Mem. Fac. Educ., Shimane Univ. (Nat. Sci.), Vol. 5, pp. 1 -- 7, December 1971.

TOPOLOGIES ON EQUICONTINUOUS FAMILIES OF MAPPINGS

By

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Introduction

It is well-known that equicontinuous families of mappings are important in Analysis through Ascoli's theorem. On the other hand, the topologies on them are fundamental to consider the problem when a locally equicontinuous group of transformations of a space becomes a locally compact transformation group. The problem will be treated in T. Karube [5] to which the present paper is a preliminary. Because a set-entourage uniformity on the family of all continuous mappings of a uniform space into itself must be the uniformity of compact-convergence under natural conditions (T. Karube [4]), we will set importace on the compact-open topology.

Notations

X : a topological space.

Y : a uniform space.

$\mathfrak{S}$: a family of subsets of X.

$\mathfrak{S}_c$: the family of all compact subsets of X.

$\mathfrak{F}(X; Y)$: the family of all mappings of X into Y.

$\mathfrak{F}$: a subfamily of $\mathfrak{F}(X; Y)$.

$\mathfrak{C}(X; Y)$: the family of all continuous mappings of X into Y.

$\mathfrak{C}$: a subfamily of $\mathfrak{C}(X; Y)$.

$\mathfrak{C}_{\mathfrak{S}}(X; Y)$: the family of all mappings of X into Y whose restriction to each set of $\mathfrak{S}$ is continuous.

τ : a topology on $\mathfrak{F}(X; Y)$.

τ_p : the point-open topology on $\mathfrak{F}(X; Y)$.

τ_c : the compact-open topology on $\mathfrak{F}(X; Y)$.

$\tau^*_{\mathfrak{S}}$: the uniform topology induced by the uniformity of $\mathfrak{S}$-convergence.

τ^*_p : the uniform topology induced by the uniformity of pointwise-convergence.

τ_c^* : the uniform topology induced by the uniformity of compact-convergence.

$T(A, B)$: the family of all $u \in \mathfrak{F}(X; Y)$ such that $u(A) \subset B$, where A and B is a given subset of X and Y respectively.

$W(A, \mathfrak{U})$: the set of all pairs (u, v) such that $u, v \in \mathfrak{F}(X; Y)$ and $(u(x), v(x)) \in \mathfrak{U}$ for any $x \in A$, where A is a given subset of X and $\mathfrak{U}$ is a given entourage of Y.

These notations will keep the meanings throughout the paper.

§ 1. Topological properties of mapping spaces.

Definition. $(\mathfrak{C}, \tau)$ is *admissible* if the mapping $(u, x) \rightarrow u(x)$ of $\mathfrak{C} \times X$ into Y is continuous with respect to the relative topology of τ to $\mathfrak{C}$ and the topologies of X and Y.

Lemma 1.1. *If X is locally compact, then $(\mathfrak{C}, \tau_c)$ is admissible.*

Proof. Since Y is a uniform space it satisfies T_3-axiom. Hence $(\mathfrak{C}, \tau_c)$ is admissible by Lemma 2.3 of S. B. Myers [7].

Lemma 1.2. (R. Arens [1]) *If $(\mathfrak{C}, \tau)$ is admissible, then τ is finer than τ_c on $\mathfrak{C}$.*

Lemma 1.3. (N. Bourbaki [3]) *$\mathfrak{C}_{\mathfrak{S}}(X; Y)$ is closed in $\mathfrak{F}(X; Y)$ under $\tau_{\mathfrak{S}}^*$.*

Definition. A topological space X is a *k'-space* (resp. *k-space*) if a subset of X intersecting each compact set (resp. each closed compact set) in a closed set is always closed.

