $2\pi\sin(\sqrt{c}\,r)/\sqrt{c}\,$) on $G(r)$. Furthermore, the lengths of all closed geodesics except integral curves of the characteristic vector field ξ on $G(r)$ are longer than $2\pi/\sqrt{c+(c/4)\cot^2(\sqrt{c}\,r/2)}$.

(5) We consider all geodesic spheres $G(r)$ $(0<r<\pi/\sqrt{c}\,)$ from the viewpoint of contact geometry (cf. [1]). $G(r)$ is a Sasakian manifold (with respect to the almost contact metric structure (ϕ,ξ,η,g) induced from the Käler structure J of $\mathbb{C}P^n(c)$) if and only if $(\sqrt{c}/2)\cot(\sqrt{c}\,r/2) = 1$. This Sasakian manifold M has automatically constant ϕ-sectional curvature $c+1$, so that it is a Sasakian space form of constant ϕ-sectional curvature $c+1$. Since an inequality $1 < \sqrt{c}/(2\sqrt{2}\,)$ leads to an inequality $c > 8$, we find that all Sasakian space forms of constant ϕ-sectional curvature $\tilde{c}$ with $\tilde{c} > 9$ are Berger spheres.

(6) We comment on the sectional curvature K of Berger spheres, i.e., geodesic spheres $G(r)$ with $\tan^2(\sqrt{c}\,r/2) > 2$ in $\mathbb{C}P^n(c)$. K satisfies sharp inequalities $\delta L \leqq K \leqq L$ for some $\delta \in (0, 1/9)$, where $L = c+(c/4)\cot^2(\sqrt{c}\,r/2)$.

References

[1] T. Adachi, M. Kameda and S. Maeda, *Geometric meaning of Sasakian space forms from the viewpoint of submanifold theory*, Kodai Math. J. **33** (2010), 383–397.

[2] T. Adachi, S. Maeda and M. Yamagishi, *Length spectrum of geodesic spheres in a non-flat complex space form*, J. Math. Soc. Japan **54** (2002), 373–408.

[3] T. Cecil and P. Ryan, *Focal sets and real hypersurfacesin complex projective space*, Trans. Amer. Math. Soc. **269** (1982), 481–499.

[4] W. Klingenberg, *Contributions to Riemannian geometry in the large*, Ann. of Math. (2) **69** (1959), 654–666.

[5] B.H. Kim, I.B. Kim and S. Maeda, *Characterizations of Berger spheres from the viewpoint of submanifold theory*, a preprint.

[6] T. Kajiwara and S. Maeda, *Sectional curvatures of geodesic spheres in a complex hyperbolic space*, Kodai Math. J. **38** (2015), 604–619.

[7] S. Maeda and K. Ogiue, *Characterizations of geodesic hyperspheres in a complex projective space by observing the extrinsic shape of geodesics*, Math. Z. **225** (1997), 537–542.

[8] R. Niebergall and P.J. Ryan, *Real hypersurfaces in complex space forms*, Tight and taut submanifolds (T.E. Cecil and S.S. Chern, eds.), Cambridge Univ. Press, 1998, 233–305.

[9] K. Okumura and S. Maeda, *Three real hypersurfaces some of whose geodesics are mapped to circles with the same curvature in a nonflat complex space form*, Geom. Dedicata **156** (2012), 71–80.

[10] Y. Tashiro and S. Tachibana, *On Fubinian and C-Fubinian manifolds*, Kōdai Math. Sem. Rep. **15** (1963), 176–183.

[11] A. Weinstein, *Distance spheres in complex projective spaces*, Proc. Amer. Math. Soc. **39** (1973), 649–650.

S. Maeda: Department of Mathematics, Saga University, 1 Honzyo, Saga, 840-8502, Japan

E-mail address: sayaki@ms.saga-u.ac.jp