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UNIFORM ASYMPTOTIC STABILITY OF TIME-VARYING DAMPED HARMONIC OSCILLATORS

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Abstract. This paper presents sufficient conditions which guarantee that the equilibrium of the damped harmonic oscillator

$$x'' + h(t)x' + \omega^2 x = 0$$

is uniformly asymptotically stable, where $h\colon [0,\infty) \to [0,\infty)$ is locally integrable. These conditions work to suppress the rapid growth of the frictional force expressed by the integral amount of the damping coefficient h. The obtained sufficient conditions are compared with known conditions for uniform asymptotic stability. Two diagrams are included to facilitate understanding of the conditions. By giving a concrete example, remaining problems are pointed out.

1. Introduction

We consider the second-order linear differential equation

$$x'' + h(t)x' + \omega^2 x = 0, \tag{1}$$

where the prime denotes d/dt, the coefficient h is a nonnegative and locally integrable function on $[0,\infty)$, and the number ω is a positive constant. The only equilibrium of (1) is the origin $(x,x') = (0,0)$. Equation (1) is often called the *damped harmonic oscillator* when h is a positive constant. Although the damped harmonic oscillator has a very simple form, there are extremely wide applications in science and engineering. Equation (1) has been studied as one of the important physical phenomenon models by many researchers.

In the qualitative theory of differential equations, the study of asymptotic stability and uniform asymptotic stability occupy very important positions. The purpose of this paper is to present some growth condition about h for the equilibrium of (1) to be uniformly asymptotically stable and to clarify the relationship between these conditions. Before advancing to the main subject, it is useful to briefly describe the history of study of the asymptotic stability of (1) and the results obtained.

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Since equation (1) is linear, if the equilibrium is attractive, then it is stable. Hence, we need only show that each solution of (1) and its derivative tend to zero as $t \to \infty$ in order to prove that the equilibrium is asymptotically stable. Many efforts have been made to find sufficient (also necessary and sufficient) conditions which guarantee that the equilibrium of (1) is asymptotically stable (for example, see [1,3,4,9–13,16,18,19,21,24–27,31]). Among them, we should mention especially the criterion given by Smith [21, Theorems 1 and 2]. Let

$$H(t) = \int_0^t h(s)ds.$$

Then the following result holds.

Theorem A. *Suppose that*

$$\text{(2)} \qquad \textit{there exists an } \underline{h} > 0 \textit{ such that } h(t) \geq \underline{h} \textit{ for } t \geq 0.$$

Then the equilibrium of (1) *is asymptotically stable if and only if*

$$\text{(S)} \qquad \int_0^\infty \frac{\int_0^t e^{H(s)}ds}{e^{H(t)}}\,dt = \infty.$$

When h satisfies condition (2), namely, h has a positive lower bound $\underline{h}$, it is often called *large damping*. Smith's condition (S) is satisfied when h has an upper bound $\overline{h}$ or $h(t) = t$. On the other hand, condition (S) is not satisfied when $h(t) = t^2$ (for the proof, see [11]). From these facts, we see that condition (S) prohibits rapid growth of the damping coefficient h. Since condition (S) is necessary and sufficient for the asymptotic stability of (1), it is not too much to say that condition (S) is very excellent. However, the weak point is that it is hard to check whether condition (S) is satisfied or not. Although Artstein and Infante [1] did not point out this fact, they gave another growth condition that guarantees the asymptotic stability of (1) as follows.

Theorem B. *Suppose that condition* (2) *holds. Then*

$$\text{(A)} \qquad \limsup_{t\to\infty} \frac{H(t)}{t^2} < \infty$$

implies that the equilibrium of (1) *is asymptotically stable.*

Artstein and Infante's condition (A) requires that H has to grow more slowly than t^2. Artstein and Infante [1] also showed that the exponent 2 of t is best possible in the sense that it cannot be replaced by $2 + \varepsilon$ for any $\varepsilon > 0$. Of course, condition (A) is not as sharp as condition (S). For example, consider $h(t) = (2+t)\log(2+t)$. Then it is clear that $H(t)/t^2$ is unbounded. Hence, condition (A) is not satisfied. However, by means of Ballieu and Peiffer's result [3, Corollary 7], we can verify that the equilibrium of (1) is asymptotically stable in this example.

The advantage of condition (A) is that it is easy to check. When an indefinite integral H of h is found, we may judge whether condition (A) is satisfied or not. By numerical computation, it may be easy to check condition (A). However, it is very hard to confirm condition (S) even with numerical computation.