Proposition 1.4. *Consider the following conditions* :

(0) $\mathfrak{C}_{\mathfrak{S}}(X; Y) = \mathfrak{C}(X; Y)$,

(1) *every point of X is interior to at least one set of $\mathfrak{S}$,*

(2) *in X, a subset intersecting each set of $\mathfrak{S}$ in a closed set is always closed,*

(3) *X is a k'-space,*

(4) *X is a k-space,*

(5) *X is a Hausdorff space which is locally compact or first countable,*

(6) *X is a locally compact Hausdorff space,*

(7) *X is a locally compact space.*

Then, $(6) \Rightarrow (5) \Rightarrow (4) \Rightarrow (3) \Rightarrow (2) \Rightarrow (0)$, $(6) \Rightarrow (7)$, *and* $(1) \Rightarrow (2)$. *If* $\mathfrak{S} = \mathfrak{S}_c$, *then* $(3) \Rightarrow (2)$, *and* $(7) \Leftrightarrow (1)$. *If X is Hausdorff, then* $(3) \Rightarrow (4)$, *and* $(7) \Rightarrow (6)$.

Proof. They are easy to see or trivial.

Corollary 1.5. *Under each condition of* (1), (2),..., (7), *where* $\mathfrak{S} = \mathfrak{S}_c$, *$\mathfrak{C}(X; Y)$ is closed in $\mathfrak{F}(X; Y)$ under any topology finer than τ_c^*.*

Lemma 1.6. (R. Arens [1]) *$\tau_c^* = \tau_c$ on $\mathfrak{C}$. And for each $u \in \mathfrak{C}$,*

$\{W(K, \mathfrak{U})(u) \cap \mathfrak{C} \mid K$ *is a compact subset of* X *and* $\mathfrak{U}$ *is an entourage of* $Y\}$ *forms a fundamental system of neighborhoods of* u *in* $\mathfrak{C}$ *under* τ_c.

Lemma 1.7. $\tau_p^* = \tau_p$ *on* $\mathfrak{F}(X; Y)$.

Proof. It is easy to see.

§ 2. Fundamental properties of equicontinuous families.

Definition. $\mathfrak{C}$ is *equicontinuous* if, for each entourage $\mathfrak{U}$ of Y and each point x of X, there exists a neighborhood U of x in X such that $(u(x), u(x')) \in \mathfrak{U}$ for any $x' \in U$ and any $u \in \mathfrak{C}$.

Lemma 2.1. *If* $\mathfrak{C}$ *is equicontinuous, then* $(\mathfrak{C}, \tau)$ *is admissible for any topology* τ *finer than* τ_p.

Proof. Since $(\mathfrak{C}, \tau_p^*)$ is admissible (cf. N. Bourbaki [3], Corollary 4 in p. 286), $(\mathfrak{C}, \tau_p)$ is admissible also.

Lemma 2.2. *If* $\mathfrak{C}$ *is equicontinuous, then* $\tau_c = \tau_p$ *on* $\mathfrak{C}$.

This is well-known. It follows from Lemmas 2.1 and 1.2 also.

Lemma 2.3. *Let* $\mathfrak{C}$ *be an equicontinuous subfamily of* $\mathfrak{F}(X; Y)$ *and* $\overline{\mathfrak{C}}$ *be the closure of* $\mathfrak{C}$ *under* $\tau_{\mathfrak{S}}^*$ *in* $\mathfrak{F}(X; Y)$. *Then* $\overline{\mathfrak{C}}$ *is contained in* $\mathfrak{C}_{\mathfrak{S}}(X; Y)$. *And if for any fixed* $x \in X$ *the mapping* $u \to u(x)$ *of* $\mathfrak{C}_{\mathfrak{S}}(X; Y)$ *into* Y *is continuous at any* $u \in \overline{\mathfrak{C}}$ *with respect to* $\tau_{\mathfrak{S}}^*$, *then* $\overline{\mathfrak{C}}$ *is equicontinuous.*

Proof. The first assertion follows from Lemma 1.3. Let x_0 be any fixed point of X, $\mathfrak{U}$ be any entourage of Y, and $\mathfrak{U}'$ be an entourage of Y such that $\overline{\mathfrak{U}'^2} \subset \mathfrak{U}$. Since $\mathfrak{C}$ is equicontinuous, there exists a neighborhood U of x_0 such that

$$(u(x_0), u(x)) \in \mathfrak{U}' \text{ for any } x \in U \text{ and any } u \in \mathfrak{C}. \tag{1}$$

