Nonoscillation of second-order linear difference systems with varying coefficients

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Abstract

This paper deals with nonoscillation problem about the non-autonomous linear difference system

$$\mathbf{x}_n = A_n \mathbf{x}_{n-1}, \qquad n = 1, 2, \ldots,$$

where A_n is a 2×2 variable matrix that is nonsingular for $n \in \mathbb{N}$. In the special case that A is a constant matrix, it is well-known that all non-trivial solutions are nonoscillatory if and only if all eigenvalues of A are positive real numbers; namely, $\det A > 0$, $\mathrm{tr} A > 0$ and $\det A/(\mathrm{tr} A)^2 \leq 1/4$. The well-known result can be said to be an analogy of ordinary differential equations. The results obtained in this paper extend this analogy result. In other words, this paper clarifies the distinction between difference equations and ordinary differential equations. Our results are explained with some specific examples. In addition, figures are attached to facilitate understanding of those examples.

Key words: Linear difference equations; Non-autonomous; Nonoscillation; Riccati transformation, Sturm's separation theorem
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1. Introduction

We consider the second-order linear time-variant system

$$\mathbf{x}_n = A_n \mathbf{x}_{n-1}, \qquad n = 1, 2, \ldots, \tag{1.1}$$

where

$$\mathbf{x}_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix} \quad \text{and} \quad A_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$$

in which the components x_n and y_n and the coefficients a_n, b_n, c_n and d_n are real numbers. It is always assumed that the matrix A_n is nonsingular for $n \in \mathbb{N}$. Needless to say, equation (1.1) has the trivial solution $\{\mathbf{x}_n\}$; that is, $(x_n, y_n) = (0, 0)$ for $n \in \mathbb{N}$. A non-trivial solution $\{\mathbf{x}_n\}$ of (1.1) is said to be *oscillatory with respect to the first* (resp., *second*) *component* if,

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for every $n \in \mathbb{N}$ there exists an $m \geq n$ such that $x_m x_{m+1} \leq 0$ (resp., $y_m y_{m+1} \leq 0$). Otherwise, it is said to be *nonoscillatory with respect to the first* (or *second*) *component*. Hence, if a non-trivial solution $\{\mathbf{x}_n\}$ of (1.1) is nonoscillatory with respect to the first (resp., second) component, then there exists an $m \in \mathbb{N}$ such that $x_n > 0$ for $n \geq m$ or $x_n < 0$ for $n \geq m$ (resp., $y_n > 0$ for $n \geq m$ or $y_n < 0$ for $n \geq m$). It is clear that if $\{\mathbf{x}_n\}$ is a solution of (1.1), then $\{-\mathbf{x}_n\}$ is also a solution of (1.1). Hence, we can assume without loss of generality that a non-trivial solution $\{\mathbf{x}_n\}$ of (1.1) which is nonoscillatory with respect to the first (resp., second) component satisfy that x_n (resp., y_n) is positive for all large n. A non-trivial solution $\{\mathbf{x}_n\}$ of (1.1) is said to be *nonoscillatory* if it is nonoscillatory with respect to the first and second components.

The purpose of this paper is to give sufficient conditions for all non-trivial solutions of (1.1) to be nonoscillatory with respect to the first (or second) component. Of course, the coefficients of the matrix A_n determine whether or not all non-trivial solutions of (1.1) are nonoscillatory with respect to the first component.

In the special case that

$$A_n \equiv A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where a, b, c and d are real constants, system (1.1) is equivalent to the second-order autonomous linear equations

$$x_{n+1} + (\det A)x_{n-1} = (\mathrm{tr}A)x_n \tag{1.2}$$

and

$$y_{n+1} + (\det A)y_{n-1} = (\mathrm{tr}A)y_n \tag{1.3}$$

for $n \in \mathbb{N}$. It is clear that if $\det A < 0$, then the characteristic equation

$$\lambda^2 - (\mathrm{tr}A)\lambda + \det A = 0$$

has two real roots of opposite signs: $\lambda_1 < 0 < \lambda_2$. Hence, two sequences $\{\lambda_1^n\}$ and $\{\lambda_2^n\}$ are an oscillatory solution and a nonoscillatory solution of (1.2) (or (1.3)), in other words, oscillatory solutions and nonoscillatory solutions coexist in equation (1.2) (or (1.3)) in the meaning of the definition described above. If $\mathrm{tr}A \leq 0 < \det A$, then all non-trivial solutions of (1.2) (or (1.3)) are oscillatory. If $\det A > 0$ and $\mathrm{tr}A > 0$, then all non-trivial solutions of (1.2) (or (1.3)) are nonoscillatory if and only if $\det A \leq (\mathrm{tr}A)^2/4$. Thus, under the above-mentioned definitions about oscillation and nonoscillation, Sturm's separation theorem holds when $\det A > 0$, but it fails to hold when $\det A < 0$.