Since condition (A) is merely a sufficient condition for the asymptotic stability of (1), it follows from condition (S). From another viewpoint, Hatvani et al. [11]

verified that condition (A) implies condition (S). They proved that condition (S) is equivalent to the discrete growth condition

$$\text{(D)} \qquad \sum_{n=1}^{\infty}\bigl(H^{-1}(nc)-H^{-1}((n-1)c)\bigr)^2=\infty \quad \text{for any } c>0,$$

where

$$H^{-1}(s)\stackrel{\text{def}}{=}\min\bigl\{t\in[0,\infty)\colon H(t)\geq s\bigr\}, \quad s\in[0,\infty),$$

provided that $H(t)$ diverges to ∞ as $t\to\infty$ (see [11, Theorem 1.1]). They also showed that condition (A) implies condition (D) in the proof of Corollary 3.7 in [11] (see also [1]). Moreover, they gave another growth condition,

$$\text{(H)} \qquad \sum_{i=N}^{\infty}\frac{1}{\int_{i-1}^{i}h(s)ds}=\infty,$$

for any fixed natural number N (their original form is slightly different) and clarified that condition (H) implies condition (D) under certain conditions including condition (2) (see [11, Corollary 3.6]). We can show that condition (A) implies condition (H) by using Artstein and Infante's result [1, Lemma] (for the proof, see Appendix). To sum up, we have the following diagram.

$$\begin{array}{ccccccc}
 & & & (2) & & & \\
 & & & \downarrow & & & \\
 & & (\mathrm{A}) & \Longrightarrow & (\mathrm{H}) & & \\
(2) & \longrightarrow & \Downarrow & \swarrow\!\!\!\!\swarrow & \longleftarrow & \lim\limits_{t\to\infty}x'(t)=0, & \text{certain conditions} \\
 & & (\mathrm{D}) & \Longleftrightarrow & (\mathrm{S}) & \Longleftrightarrow\ [\mathrm{AS}] & \Uparrow \\
 & & & & \uparrow & \uparrow & (2) \\
\lim\limits_{t\to\infty}H(t)=\infty & & \Longleftarrow & & (2) & &
\end{array}$$

FIGURE 1. The marks "⟶", "⟹", "⟺" and [AS] mean "addition to", "implies", "if and only if" and the asymptotic stability of (1), respectively.

The equilibrium of (1) is said to be *asymptotically stable* [AS] if

$$\lim_{t\to\infty}x(t)=\lim_{t\to\infty}x'(t)=0$$

for every solution x of (1). As is well known, the equilibrium is not necessarily uniformly asymptotically stable even if it is [AS]. We need to check that each solution of (1) and its derivative converge to zero with the speed of the same level in order to prove that the equilibrium is uniformly asymptotically stable. In this sense, we need to pay close attention to the analysis of uniform asymptotic stability.

To strictly describe definitions, we give some notation. Let $\mathbf{x}(t)=(x(t),x'(t))$ and $\mathbf{x}_0\in\mathbb{R}^2$, and let $\|\cdot\|$ be any suitable norm. We denote the solution of (1) through $(t_0,\mathbf{x}_0)$ by $\mathbf{x}(t;t_0,\mathbf{x}_0)$. The uniqueness of solutions of (1) is guaranteed for the initial value problem.

The equilibrium is said to be *eventually uniformly stable* [EvUS] if for any $\varepsilon>0$, there exist an $\alpha(\varepsilon)\geq 0$ and a $\delta(\varepsilon)>0$ such that $\|\mathbf{x}_0\|<\delta$ and $t_0\geq\alpha$ imply that $\|\mathbf{x}(t;t_0,\mathbf{x}_0)\|<\varepsilon$ for all $t\geq t_0$. If we can choose $\alpha(\varepsilon)=0$, the equilibrium is said to be *uniformly stable* [US]. The equilibrium is said to be *eventually uniformly* *attractive* [EvUA] if there exist an $\alpha_0 \geq 0$ and a $\delta_0 > 0$, and if for every $\eta > 0$ there is a $T(\eta) > 0$ such that $t_0 \geq \alpha_0$ and $\|\mathbf{x}_0\| < \delta_0$ imply that $\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| < \eta$ for all $t \geq t_0 + T(\eta)$. If we can choose $\alpha_0 = 0$, the equilibrium is said to be *uniformly attractive* [UA]. The equilibrium is *eventually uniformly asymptotically stable* [EvUAS] if it is [EvUS] and [EvUA]. The equilibrium is *uniformly asymptotically stable* [UAS] if it is [US] and [UA]. With respect to the various definitions of stability, the reader may refer to the books [2, 5, 6, 15, 20, 32] for example.

It is well known that the equilibrium of (1) is uniformly asymptotically stable if and only if it is *exponentially asymptotically stable* [ExpAS]; namely, there exists a $\kappa > 0$ and, for any $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that $t_0 \geq 0$ and $\|\mathbf{x}_0\| < \delta(\varepsilon)$ imply that $\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| < \varepsilon \exp(-\kappa(t - t_0))$ for all $t \geq t_0$. If the equilibrium of (1) is [ExpAS], then the existence of a good Lyapunov function $V(\cdot,\cdot)\colon [0,\infty)\times\mathbb{R}^n \to \mathbb{R}$ that satisfies the following conditions is guaranteed:

(i) $a(\|\mathbf{x}\|) \leq V(t,\mathbf{x}) \leq b(\|\mathbf{x}\|)$,

(ii) $\dot{V}_{(1)}(t,\mathbf{x}) \leq -\,c(\|\mathbf{x}\|)$ or $\dot{V}_{(1)}(t,\mathbf{x}) \leq -\,d\,V(t,\mathbf{x})$,