By our assumption, for any $u' \in \overline{\mathfrak{C}}$ there exists a $\tau_{\mathfrak{S}}^*$-neighborhood W of u' in $\mathfrak{C}_{\mathfrak{S}}(X; Y)$ such that

$$(u'(x_0), u(x_0)) \in \mathfrak{U}' \text{ for any } u \in W. \tag{2}$$

For any directed set $\{u_\alpha\} \subset \mathfrak{C}$ converging to u', there is a α_0 such that $u_\alpha \in W$ for any $\alpha \geqq \alpha_0$. Then from (2) we have

$$(u'(x_0), u_\alpha(x_0)) \in \mathfrak{U}' \text{ for any } \alpha \geqq \alpha_0. \tag{3}$$

From (1) and (3), we have

$(u'(x_0), u_\alpha(x)) \in \mathfrak{U}'^2$ for any $\alpha \geqq \alpha_0$ and any $x \in U$.

Hence $(u'(x_0),\ u'(x)) \in \overline{\mathfrak{U}'^2} \subset \mathfrak{U}$ for any $x \in U$.
And so $(u'(x_0),\ u'(x)) \in \mathfrak{U}$ for any $x \in U$ and any $u' \in \overline{\mathfrak{C}}$.

Lemma 2.4. *Let $\mathfrak{S}$ be a family of subsets of X that covers X, $\mathfrak{C}$ an equicontinuous subfamily of $\mathfrak{F}(X;Y)$, and $\overline{\mathfrak{C}}$ the closure of $\mathfrak{C}$ under $\tau^*_{\mathfrak{S}}$ in $\mathfrak{F}(X;Y)$. Then $\overline{\mathfrak{C}}$ is equicontinuous.*

Proof. Since $(\mathfrak{C}_{\mathfrak{S}}(X;Y),\ \tau^*_{\mathfrak{S}})$ is jointly continuous on each member of $\mathfrak{S}$ (cf. e. g. pp. 228, 229 of J. L. Kelly [6]) and $\mathfrak{S}$ covers X, the condition in Lemma 2.3 is satisfied.

By Lemm 1.7 and Lemma 2.4 we have the following

Proposition 2.5. *Let $\mathfrak{C}$ be an equicontinuous subfamily of $\mathfrak{F}(X;Y)$ and $\overline{\mathfrak{C}}$ be the closure of $\mathfrak{C}$ in $\mathfrak{F}(X;Y)$ under any topology finer than τ_p on $\mathfrak{F}(X;Y)$. Then $\overline{\mathfrak{C}}$ is equiconinuous.*

Lemma 2.6. *Let $\mathfrak{C}$ be an equicontinuous subfamily of $\mathfrak{F}(X;Y)$ and τ any topology finer than τ_p on $\mathfrak{F}(X;Y)$. Then the closure of $\mathfrak{C}$ in $\mathfrak{F}(X;Y)$ under τ coincides with the closure of $\mathfrak{C}$ in $\mathfrak{C}(X;Y)$ under τ.*

Proof. Note that the closure of $\mathfrak{C}$ in $\mathfrak{F}(X;Y)$ under τ is contained in $\mathfrak{C}(X;Y)$ by Proposition 2.5.

Proposition 2.7. *Let $\mathfrak{C}$ be an equicontinuous subfamily of $\mathfrak{F}(X;Y)$. Then the closures of $\mathfrak{C}$ in $\mathfrak{C}(X;Y)$ under τ_c, τ^*_c, τ_p, τ^*_p, and the closures of $\mathfrak{C}$ in $\mathfrak{F}(X;Y)$ under τ_c, τ^*_c, τ_p, τ^*_p, are all equal, and equicontinuous.*

Proof. Use Lemmas 1.6, 1.7, 2.2, 2.6, and Proposition 2.5.

§ 3. Equicontinuity and compactness.

We will generalize Ascoli's theorem.

Theorem 3.1. *Consider the following two conditions :*

i) *$\mathfrak{C}$ is equicontinuous and $\overline{\mathfrak{C}(x)}$ is compact for each $x \in X$, and*

ii) *the closure of $\mathfrak{C}$ in $\mathfrak{C}(X;Y)$ under τ_c is compact.*

Then i) *implies* ii). *If X is locally compact, then* ii) *implies* i).