Remark 1.1. Equations (1.2) and (1.3) are contained in the self-adjoint second-order difference equation

$$\Delta(e_n \Delta x_{n-1}) + f_n x_n = 0$$

with $e_n \neq 0$. Note that e_n is not necessarily of one sign (for example, see [2]). The following definitions different from the above are often made for oscillation and nonoscillation (refer to [4, 6, 16, 17]). An interval $(m, m+1]$ is said to contain a *generalized zero* of a solution $\{x_n\}$ if $x_m \neq 0$ and $e_m x_m x_{m+1} \leq 0$. A non-trivial solution is called *oscillatory* if it has infinitely many generalized zeros. In the opposite case, a non-trivial solution is called *nonoscillatory*. It is known that the so-called Reid roundabout theorem, Sturm's comparison theorem, and Sturm's separation theorem hold under the definition focused on the sign of $e_n x_n x_{n+1}$. Hence, all non-trivial solutions are either oscillatory or nonoscillatory. However, these pieces of information are different topics from this paper and will not be used hereafter.

Let us get back to the subject. If there exists a subsequence $\{n_k\}$ such that $b_{n_k} = 0$, then it follows from system (1.1) that $x_{n_k} = a_{n_k} x_{n_k - 1}$. Hence, it is clear that if $a_{n_k} \leq 0$, then x_{n_k-1} and x_{n_k} do not have the same signs. This means that a_n has to be positive for all sufficiently large $n \in \mathbb{N}$ in order for all non-trivial solutions of (1.1) to be nonoscillatory with respect to the first component. In other words, when discussing whether all non-trivial solutions of (1.1) are nonoscillatory with respect to the first component or not, it is important to analyze the asymptotic behavior of solutions when b_n is not zero. Since nonoscillation for the second component is also the same situation as that for the first component, we assume that b_n and c_n are not zero for all sufficiently large $n \in \mathbb{N}$ in this paper.

From consideration about the case that A_n is a 2×2 constant matrix, to achieve our purpose, it is natural to assume that

$$\det A_n > 0, \tag{1.4}$$

$$\frac{a_{n+1}b_n + b_{n+1}d_n}{b_n} > 0, \tag{1.5}$$

and

$$\frac{a_n c_{n+1} + c_n d_{n+1}}{c_n} > 0 \tag{1.6}$$

for all sufficiently large $n \in \mathbb{N}$. We can rewrite system (1.1) as

$$x_{n+1} + \frac{b_{n+1}}{b_n}(\det A_n) x_{n-1} = \frac{a_{n+1}b_n + b_{n+1}d_n}{b_n} x_n \tag{1.7}$$

and

$$y_{n+1} + \frac{c_{n+1}}{c_n}(\det A_n) y_{n-1} = \frac{a_n c_{n+1} + c_n d_{n+1}}{c_n} y_n \tag{1.8}$$

for all sufficiently large $n \in \mathbb{N}$.

Hooker et al. [10, 11, 14] have considered the second-order linear difference equation

$$\gamma_n x_{n+1} + \gamma_{n-1} x_{n-1} = \beta_n x_n, \qquad n = 1, 2, \dots, \tag{1.9}$$

and presented some conditions which guarantee that all non-trivial solutions of (1.9) are nonoscillatory (they also gave sufficient conditions for all non-trivial solutions of (1.9) to be oscillatory). Their typical and fundamental result on nonoscillation is as follows (see also the books [1, Chap. 6], [5, Chap. 7], [12, Chap. 6]).

Theorem A. *If* $\beta_n\gamma_n > 0$ *and* $\gamma_n^2/(\beta_n\beta_{n+1}) \le 1/4$ *for all sufficiently large* $n \in \mathbb{N}$, *then all non-trivial solutions of* (1.9) *are nonoscillatory*.

The constant $1/4$ often appears as a critical value that divides oscillation and nonoscillation of solutions of second-order linear differential equations (for example, see [9, 13, 15, 19]). This critical value is called an *oscillation constant*. We can also find researches on the oscillation constant of second-order difference equations in [3, 7, 8, 18] and the references cited therein. In that sense, it is not too much to say that Theorem A is an analogy of results of ordinary differential equations.

Using Theorem A, we obtain the following results (see Section 2 for the proof).

Theorem B. *Assume* (1.4) *and* (1.5). *If* $b_n/b_{n+1} > 0$ *and*

$$\frac{b_n b_{n+2}\det A_{n+1}}{(a_{n+1}b_n + b_{n+1}d_n)(a_{n+2}b_{n+1} + b_{n+2}d_{n+1})} \le \frac{1}{4}$$

for n *sufficiently large, then all non-trivial solutions of* (1.1) *are nonoscillatory with respect to the first component*.

Theorem C. *Assume* (1.4) *and* (1.6). *If* $c_n/c_{n+1} > 0$ *and*

$$\frac{c_n c_{n+2}\det A_{n+1}}{(a_n c_{n+1} + c_n d_{n+1})(a_{n+1}c_{n+2} + c_{n+1}d_{n+2})} \le \frac{1}{4}$$

for n *sufficiently large, then all non-trivial solutions of* (1.1) *are nonoscillatory with respect to the second component.*

If $A_n \equiv A$, then

$$\begin{aligned}&\frac{b_n b_{n+2}\det A_{n+1}}{(a_{n+1}b_n + b_{n+1}d_n)(a_{n+2}b_{n+1} + b_{n+2}d_{n+1})}\\ &\quad = \frac{\det A}{(\mathrm{tr}A)^2} = \frac{c_n c_{n+2}\det A_{n+1}}{(a_n c_{n+1} + c_n d_{n+1})(a_{n+1}c_{n+2} + c_{n+1}d_{n+2})}.\end{aligned}$$

Hence, all non-trivial solutions of (1.1) are nonoscillatory with respect to the first component if and only if those are nonoscillatory with respect to the second component.