(iii) $|V(t,\mathbf{x}_1) - V(t,\mathbf{x}_2)| \leq f(t)\|\mathbf{x}_1 - \mathbf{x}_2\|$

on $[0,\infty) \times \mathbb{R}^n$, where a, b and c are continuous increasing and positive definite functions, d is a positive constant and f is a positive suitable function (this is called a converse theorem on [UAS]). However, if the zero solution of (1) is merely only asymptotically stable, such a good Lyapunov function does not necessarily exist. This is a big difference between [UAS] and [AS]. By using the converse theorem on [UAS], we can show that the uniform asymptotic stability is maintained even if a small perturbation term is added to equation (1). Small errors cannot be ignored in model design. For this reason, it is necessary to consider the perturbation problem in actual phenomena analysis. From this point of view, the study of [UAS] is very important.

In this paper, we use the terminology "uniformly with respect to $\sigma \geq 0$". This means the following: Let $f_\sigma : \mathbb{R} \to \mathbb{R}$ be a family of functions parametrized by $\sigma \geq 0$. We say that $\lim_{t\to\infty} f_\sigma(t) = \infty$ uniformly with respect to $\sigma \geq 0$ if, and only if, for any $M > 0$ there exists a $T \geq 0$ such that $\sigma \geq 0$ and $t \geq T$ imply $f_\sigma(t) \geq M$. We also use the symbol $[c]$ to mean the greatest integer that is less than or equal to a real number c.

Sugie and Onitsuka [30, Theorem 1.1] gave the following result.

Theorem C. *Suppose that*

$$\liminf_{t\to\infty} \int_t^{t+d} h(s)ds > 0 \quad \textit{for every } d > 0. \tag{3}$$

If

$$\lim_{t\to\infty} \int_\sigma^{t+\sigma} \frac{\int_\sigma^s e^{H(\tau)}d\tau}{e^{H(s)}}\, ds = \infty \quad \textit{uniformly with respect to } \sigma \geq 0, \tag{4}$$

then the equilibrium of (1) *is uniformly asymptotically stable.*

When condition (3) holds, the damping coefficient h is said to be *integrally positive*. The concept of the integral positivity was introduced by Matrosov [14] (see also [7–9, 17, 24, 28, 29]). It is obvious that condition (2) implies condition (3). However, the converse is not always true. Integrally positive functions are allowed to have an infinite number of zeros. A typical example of integrally positive functions is $\sin^2 t$.

Let us compare condition (4) with condition (S). We notice that both are double integrals of $\exp(H(\tau) - H(s))$. Condition (4) requires that this double integral diverges uniformly with respect to σ. On the other hand, condition (S) only needs to diverge when $\sigma = 0$. Hence, we may say that (4) is a uniform divergence condition. A growth condition similar to condition (4) was first presented by Hatvani [9, Theorem 2.5] as a sufficient condition for the zero solution of a certain two-dimensional linear system to become asymptotically stable.

By the same method as in the proof of Theorem C, we can obtain the following result.

Theorem D. *Suppose that condition* (3) *holds. If*

$$\text{(SU)}\qquad \begin{array}{l} \textit{there exists an } m \geq 0 \textit{ such that } \displaystyle\lim_{t\to\infty} \int_\sigma^{t+\sigma} \frac{\int_\sigma^s e^{H(\tau)}d\tau}{e^{H(s)}}\, ds = \infty \\ \textit{uniformly with respect to } \sigma \geq m, \end{array}$$

then the equilibrium of (1) *is eventually uniformly asymptotically stable.*

For nonlinear differential equations, the concept of [EvUAS] is different from that of [UAS]. Of course, the equilibrium is uniformly asymptotically stable, so it is eventually uniformly asymptotically stable; namely, [UAS] implies [EvUAS]. Strauss and Yorke [22, Lemma 2.7] gave a necessary and sufficient condition for the converse to be true as follows (see also [23]).

Theorem E. *Suppose that the equilibrium is eventually uniformly asymptotically stable. Then it is uniformly asymptotically stable if and only if the zero function is a unique solution defined on the interval* $[t_0, \infty)$.

Since equation (1) is linear, from Theorem E it turns out that [EvUAS] is equivalent to [UAS]. Hence, condition (SU) is a growth condition on uniform asymptotic stability. In this paper, based on the uniform divergence condition (SU), we intend to present other growth conditions on uniform asymptotic stability and give a correlation diagram showing their relation.