Proof. i) implies ii) : By Tychonoff's theorem, the topological product of $\{\overline{\mathfrak{C}(x)} \mid x \in X\}$ is compact under the relative product topology. Since the set contains the τ_p-closure $\mathrm{Cl}_p\mathfrak{C}$ of $\mathfrak{C}$ in $\mathfrak{F}(X;Y)$, $\mathrm{Cl}_p\mathfrak{C}$ is τ_p-compact. $\mathrm{Cl}_p\mathfrak{C}$ is equicontinuous by Proposition 2.5, and so τ_p coincides with τ_c on $\mathrm{Cl}_p\mathfrak{C}$ by Lemma 2.2. Hence $\mathrm{Cl}_p\mathfrak{C}$ is τ_c-compact. On the other hand, the τ_c-closure $\mathrm{Cl}_c\mathfrak{C}$ of $\mathfrak{C}$ in $\mathfrak{F}(X;Y)$ is contained in $\mathrm{Cl}_p\mathfrak{C}$. Hence $\mathrm{Cl}_c\mathfrak{C}$ is τ_c-compact. Consequently from the fact that $\mathrm{Cl}_c\mathfrak{C}$ coincides with the τ_c-closure of $\mathfrak{C}$ in $\mathfrak{C}(X;Y)$, the condition ii) follows.

ii) implies i) : It is shown first that $\overline{\mathfrak{C}}$ is equicontinuous as follows. Let x_0 be any fixed point of X. For any given entourage $\mathfrak{U}$ of Y take a symmetric entourage $\mathfrak{U}_1$ of Y such that $\mathfrak{U}_1^2 \subset \mathfrak{U}$. Since $(\overline{\mathfrak{C}}, \tau_c)$ is admissible (cf. Lemma 1.1), for each $u \in \overline{\mathfrak{C}}$ there exist a τ_c-neighborhood $W(u)$ of u in $\mathfrak{C}(X; Y)$ and a neighborhood U_u of x_0 such that

$(u(x_0), u'(x)) \in \mathfrak{U}_1$ for any $u' \in W(u)$ and any $x \in U_u$.

Since $\overline{\mathfrak{C}}$ is compact, we can choose a finite covering $\{W(u_i) \mid i=1, 2, \dots, n\}$ of $\overline{\mathfrak{C}}$. Put $U = \bigcap_{i=1}^{n} U_{u_i}$. Then

$(u'(x_0), u'(x)) \in \mathfrak{U}$ for any $u' \in \overline{\mathfrak{C}}$ and any $x \in U$.

Consequently $\overline{\mathfrak{C}}$ is equicontinuous, a fortiori $\mathfrak{C}$ is equicontinuous. On the other hand, $\mathfrak{C}(X; Y)$ is admissible and $\overline{\mathfrak{C}}$ is compact under τ_c. Hence $\overline{\mathfrak{C}}(x)$ is compact. Since the closure of a compact set in a uniform space is compact, $\overline{\overline{\mathfrak{C}}(x)}$ is compact. And so $\overline{\mathfrak{C}(x)}$ is compact. Q. E. D.

Remark. Several generalizations of Ascoli's theorem are well-known under some additional conditions on X and Y (cf. e. g. J. L. Kelley [6], and H. Schubert [8]). Moreover the condition ii) implies the condition i) in the case where

a) X is first countable, or

b) X is a Hausdorff k-space and Y is a Hausdorff uniform space.

This is a generalization of a result of S. B. Myers [7].

Definition. A uniform space Y is *uniformly locally complete* if there exists an entourage $\mathfrak{U}$ in Y such that $\mathfrak{U}(y)$ is complete for any $y \in Y$.

Lemma 3.2. *Let X be a connected space, and Y be a locally compact and uniformly locally complete uniform space. If $\mathfrak{C}$ is equicontinuous and $\overline{\mathfrak{C}(x_0)}$ is compact for at least one $x_0 \in X$, then $\overline{\mathfrak{C}(x)}$ is compact for each $x \in X$.*