In this paper, we would like to clarify a distinction between difference equations and ordinary differential equations. For simplicity, let

$$q_n = \frac{b_n b_{n+2}\det A_{n+1}}{(a_{n+1}b_n + b_{n+1}d_n)(a_{n+2}b_{n+1} + b_{n+2}d_{n+1})}$$

and

$$r_n = \frac{c_n c_{n+2}\det A_{n+1}}{(a_n c_{n+1} + c_n d_{n+1})(a_{n+1}c_{n+2} + c_{n+1}d_{n+2})}.$$

We can notice that two positive sequences $\{q_n\}$ and $\{r_n\}$ play important roles in Theorems B and C, respectively. Theorems B and C demand that each term of the sequences is less than or equal to 1/4. However, it is thought that this demand is very strong. To weaken the condition that the sequences should satisfy, we pay our attention to a weighted sum of two adjacent terms of the sequences.

We choose an $N \in \mathbb{N}$ arbitrarily. Let p_i $(1 \le i \le N)$ be any real number that is larger than 1 and let p_i^* be the conjugate number of p_i; namely,

$$\frac{1}{p_i} + \frac{1}{p_i^*} = 1. \tag{1.10}$$

Then p_i^* is also greater than 1. We regard p_{N+1} as p_1. Then we have the following results.

Theorem 1.1. *Assume* (1.4) *and* (1.5). *Suppose that $b_n/b_{n+1} > 0$ for n sufficiently large, and*

$$p_i^* q_{2N(k-1)+2i-1} + p_{i+1} q_{2N(k-1)+2i} \le 1 \tag{1.11}$$

for $i = 1, 2, \ldots, N$ and k sufficiently large. Then all non-trivial solutions of (1.1) *are nonoscillatory with respect to the first component*.

Theorem 1.2. *Assume* (1.4) *and* (1.6). *Suppose that $c_n/c_{n+1} > 0$ for n sufficiently large, and*

$$p_i^* r_{2N(k-1)+2i-1} + p_{i+1} r_{2N(k-1)+2i} \le 1 \tag{1.12}$$

for $i = 1, 2, \ldots, N$ and k sufficiently large. Then all non-trivial solutions of (1.1) *are nonoscillatory with respect to the second component*.

To compare Theorems 1.1 and 1.2 with Theorems B and C, respectively, we choose 1 as $N \in \mathbb{N}$. Let $p_1 = p_1^* = 2$. Then the following corollaries hold.

Corollary 1.3. *Assume* (1.4) *and* (1.5). *Suppose that $b_n/b_{n+1} > 0$ for n sufficiently large, and*

$$2q_{2k-1} + 2q_{2k} \le 1$$

for k sufficiently large. Then all non-trivial solutions of (1.1) *are nonoscillatory with respect to the first component*.

Corollary 1.4. *Assume* (1.4) *and* (1.6). *Suppose that $c_n/c_{n+1} > 0$ for n sufficiently large, and*

$$2r_{2k-1} + 2r_{2k} \le 1$$

for k sufficiently large. Then all non-trivial solutions of (1.1) *are nonoscillatory with respect to the second component*.

Although Corollary 1.3 (or Corollary 1.4) has a pretty simple form, it contains Theorem B (or Theorem C) completely.

2. Transforming from system (1.1) to equation (1.9)

As was mentioned in Section 1, system (1.1) is equivalent to equation (1.7). Moreover, equation (1.7) is transformed into equation (1.9) with

$$\beta_n = \frac{(a_{n+1}b_n + b_{n+1}d_n)\det A_1}{b_n b_{n+1}\prod_{j=1}^{n}\det A_j} \quad \text{and} \quad \gamma_n = \frac{\det A_1}{b_{n+1}\prod_{j=1}^{n}\det A_j}.$$

Here, we regard $\prod_{j=1}^{0}\det A_j$ as 1. In fact, we have

$$\gamma_n \frac{b_{n+1}}{b_n}\det A_n = \frac{\det A_1}{b_n\prod_{j=1}^{n-1}\det A_j} = \gamma_{n-1}$$

and

$$\gamma_n \frac{a_{n+1}b_n + b_{n+1}d_n}{b_n} = \frac{\det A_1(a_{n+1}b_n + b_{n+1}d_n)}{b_{n+1}(\prod_{j=1}^{n}\det A_j)\, b_n} = \beta_n.$$

Since $b_n/b_{n+1} > 0$ for $n \in \mathbb{N}$, we see that b_n and b_{n+1} have the same sign. Hence, if assumptions (1.4) and (1.5) holds, then $\beta_n\gamma_n > 0$ for n sufficiently large. Since

$$\frac{\gamma_n^2}{\beta_n\beta_{n+1}} = \frac{b_n b_{n+2}\det A_{n+1}}{(a_{n+1}b_n + b_{n+1}d_n)(a_{n+2}b_{n+1} + b_{n+2}d_{n+1})},$$

Theorem B follows from Theorem A.