2. Conditions for suppressing the rapid growth of h

As mentioned in Section 1, Hatvani et al. [11] presented the discrete growth condition for the asymptotic stability of (1), which is equivalent to Smith's condition (S). Inspired by this result, Sugie and Onitsuka [30, Theorem 4.2] gave the discrete growth condition

$$\text{(5)}\quad \lim_{n\to\infty} \sum_{i=N}^{n+N} \left(H^{-1}(i) - H^{-1}(i-1)\right)^2 = \infty \quad \text{uniformly with respect to } N \in \mathbb{N},$$

for uniform asymptotic stability of (1) and proved that condition (5) implies the uniform divergence condition (4) under the assumption (3). Using the same idea, we can show that

$$\text{(DU)}\qquad \begin{array}{l} \text{there exists an } N^* \in \mathbb{N} \text{ such that } \displaystyle\lim_{n\to\infty} \sum_{i=N}^{n+N} \left(H^{-1}(i) - H^{-1}(i-1)\right)^2 = \infty \\ \text{uniformly with respect to } N \geq N^* \end{array}$$

implies condition (SU). We may regard condition (DU) as a discrete version of (D).

Unfortunately, in general, it is not so easy to check whether a given h satisfies conditions (SU) and (DU). In this section, we propose other growth conditions corresponding to conditions (A) and (H) given by Artstein and Infante [1] and Hatvani et al. [11], respectively. We also reveal implications between conditions (SU), (DU) and these new growth conditions.

Theorem 1. *Suppose that* $\lim_{t\to\infty} H(t) = \infty$. *If*

$$\text{(AU)} \qquad \begin{gathered} \textit{there exists an } \varepsilon_0 > 0 \textit{ and an } m \geq 0 \textit{ such that} \\ \limsup_{t\to\infty} \frac{1}{t^{2-\varepsilon_0}} \int_\tau^{t+\tau} h(s)ds < \infty \textit{ uniformly with respect to } \tau \geq m, \end{gathered}$$

then condition (DU) *holds.*

Remark 1. If condition (3) is satisfied, then $\lim_{t\to\infty} H(t) = \infty$. Condition (AU) is a uniform convergence version of condition (A) of Artstein and Infante [1].

Theorem 2. *Suppose that*

$$\text{(6)} \qquad \textit{there exists an } N \in \mathbb{N} \textit{ such that } a_n \stackrel{\text{def}}{=} \int_{n-1}^{n} h(s)ds > 0 \textit{ for all } n \geq N.$$

If condition (AU) *is satisfied, then*

$$\text{(HU)} \qquad \begin{gathered} \textit{there exists an } N^* \in \mathbb{N} \textit{ such that } \lim_{n\to\infty} \sum_{i=N}^{n+N} \frac{1}{\int_{i-1}^{i} h(s)ds} = \infty \\ \textit{uniformly with respect to } N \geq N^*. \end{gathered}$$

Remark 2. Condition (3) implies condition (6). Condition (HU) is a uniform divergence version of condition (H) of Hatvani et al. [11].

Theorem 3. *Suppose that*

$$\text{(7)} \qquad \begin{gathered} \textit{there exists a } T > 0 \textit{ such that } 1/h \textit{ is a bounded function on } [T, \infty) \\ \textit{and } (1/h)' \textit{ is a function on } [T, \infty) \textit{ that is bounded from above.} \end{gathered}$$

Then condition (HU) *yields condition* (SU).

Combining Theorems 1, 2 and 3 with Theorems D and E, we can give the diagram for [UAS] shown in Figure 2:

$$\begin{array}{cccccc}
 & (3) & \Longrightarrow & (6) & & \\
 & & & \downarrow & & (2) \Longrightarrow (3) \\
 & (\text{AU}) & \Longrightarrow & (\text{HU}) & & \Uparrow \\
\lim_{t\to\infty} H(t) = \infty \longrightarrow & \Downarrow & & \Downarrow & \longleftarrow & (7) \\
\Uparrow & (\text{DU}) & \Longrightarrow & (\text{SU}) & \Longrightarrow & [\text{UAS}] \\
(3) & & \uparrow & & \uparrow & \\
 & & (3) & & (3) &
\end{array}$$

FIGURE 2. The marks "$\longrightarrow$", "$\Longrightarrow$", "$\Longleftrightarrow$" and [UAS] mean "addition to", "implies", "if and only if" and the uniform asymptotic stability of (1), respectively.

3. Proofs

Proof of Theorem 1. Let $t_0 = 0$ and $t_n = H^{-1}(n)$ for all $n \in \mathbb{N}$. Since $\lim_{t\to\infty} H(t) = \infty$, the sequence $\{t_n\}$ is increasing and diverges to ∞ as $n \to \infty$. Hence, we can choose an $N^* \in \mathbb{N}$ so that

$$t_{N^*-1} \geq m.$$

Define $\Delta t_n = t_n - t_{n-1}$. In order to show condition (DU), it suffices to show that for any $L > 0$ there exists an $M(L) \in \mathbb{N}$ such that $n \geq M$ implies that

$$\sum_{i=N}^{n+N} (\Delta t_i)^2 > L$$

for any $N \geq N^*$. From (AU) it follows that

$$\begin{aligned}&\text{there exists a } K > 0 \text{ and a } T > 0 \text{ such that } \tau \geq m\\ &\text{implies that } \int_\tau^{t+\tau} h(s)ds < Kt^{2-\varepsilon_0} \ \text{ for } t \geq T.\end{aligned} \tag{8}$$