Proof. Let $\overline{\mathfrak{C}}$ be the closure of $\mathfrak{C}$ in $\mathfrak{C}(X; Y)$ under τ_c. Since $\overline{\mathfrak{C}}$ is equicontinuous (cf. Proposition 2.5), $(\overline{\mathfrak{C}}, \tau_c)$ is admissible (cf. Lemma 2.1). Hence $\overline{\mathfrak{C}}(x_0)$ has compact closure. Let E be the set of all $x \in X$ such that $\overline{\mathfrak{C}}(x)$ has compact closure. E is not empty. If Y is locally compact, then E is open. In fact, let $\tilde{x}$ be any point of E. There exists a compact neighborhood V of $\overline{\overline{\mathfrak{C}}(\tilde{x})}$, and an entourage $\mathfrak{U}$ of Y such that $\mathfrak{U}[\overline{\overline{\mathfrak{C}}(\tilde{x})}] \subset V$. Since $\overline{\mathfrak{C}}$ is equicontinuous, there exists a neighborhood U of $\tilde{x}$ such that

$(u(\tilde{x}), u(x)) \in \mathfrak{U}$ for any $u \in \overline{\mathfrak{C}}$ and any $x \in U$.

And so $\overline{\mathfrak{C}}(x) \subset V$ for any $x \in U$. Since V is compact, $\overline{\overline{\mathfrak{C}}(x)}$ is compact for any $x \in U$ i. e. $U \subset E$. If Y is uniformly locally complete, then E is closed also.

In fact, it is shown as follows that $\overline{\mathfrak{C}}(\tilde{x})$ is totally bounded for any $\tilde{x} \in \bar{E}$. Let $\mathfrak{U}_0$ be any entourage of Y, and $\mathfrak{U}$ be a symmetric entourage of Y such that $\mathfrak{U}^4 \subset \mathfrak{U}_0$. Since $\overline{\mathfrak{C}}$ is equicontinuous, there exists a neighborhood U of $\tilde{x}$ such that

$(u(\tilde{x}),\ u(x)) \in \mathfrak{U}$ for any $u \in \overline{\mathfrak{C}}$ and any $x \in U$.

There exists a point x_1 in $U \cap E$. As $\overline{\mathfrak{C}}(x_1)$ has compact closure, there exist finite number of points $y_1,\ldots, y_n$ in $\overline{\overline{\mathfrak{C}}(x_1)}$ such that $\{\mathfrak{U}(y_i) \mid i = 1,\ldots, n\}$ cover $\overline{\overline{\mathfrak{C}}(x_1)}$. Then $\overline{\mathfrak{C}}(\tilde{x})$ is covered by finite number of $\mathfrak{U}^4$-small and so $\mathfrak{U}_0$-small sets $\mathfrak{U}^2(y_i)\,(i = 1,\ldots, n)$. Now we may suppose without loss of generality that $\mathfrak{U}_0(y)$ is complete for any $y \in Y$. Then from the fact that $\overline{\overline{\mathfrak{C}}(\tilde{x})} \subset \bigcup_{i=1}^{n} \overline{\mathfrak{U}_0(y_i)}$, we can show that $\overline{\overline{\mathfrak{C}}(\tilde{x})}$ is complete. Consequently $\overline{\overline{\mathfrak{C}}(\tilde{x})}$ is compact i. e. $\tilde{x} \in E$. The set E is open, closed, and non-empty in the connected space. Therefore E coincides with X.

Corollary 3.3. *If X is a locally compact connected space, and Y is a locally compact and uniformly locally complete uniform space, then the following two conditions are equivalent :*

i) *$\mathfrak{C}$ is equicontinuous and $\overline{\mathfrak{C}(x_0)}$ is compact for at least one $x_0 \in X$, and*

ii) *the closure of $\mathfrak{C}$ in $\mathfrak{C}(X\,;\,Y)$ under τ_c is compact.*

This corollary is more convenient than Theorem 3.1 to make compact topological transformation groups of many equicontinuous transformation groups by means of the compact-open topology.

Proposition 3.4. *Let X be a connected space, Y be a locally compact space, and $\mathfrak{C}$ a subfamily of $\mathfrak{C}(X\,;\,Y)$. If $\mathfrak{C}(x)$ has compact closure for at least one $x \in X$, and if there exists a uniformly locally complete compatible uniformity of Y under which $\mathfrak{C}$ is equicontinuous, then $\mathfrak{C}$ is equicontinuous for any compatible uniformity of Y.*

Proof. It is similar to that of S. B. Myers [7].

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