By the same manner, from equation (1.8) we obtain

$$\frac{\det A_1}{c_{n+1}\prod_{j=1}^{n}\det A_j}y_{n+1} + \frac{\det A_1}{c_n\prod_{j=1}^{n-1}\det A_j}y_{n-1} = \frac{(a_n c_{n+1} + c_n d_{n+1})\det A_1}{c_n c_{n+1}\prod_{j=1}^{n}\det A_j}y_n \tag{2.1}$$

for $n \in \mathbb{N}$. This difference equation has the same form as equation (1.9). Comparing the coefficients of both equations, we see that

$$\beta_n = \frac{(a_n c_{n+1} + c_n d_{n+1})\det A_1}{c_n c_{n+1}\prod_{j=1}^{n}\det A_j} \quad \text{and} \quad \gamma_n = \frac{\det A_1}{c_{n+1}\prod_{j=1}^{n}\det A_j}.$$

Since $c_n/c_{n+1} > 0$ for $n \in \mathbb{N}$, we see that c_n and c_{n+1} have the same sign. Hence, under the assumptions (1.4) and (1.6), the coefficients $\beta_n\gamma_n$ are positive for n sufficiently large. Since

$$\frac{\gamma_n^2}{\beta_n\beta_{n+1}} = \frac{c_n c_{n+2}\det A_{n+1}}{(a_n c_{n+1} + c_n d_{n+1})(a_{n+1}c_{n+2} + c_{n+1}d_{n+2})},$$

Theorem C follows from Theorem A.

3. Transforming into a difference equation of Riccati type

To prove Theorems 1.1 and 1.2, we should first note that Sturm's separation theorem holds for the difference equation

$$\frac{\det A_1}{b_{n+1}\prod_{j=1}^{n}\det A_j}x_{n+1}+\frac{\det A_1}{b_n\prod_{j=1}^{n-1}\det A_j}x_{n-1}=\frac{(a_{n+1}b_n+b_{n+1}d_n)\det A_1}{b_nb_{n+1}\prod_{j=1}^{n}\det A_j}x_n \tag{3.1}$$

and equation (2.1), provided that $b_n/b_{n+1}>0$, $c_n/c_{n+1}>0$ and $\det A_n>0$ for n sufficiently large. About the proof of Sturm's separation theorem concerning linear difference equations, see [5, pp. 321–322] for example. From Sturm's separation theorem it follows that if equation (3.1) (or (2.1)) has one non-trivial solution that is nonoscillatory, then all its non-trivial solutions are nonoscillatory.

Suppose that system (1.1) has a non-trivial solution $\{\mathbf{x}_n\}$ which is nonoscillatory with respect to the first component. Note that the first component $\{x_n\}$ is a nonoscillatory solution of (3.1). We can find an $m\in\mathbb{N}$ such that $x_n>0$ for $n\geq m$. Recall that

$$q_n=\frac{b_nb_{n+2}\det A_{n+1}}{(a_{n+1}b_n+b_{n+1}d_n)(a_{n+2}b_{n+1}+b_{n+2}d_{n+1})}.$$

Define

$$z_n=\frac{(a_{n+2}b_{n+1}+b_{n+2}d_{n+1})x_{n+1}}{b_{n+2}\det A_{n+1}x_n}$$

for $n\geq m$. Then it follows from (1.4) and (1.5) that $q_n>0$ for n sufficiently large. Since $b_n/b_{n+1}>0$ for n sufficiently large, we see that $z_n>0$. The sequence $\{z_n\}$ satisfies the first-order non-linear difference equation

$$q_nz_n+\frac{1}{z_{n-1}}=1,\qquad n=m+1,m+2,\ldots. \tag{3.2}$$

Equation (3.2) is often called a difference equation of *Riccati* type. From Riccati transformation we see that a nonoscillatory solution $\{x_n\}$ of (3.1) corresponds to a positive solution $\{z_n\}$ of (3.2) and the converse is also true. Hence, Sturm's separation theorem guarantees that all non-trivial solutions of (3.1) are nonoscillatory if and only if there exists an integer $\ell\geq m$ such that equation (3.2) has a solution $\{z_n\}$ satisfying $z_n>0$ for all $n\geq\ell$. We therefore have only to find a positive solution of (3.2) to prove that all non-trivial solutions of (1.1) are nonoscillatory with respect to the first component; namely, Theorem 1.1.

To prove Theorem 1.2, we can use the same procedure as that of the above-mentioned. Suppose that system (1.1) has a non-trivial solution $\{\mathbf{x}_n\}$ of (1.1) which is nonoscillatory with respect to the second component. Then there exists an $m\in\mathbb{N}$ such that $y_n>0$ for $n\geq m$. Hence, we can define

$$w_n=\frac{(a_{n+1}c_{n+2}+c_{n+1}d_{n+2})y_{n+1}}{c_{n+2}\det A_{n+1}y_n}$$

for $n \geq m$. It is easy to check that the sequence $\{w_n\}$ satisfies the Riccati difference equation

$$r_n w_n + \frac{1}{w_{n-1}} = 1, \qquad n = m+1, m+2, \dots, \tag{3.3}$$

where

$$r_n = \frac{c_n c_{n+2} \det A_{n+1}}{(a_n c_{n+1} + c_n d_{n+1})(a_{n+1} c_{n+2} + c_{n+1} d_{n+2})}.$$

Hence, by virtue of Sturm's separation theorem, we need only to find a positive solution of (3.3) to prove Theorem 1.2.