For any $L > 0$, let

$$M(L) = \max\left\{1,\ \left[KT^{2-\varepsilon_0}\right],\ \left[K^{2/\varepsilon_0} L^{(2-\varepsilon_0)/\varepsilon_0}\right]\right\} \in \mathbb{N}. \tag{9}$$

Suppose that there exists an $N_0 \in \mathbb{N}$ with $N_0 \geq N^*$ such that

$$t_{N_0+M} < t_{N_0-1} + T.$$

Then, since H is an increasing function on $[0,\infty)$ and $H(t_n) = n$ for all $n \in \mathbb{N}$, we see that

$$\begin{aligned} N_0 + M = H(t_{N_0+M}) &\leq H(t_{N_0-1} + T)\\ &= H(t_{N_0-1}) + \int_{t_{N_0-1}}^{t_{N_0-1}+T} h(s)ds\\ &= N_0 - 1 + \int_{t_{N_0-1}}^{t_{N_0-1}+T} h(s)ds. \end{aligned}$$

Since $\{t_n\}$ is an increasing sequence, we see that $t_{N_0-1} \geq t_{N^*-1} \geq m$. Using (8) with $\tau = t_{N_0-1}$ and $t = T$, we obtain

$$\int_{t_{N_0-1}}^{t_{N_0-1}+T} h(s)ds < KT^{2-\varepsilon_0}.$$

Hence, we have

$$M < KT^{2-\varepsilon_0} - 1.$$

This contradicts (9). We therefore conclude that

$$t_{N+M} \geq t_{N-1} + T \quad \text{for any } N \geq N^*. \tag{10}$$

We can rewrite (8) as

$$\int_\tau^t h(s)ds < K(t-\tau)^{2-\varepsilon_0} \quad \text{for } \ t \geq \tau + T. \tag{11}$$

Using (10) and (11), we get

$$\int_{t_{N-1}}^{t_{N+M}} h(s)ds < K(t_{N+M} - t_{N-1})^{2-\varepsilon_0} \quad \text{for any } N \geq N^*.$$

Hence, we have

$$\frac{M+1}{(t_{N+M}-t_{N-1})^{2-\varepsilon_0}} = \frac{H(t_{N+M})-H(t_{N-1})}{(t_{N+M}-t_{N-1})^{2-\varepsilon_0}} = \frac{1}{(t_{N+M}-t_{N-1})^{2-\varepsilon_0}}\int_{t_{N-1}}^{t_{N+M}} h(s)ds < K;$$

that is,

$$(t_{N+M}-t_{N-1})^{2-\varepsilon_0} > \frac{M+1}{K} \quad \text{for any } N \ge N^*. \tag{12}$$

By means of the Cauchy-Bunyakovski-Schwarz inequality, we have

$$\left(\sum_{i=N}^{M+N} \Delta t_i\right)^2 \le \sum_{i=N}^{M+N} 1^2 \sum_{i=N}^{M+N} (\Delta t_i)^2 = (M+1)\sum_{i=N}^{M+N} (\Delta t_i)^2$$

for any $N \ge N^*$. Hence, it follows from (12) that $n \ge N$ implies that

$$\begin{aligned}\sum_{i=N}^{n+N} (\Delta t_i)^2 \ge \sum_{i=N}^{M+N} (\Delta t_i)^2 &\ge \frac{1}{M+1}\left(\sum_{i=N}^{M+N} \Delta t_i\right)^2 \\ &= \frac{1}{M+1}\left(\left(t_{N+M}-t_{N-1}\right)^{2-\varepsilon_0}\right)^{2/(2-\varepsilon_0)} \\ &> \frac{1}{M+1}\left(\frac{M+1}{K}\right)^{2/(2-\varepsilon_0)} = \frac{(M+1)^{\varepsilon_0/(2-\varepsilon_0)}}{K^{2/(2-\varepsilon_0)}}\end{aligned}$$

for any $N \ge N^*$. From (9) it turns out that

$$M > K^{2/\varepsilon_0} L^{(2-\varepsilon_0)/\varepsilon_0} - 1.$$

Hence, we obtain

$$\sum_{i=N}^{n+N} (\Delta t_i)^2 > L,$$

which is our desired estimate. This completes the proof. □

Proof of Theorem 2. Note that condition (8) holds as in the proof of Theorem 1. For any $L > 0$, let

$$M(L) = \max\left\{1,\ [T],\ \left[(KL)^{1/\varepsilon_0}\right]\right\} \in \mathbb{N}. \tag{13}$$

From (6), we can find an $N_* \in \mathbb{N}$ so that $a_n > 0$ for $n \ge N_*$. Hence, we have

$$(M+1)^2 = \left(\sum_{i=N}^{M+N} 1\right)^2 = \left(\sum_{i=N}^{M+N} \sqrt{a_i}\,\frac{1}{\sqrt{a_i}}\right)^2$$

for any $N \ge N_*$. Using the Cauchy-Bunyakovski-Schwarz inequality, we obtain

$$\left(\sum_{i=N}^{M+N} \sqrt{a_i}\,\frac{1}{\sqrt{a_i}}\right)^2 \le \sum_{i=N}^{M+N} a_i \sum_{i=N}^{M+N} \frac{1}{a_i} \quad \text{for any } N \ge N_*.$$