Since the proof of Theorem 1.2 is essentially the same as that of Theorem 1.1, we prove only Theorem 1.1

Proof of Theorem 1.1. From condition (1.11) we see that there exists a $K \in \mathbb{N}$ such that

$$p_{i+1} \leq \frac{1}{q_{2N(k-1)+2i}} \left(1 - p_i^* q_{2N(k-1)+2i-1}\right) \tag{3.4}$$

for $i = 1, 2, \dots, N$ and $k \geq K$. We choose a solution $\{z_n\}$ of (3.2) satisfying $z_{2N(K-1)} \geq p_1 > 1$. By (1.10) and (3.2), we have

$$\begin{aligned} z_{2N(K-1)+1} &= \frac{1}{q_{2N(K-1)+1}} \left(1 - \frac{1}{z_{2N(K-1)}}\right) \\ &\geq \frac{1}{q_{2N(K-1)+1}} \left(1 - \frac{1}{p_1}\right) = \frac{1}{p_1^* q_{2N(K-1)+1}} > 0. \end{aligned}$$

Hence, by (3.4) with $i = 1$ and $k = K$, we obtain

$$\begin{aligned} z_{2N(K-1)+2} &= \frac{1}{q_{2N(K-1)+2}} \left(1 - \frac{1}{z_{2N(K-1)+1}}\right) \\ &\geq \frac{1}{q_{2N(K-1)+2}} \left(1 - p_1^* q_{2N(K-1)+1}\right) \geq p_2 > 1. \end{aligned}$$

Similarly, if $z_{2N(K-1)+2i-2} \geq p_i$ $(i = 2, 3, \dots, N)$, then

$$\begin{aligned} z_{2N(K-1)+2i-1} &= \frac{1}{q_{2N(K-1)+2i-1}} \left(1 - \frac{1}{z_{2N(K-1)+2i-2}}\right) \\ &\geq \frac{1}{q_{2N(K-1)+2i-1}} \left(1 - \frac{1}{p_i}\right) = \frac{1}{p_i^* q_{2N(K-1)+2i-1}} > 0, \end{aligned}$$

$$\begin{aligned} z_{2N(K-1)+2i} &= \frac{1}{q_{2N(K-1)+2i}} \left(1 - \frac{1}{z_{2N(K-1)+2i-1}}\right) \\ &\geq \frac{1}{q_{2N(K-1)+2i}} \left(1 - p_i^* q_{2N(K-1)+2i-1}\right) \geq p_{i+1} > 1. \end{aligned}$$

By mathematical induction, we conclude that

$$z_n \geq \begin{cases} \dfrac{1}{p_i^* q_n} & \text{if } n = 2N(K-1)+2i-1 \\ p_{i+1} & \text{if } n = 2N(K-1)+2i \end{cases}$$

for $i = 1, 2, \ldots, N$. In particular $(i = N)$, we get

$$z_{2NK-1} \geq \frac{1}{p_N^* q_{2NK-1}} > 0 \quad \text{and} \quad z_{2NK} \geq p_{N+1} = p_1 > 1.$$

Using (3.4) with $k = K+1$ and repeating the same procedure, we have

$$z_{2NK+1} = \frac{1}{q_{2NK+1}}\left(1 - \frac{1}{z_{2NK}}\right) \geq \frac{1}{q_{2NK+1}}\left(1 - \frac{1}{p_1}\right) = \frac{1}{p_1^* q_{2NK+1}} > 0,$$
$$z_{2NK+2} = \frac{1}{q_{2NK+2}}\left(1 - \frac{1}{z_{2NK+1}}\right) \geq \frac{1}{q_{2NK+2}}(1 - p_1^* q_{2NK+1}) \geq p_2 > 1$$

and

$$z_{2NK+2i-1} = \frac{1}{q_{2NK+2i-1}}\left(1 - \frac{1}{z_{2NK+2i-2}}\right) \geq \frac{1}{q_{2NK+2i-1}}\left(1 - \frac{1}{p_i}\right) = \frac{1}{p_i^* q_{2NK+2i-1}} > 0,$$
$$z_{2NK+2i} = \frac{1}{q_{2NK+2i}}\left(1 - \frac{1}{z_{2NK+2i-1}}\right) \geq \frac{1}{q_{2NK+2i}}(1 - p_1^* q_{2NK+2i-1}) \geq p_{i+1} > 1$$

for $i = 1, 2, \ldots, N$. To sum up, we obtain

$$z_n \geq \begin{cases} \dfrac{1}{p_i^* q_n} & \text{if } n = 2NK+2i-1 \\ p_{i+1} & \text{if } n = 2NK+2i. \end{cases}$$

Similarly, the following relation holds:

$$z_n \geq \begin{cases} \dfrac{1}{p_i^* q_n} & \text{if } n = 2N(k-1)+2i-1 \\ p_{i+1} & \text{if } n = 2N(k-1)+2i \end{cases}$$

for $i = 1, 2, \ldots, N$ and $k \geq K$. Hence, the sequence $\{z_n\}$ is a positive solution of (3.2). We therefore conclude that all non-trivial solutions of (1.1) are nonoscillatory with respect to the first component. This completes the proof. □

In Theorems 1.1 and 1.2, we have focused on a weighted sum of each odd-numbered term and the next term of the sequences $\{q_n\}$ and $\{r_n\}$, respectively. As can be seen from the proof of Theorem 1.1, we can exchange condition (1.11) (resp., (1.12)) with a condition about a weighted sum of each even-numbered term and the next term of the sequence $\{q_n\}$ $\big($resp., $\{r_n\}\big)$ as follows.

Theorem 3.1. *Assume* (1.4) *and* (1.5). *Suppose that* $b_n/b_{n+1} > 0$ *for n sufficiently large, and*

$$p_i^* q_{2N(k-1)+2i} + p_{i+1} q_{2N(k-1)+2i+1} \leq 1 \tag{3.5}$$

for $i = 1,2,\ldots,N$ *and k sufficiently large. Then all non-trivial solutions of* (1.1) *are nonoscillatory with respect to the first component.*

Theorem 3.2. *Assume* (1.4) *and* (1.6). *Suppose that* $c_n/c_{n+1} > 0$ *for n sufficiently large, and*

$$p_i^* r_{2N(k-1)+2i} + p_{i+1} r_{2N(k-1)+2i+1} \leq 1$$

for $i = 1,2,\ldots,N$ *and k sufficiently large. Then all non-trivial solutions of* (1.1) *are nonoscillatory with respect to the second component.*

4. Periodic linear systems

To illustrate our results stated in Section 1, we give some examples in this section.

Example 4.1. Consider system (1.1) with

$$A_1 = \begin{pmatrix} 2 & 1 \\ \sqrt{6}/4 & \sqrt{6}/6 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} \sqrt{6}/6 & 1 \\ -\sqrt{3}/3 & 2\sqrt{2} \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 2\sqrt{2} & 1 \\ -8\sqrt{2}/7 & -3/7 \end{pmatrix}, \qquad A_4 = \begin{pmatrix} 1 & 1 \\ 16 & 23 \end{pmatrix},$$

and $A_{n+4} = A_n$ for $n \in \mathbb{N}$. Then all non-trivial solutions are nonoscillatory with respect to the first component.

In this example, it is clear that

$$\det A_1 = \frac{\sqrt{6}}{12}, \quad \det A_2 = \sqrt{3}, \quad \det A_3 = \frac{2}{7}\sqrt{2}, \quad \det A_4 = 7,$$

and $\det A(n+4) = \det A_n$ for $n \in \mathbb{N}$. Hence, assumption (1.4) is satisfied. It is obvious that assumption (1.5) is also satisfied, because a_n, b_n and d_n are positive numbers. In addition, we have

$$q_n = \frac{b_n b_{n+2} \det A_{n+1}}{(a_{n+1}b_n + b_{n+1}d_n)(a_{n+2}b_{n+1} + b_{n+2}d_{n+1})} = \begin{cases} 0.375 & \text{if } n = 4k-3 \\ 0.125 & \text{if } n = 4k-2 \\ 0.49 & \text{if } n = 4k-1 \\ 0.01 & \text{if } n = 4k \end{cases}$$

with $k \in \mathbb{N}$. Hence,

$$q_{4k-3} + q_{4k-2} = 0.5$$

and

$$q_{4k-1} + q_{4k} = 0.5.$$

This means that condition (1.11) holds for $N = 2$, $i = 1,2$ and $p_1 = p_2 = 2$. Thus, by Theorem 1.1, all non-trivial solutions of (1.1) are nonoscillatory with respect to the first component.

Theorem B is not available for Example 4.1 because $q_{4k-3} > 1/4$ and $q_{4k-1} > 1/4$ for $k \in \mathbb{N}$.

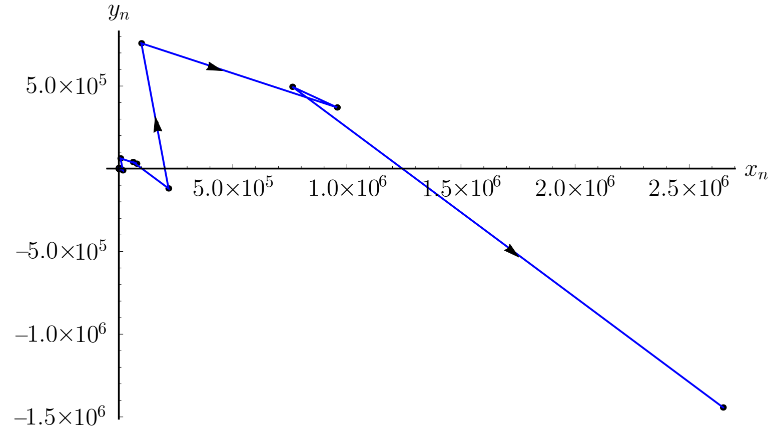

Figure 1: This line graph displays the motion of a solution $\{\mathbf{x}_n\}$ of (1.1). The initial condition of the solution is $\mathbf{x}_0 = {}^t(1,1)$.