Let

$$N^* = \max\left\{m+1,\ N_*\right\}.$$

From (13) it turns out that $M > T - 1$. It is obvious that $N - 1 \geq N^* - 1 \geq m$ for $N \geq N^*$. Hence, we can use inequality (8) with $t = M + 1 > T$ and $\tau = N - 1$ for any $N \geq N^*$ and get

$$\sum_{i=N}^{M+N} a_i = \int_{N-1}^{M+N} h(s)ds < K(M+1)^{2-\varepsilon_0}$$

for $N \geq N^*$. We therefore conclude that

$$\sum_{i=N}^{M+N} \frac{1}{a_i} > \frac{1}{K}(M+1)^{\varepsilon_0} \quad \text{for any } N \geq N^*.$$

Using (13) again, we obtain

$$\sum_{i=N}^{n+N} \frac{1}{\int_{i-1}^{i} h(s)ds} \geq \sum_{i=N}^{M+N} \frac{1}{a_i} > \frac{1}{K}(M+1)^{\varepsilon_0} > L$$

for any $n \geq M$ and $N \geq N^*$, namely, condition (HU). □

By using the same method of Sugie and Onitsuka [30, Theorem 3.1], we can prove the following result (we omit the proof).

Lemma 4. *Suppose that there exists a function $k\colon [0,\infty) \to [0,\infty)$ and a $T > 0$ such that*

$$\begin{cases} 0 \leq h(t) \leq k(t) \ \ \textit{for } t \geq T, \\ \dfrac{1}{k} \ \textit{is a bounded function on } [T,\infty), \textit{ and} \\ \left(\dfrac{1}{k}\right)' \ \textit{is a function on } [T,\infty) \textit{ that is bounded from above.} \end{cases}$$

If

$$\begin{aligned} &\textit{there exists an } m \geq 0 \textit{ such that } \lim_{t\to\infty} \int_{\sigma}^{t+\sigma} \frac{1}{k(s)}\, ds = \infty \\ &\textit{uniformly with respect to } \sigma \geq m. \end{aligned}$$

then condition (SU) *holds.*

Remark 3. As can be seen from the condition (13), the damping coefficient h need not even be differentiable.

Proof of Theorem 3. In order to prove Theorem 3, it suffices, in view of Lemma 4, to show that

$$\begin{aligned} &\text{there exists an } m \geq 0 \text{ such that } \lim_{t\to\infty} \int_{\sigma}^{t+\sigma} \frac{1}{h(s)}\, ds = \infty \\ &\text{uniformly with respect to } \sigma \geq m. \end{aligned} \tag{14}$$

From (7), we can find a $c > 0$ and $T^* > 0$ such that

$$\frac{1}{h(t)} \leq c \quad \text{for } \ t \geq T^*.$$

Hence, we see that $h(t) \geq 1/c > 0$ for all $t \geq T^*$. Let $i^* = [T^*] + 2 \in \mathbb{N}$. Then, by the Cauchy-Bunyakovski-Schwarz inequality, we obtain

$$1 = \left(\int_{i-1}^{i} ds\right)^2 = \left(\int_{i-1}^{i} \sqrt{h(s)}\,\frac{1}{\sqrt{h(s)}}\,ds\right)^2 \leq a_i \int_{i-1}^{i} \frac{1}{h(s)}\,ds,$$

where $\{a_i\}$ is the sequence given in Theorem 2. Hence, we have

$$\int_{i-1}^{i} \frac{1}{h(s)}\,ds \geq \frac{1}{a_i} \quad \text{for all } \ i \geq i^*.$$

Let $N^{**} = \max\left\{N^*, i^*\right\}$. Then, from condition (HU) it turns out that

for any $L > 0$ there exists an $M(L) \in \mathbb{N}$ such that $N \geq N^{**}$ implies that

$$\int_{N-1}^{M+N} \frac{1}{h(s)}\,ds = \sum_{i=N}^{M+N} \int_{i-1}^{i} \frac{1}{h(s)}\,ds \geq \sum_{i=N}^{M+N} \frac{1}{a_i} > L. \tag{15}$$

Let $m = N^{**} - 1$. Then, for any $\sigma \geq m$, there exists an $N \in \mathbb{N}$ so that

$$N - 2 < \sigma \leq N - 1.$$

It is clear that $N \geq N^{**}$. Let $T(L) = M(L) + 2$. Then, using (15), we get

$$\int_{\sigma}^{t+\sigma} \frac{1}{h(s)}\,ds \geq \int_{N-1}^{T+N-2} \frac{1}{h(s)}\,ds = \int_{N-1}^{M+N} \frac{1}{h(s)}\,ds > L$$

for $t \geq T$. Hence, condition (14) is satisfied. □

4. Discussion

As shown in Figure 2, condition (AU) is the most concise condition which guarantees that the equilibrium of (1) is uniformly asymptotically stable. By contrast, the condition that is harder to check is condition (SU), which includes other conditions. In particular, it is difficult to judge whether the divergence of the double integral in condition (SU) is uniform with respect to σ even if it diverges to ∞. We give a simple example to show this situation.

Example 1. Consider equation (1) with

$$h(t) = \begin{cases} 1 + n & \text{if } \ n - 1/n \leq t \leq n, \\ 1 & \text{if } \ n < t < n + 1 - 1/(n+1) \end{cases}$$

for each $n \in \mathbb{N}$. Then the equilibrium is uniformly asymptotically stable.

It is clear that condition (2) is satisfied with $\underline{h} = 1$. Hence, condition (3) is also satisfied. We can easily calculate the integral H as follows:

$$s = H(t) = \begin{cases} (1+n)\,t - n(n-1) & \text{if } \ n - 1/n \leq t \leq n, \\ t + n & \text{if } \ n < t < n + 1 - 1/(n+1) \end{cases}$$

for each $n \in \mathbb{N}$ (see Figure 3). Hence, H is a strictly increasing function on $[0, \infty)$ and $\lim_{t\to\infty} H(t) = \infty$. Since

for any $\tau \geq 0$ and any $t \geq 0$, there exists an $n \in \mathbb{N}$ and an $m \in \mathbb{N}$

such that $\ n - 1 \leq \tau < n \ $ and $\ m - 1 \leq t < m$,

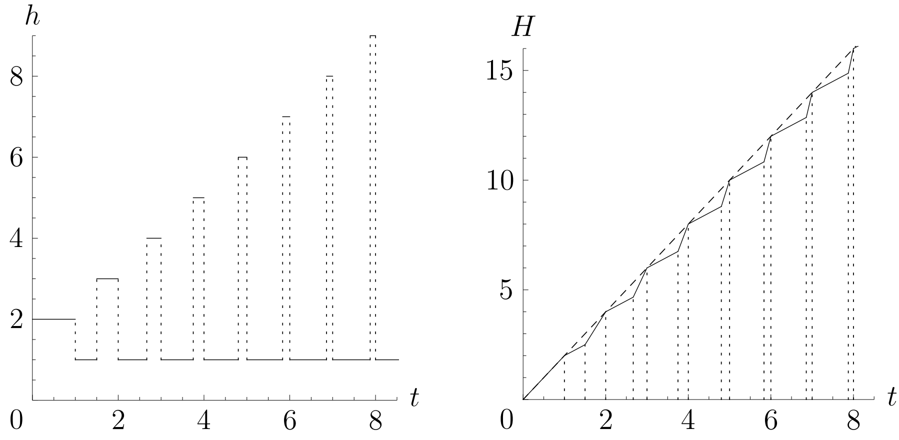

FIGURE 3. The graphs of the functions h and H

we see that

$$\begin{aligned}\int_{\tau}^{t+\tau} h(s)ds &= H(t+\tau) - H(\tau) < H(m+n) - H(n-1)\\ &= 2(m+n) - 2(n-1) = 2(m+1) < 2(t+2).\end{aligned}$$

Let $\varepsilon_0 = 1$ and $m = 0$. Then, we see that $\tau \geq m$ implies that

$$\frac{1}{t^{2-\varepsilon_0}}\int_{\tau}^{t+\tau} h(s)ds < \frac{2(t+2)}{t} \leq 3 \quad \text{for } \ t \geq 4.$$

This means that condition (AU) holds. Hence, from the diagram for [UAS] shown in Figure 2, we see that the equilibrum is uniformly asymptotically stable.

Because condition (AU) is satisfied, conditions (DU), (HU) and (SU) are also satisfied as can be seen from the diagram for [UAS]. In this example, we can directly verify that conditions (DU) and (HU) hold. However, it would be difficult to show that condition (SU) is satisfied.

Since H is a strictly increasing function diverging to ∞, the function H^{-1} is the inverse function of H. We can obtain the inverse function H^{-1} by a straightforward calculation as follows:

$$t = H^{-1}(s) = \begin{cases} \dfrac{s + n(n-1)}{1+n} & \text{if } \ 2n - 1 - 1/n \leq s \leq 2n,\\[2ex] s - n & \text{if } \ 2n < s < 2n + 1 - 1/(n+1) \end{cases}$$

for each $n \in \mathbb{N}$ (see Figure 4). Hence, we see that

$$H^{-1}(2n) = \frac{2n + n(n-1)}{1+n} = n; \tag{16}$$

$$H^{-1}(2n-1) = \frac{2n - 1 + n(n-1)}{1+n} = n - \frac{1}{1+n}.$$

For any $N \in \mathbb{N}$ and $n \in \mathbb{N}$, let

$$P = \left[\frac{N+3}{2}\right] \in \mathbb{N} \quad \text{and} \quad p = \left[\frac{n}{2}\right] - 1.$$