Figure 1 shows that system (1.1) has a solution which is nonoscillatory with respect to the first component. The first component x_n is monotone increasing and tends to ∞ as $n \to \infty$. Recall that if there is a non-trivial solution which is nonoscillatory with respect to the first component, then all non-trivial solutions are nonoscillatory with respect to the first component. However, this solution is oscillatory with respect to the second component. Hence, by Sturm's separation theorem, all non-trivial solutions of (1.1) are oscillatory with respect to the second component.

The next example provides system (1.1) whose all non-trivial solutions are nonoscillatory with respect to both components.

Example 4.2. Consider system (1.1) with

$$A_1 = \begin{pmatrix} 5/14 & 1 \\ 1/2 & 21 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 4 & 1 \\ \sqrt{6}/12 & \sqrt{6}/24 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 7\sqrt{6}/24 & 1 \\ \sqrt{3}/6 & 2\sqrt{2} \end{pmatrix}, \qquad A_4 = \begin{pmatrix} 2\sqrt{2} & 1 \\ \sqrt{2}/7 & 3/14 \end{pmatrix},$$

and $A_{n+4} = A_n$ for $n \in \mathbb{N}$. Then all non-trivial solutions are nonoscillatory.

Since

$$\det A_1 = 7, \quad \det A_2 = \frac{\sqrt{6}}{12}, \quad \det A_3 = \sqrt{3}, \quad \det A_4 = \frac{2}{7}\sqrt{2},$$

and $\det A(n+4) = \det A_n$ for $n \in \mathbb{N}$, assumption (1.4) is satisfied. It is clear that assumption (1.5) is also satisfied. We can easily check that

$$q_n = \frac{b_n b_{n+2} \det A_{n+1}}{(a_{n+1}b_n + b_{n+1}d_n)(a_{n+2}b_{n+1} + b_{n+2}d_{n+1})} = \begin{cases} 0.01 & \text{if } n = 4k-3 \\ 0.375 & \text{if } n = 4k-2 \\ 0.125 & \text{if } n = 4k-1 \\ 0.49 & \text{if } n = 4k \end{cases}$$

with $k \in \mathbb{N}$. Hence, condition (3.5) holds for $N = 2$, $i = 1,2$ and $p_1 = p_2 = 2$. In fact,

$$q_{4k-2} + q_{4k-1} = 0.5$$

and

$$q_{4k} + q_{4k+1} = 0.5.$$

Thus, by Theorem 3.1, all non-trivial solutions of (1.1) are nonoscillatory with respect to the first component.

It is obvious that assumption (1.6) is also satisfied, because a_n, c_n and d_n are positive numbers. It is also easy to check that

$$r_n = \frac{c_n c_{n+2} \det A_{n+1}}{(a_n c_{n+1} + c_n d_{n+1})(a_{n+1}c_{n+2} + c_{n+1}d_{n+2})} = \begin{cases} 0.137\cdots & \text{if } n = 4k-3 \\ 0.200\cdots & \text{if } n = 4k-2 \\ 0.040\cdots & \text{if } n = 4k-1 \\ 0.411\cdots & \text{if } n = 4k \end{cases}$$

with $k \in \mathbb{N}$. Hence, condition (1.12) holds for $N = 2$, $i = 1,2$ and $p_1 = p_2 = 2$. In fact,

$$r_{4k-3} + r_{4k-2} = 0.337\cdots < 0.5$$

and

$$r_{4k-1} + r_{4k} = 0.452\cdots < 0.5.$$

Thus, by Theorem 1.2, all non-trivial solutions of (1.1) are nonoscillatory with respect to the second component.

However, since $q_{4k-2} > 1/4$, $q_{4k} > 1/4$ and $r_{4k} > 1/4$ for $k \in \mathbb{N}$, Theorems B and C are not applicable for Example 4.2.

From Figure 2 we see that there is a non-trivial solution of (1.1) which is nonoscillatory with respect to the first and second components. Both components x_n and y_n are monotone increasing and tend to ∞ as $n \to \infty$. Sturm's separation theorem guarantees that all non-trivial solutions of (1.1) are nonoscillatory.

In Examples 4.1 and 4.2, system (1.1) was periodic one with period 4. We next give a periodic linear difference system with larger size period.

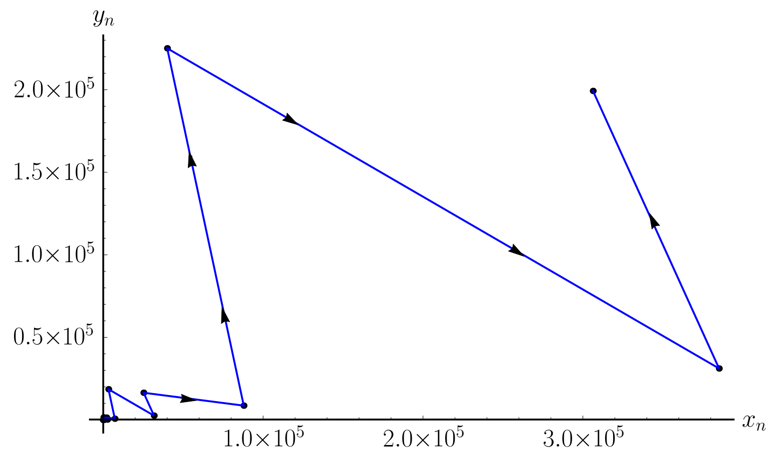

Figure 2: This line graph displays the motion of a solution $\{\mathbf{x}_n\}$ of (1.1). The initial condition of the solution is $\mathbf{x}_0 = {}^t(1,1)$.

Example 4.3. Consider system (1.1) with

$$A_1 = \begin{pmatrix} 6 & 1 \\ 1 & 1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 9 & 1 \\ 0 & 9 \end{pmatrix}, \qquad A_3 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad A_4 = \begin{pmatrix} 9 & 1 \\ -4 & 9 \end{pmatrix},$$

$$A_5 = \begin{pmatrix} 1 & 1 \\ 5/9 & 5 \end{pmatrix}, \qquad A_6 = \begin{pmatrix} 5 & 1 \\ 1 & 1 \end{pmatrix}, \qquad A_7 = \begin{pmatrix} 9 & 1 \\ -1 & 7 \end{pmatrix}, \qquad A_8 = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix},$$

and $A_{n+8} = A_n$ for $n \in \mathbb{N}$. Then all non-trivial solutions are nonoscillatory with respect to the first component.

Since

$$\det A_1 = 5, \qquad \det A_2 = 81, \qquad \det A_3 = 1, \qquad \det A_4 = 85,$$
$$\det A_5 = 40/9, \qquad \det A_6 = 4, \qquad \det A_7 = 64, \qquad \det A_8 = 10,$$

and $\det A_{n+8} = \det A_n$ for $n \in \mathbb{N}$, assumptions (1.4) is satisfied. It is clear that assumption (1.5) is also satisfied. We can check that

$$q_n = \frac{b_n b_{n+2} \det A_{n+1}}{(a_{n+1}b_n + b_{n+1}d_n)(a_{n+2}b_{n+1} + b_{n+2}d_{n+1})} = \begin{cases} 0.81 & \text{if } n = 8k-7 \\ 0.01 & \text{if } n = 8k-6 \\ 0.85 & \text{if } n = 8k-5 \\ 0.0\dot{4} & \text{if } n = 8k-4 \\ 0.04 & \text{if } n = 8k-3 \\ 0.64 & \text{if } n = 8k-2 \\ 0.10 & \text{if } n = 8k-1 \\ 0.05 & \text{if } n = 8k \end{cases}$$

with $k \in \mathbb{N}$. Hence, condition (3.5) holds for $N = 4$, $i = 1,2,3,4$ and $p_1 = p_2 = 10$, $p_3 = p_4 = 5/4$. In fact, since $p_1^* = p_2^* = 10/9$ and $p_3^* = p_4^* = 5$, we see that

$$p_1^* q_{8k-7} + p_2 q_{8k-6} = 1,$$
$$p_2^* q_{8k-5} + p_3 q_{8k-4} = 1,$$
$$p_3^* q_{8k-3} + p_4 q_{8k-2} = 1,$$
$$p_4^* q_{8k-1} + p_1 q_{8k} = 1.$$

Thus, by Theorem 3.1, all non-trivial solutions of (1.1) are nonoscillatory with respect to the first component.

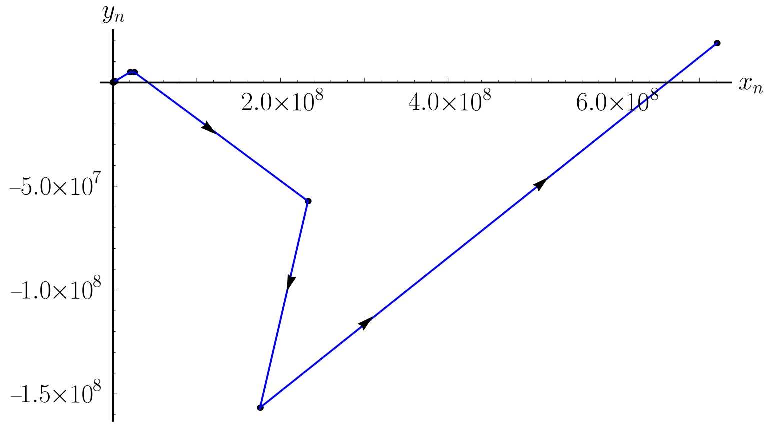

Figure 3: This line graph displays the motion of a solution $\{\mathbf{x}_n\}$ of (1.1). The initial condition of the solution is $\mathbf{x}_0 = {}^t(1,1)$.

As drawn in Figure 3, there is a non-trivial solution of (1.1) which is nonoscillatory with respect to the first component. Hence, from Sturm's separation theorem it turns that all non-trivial solutions of (1.1) are nonoscillatory with respect to the first component.

We mentioned the periodic system so that it might be easy to carry out a simulation in this section. Finally, the reader should note that our results can be applied even to non-periodic linear difference systems.

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