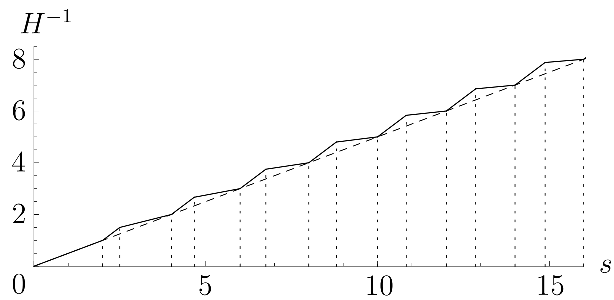

FIGURE 4. The graph of the inverse function H^{-1} of H

Then it follows that $2P-3 \le N < 2P-1$ and $2(p+P)-1 \le n+N$. Hence, we have

$$\sum_{i=N}^{n+N} \big(H^{-1}(i)-H^{-1}(i-1)\big)^2 > \sum_{i=2P-1}^{2(p+P)-1} \big(H^{-1}(i)-H^{-1}(i-1)\big)^2$$
$$= \sum_{j=P}^{p+P} \big(H^{-1}(2j-1)-H^{-1}(2j-2)\big)^2.$$

It turns out from (16) that

$$\sum_{j=P}^{p+P} \big(H^{-1}(2j-1)-H^{-1}(2j-2)\big)^2 = \sum_{j=P}^{p+P} \left(j-\frac{1}{1+j}-(j-1)\right)^2$$
$$= \sum_{j=P}^{p+P} \left(\frac{j}{1+j}\right)^2 \ge \sum_{j=P}^{p+P} \frac{1}{4} = \frac{1}{4}(p+1)$$
$$= \frac{1}{4}\left[\frac{n}{2}\right] > \frac{n-2}{8}.$$

We therefore conclude that for any $L>0$, there exists an $M(L)=8L+2$ such that $n \ge M$ implies that

$$\sum_{i=N}^{n+N} \big(H^{-1}(i)-H^{-1}(i-1)\big)^2 > \frac{n-2}{8} > \frac{M-2}{8} = L;$$

that is, condition (DU) is satisfied with $N^*=1$.

Recall that

$$a_n = \int_{n-1}^{n} h(s)ds.$$

In this example, it is clear that $a_n = 2$ for all $n \in \mathbb{N}$. Hence, we obtain

$$\sum_{i=N}^{n+N} \frac{1}{\int_{i-1}^{i} h(s)ds} = \frac{n+1}{2}.$$

This means that condition (HU) is satisfied with $N^*=1$. Note that the damping coefficient h is piecewise continuous but not continuous. Since the differentiability of h is necessary to apply Theorem 3, we cannot show that the equilibrium is uniformly asymptotically stable in Example 1 only by satisfying the condition (HU). Hence, there is room for improvement in the assumption of Theorem 3.

Appendix

We can improve the diagram for [AS] given in Section 1 as follows (compare with Figure 1):

$$\begin{array}{cccccccc}
 & & & (3) & & & & \\
 & & & \downarrow & & & & \\
 & & (A) & \Longrightarrow & (H) & & & \\
(3) & \longrightarrow & \Downarrow & \swarrow & \longleftarrow & (3) & \Longleftarrow & (2) \\
 & & (D) & \Longleftrightarrow & (S) & \Longleftrightarrow & [AS] & \\
 & & & \uparrow & & \uparrow & & \\
\multicolumn{8}{c}{\lim_{t\to\infty} H(t) = \infty \Longleftarrow (3)}
\end{array}$$

FIGURE 5. The marks "$\longrightarrow$", "$\Longrightarrow$", "$\Longleftrightarrow$" and [AS] mean "addition to", "implies", "if and only if" and the asymptotic stability of (1), respectively.

Because of limitations of space, we prove only the following relationship here.

Proposition 5. *Suppose that condition* (3) *holds. Then condition* (A) *implies condition* (H).

To prove Proposition 5, we need the following lemma, which is obtained by using an idea of Artstein and Infante [1].

Lemma 6. *Let* $\{a_n\}$ *be a sequence. If*

$$\begin{gathered}
\textit{there exist a } K > 0 \textit{ and an } m \in \mathbb{N} \textit{ such that} \\
a_n > 0 \ \textit{ for } n \geq m \ \textit{ and } \sum_{i=m}^{\ell} a_i \leq K\ell^2 \ \textit{ for } \ell \geq m,
\end{gathered}$$

then

$$\sum_{i=m}^{\infty} \frac{1}{a_i} = \infty.$$

Proof. For any fixed integer $n \geq m$, let $b_j = a_{2^n+j} > 0$ with $j = m, \dots, 2^n$. Then, by assumption, we have

$$\sum_{j=m}^{2^n} b_j = \sum_{j=m}^{2^n} a_{2^n+j} = \sum_{i=2^n+m}^{2^{n+1}} a_i < \sum_{i=m}^{2^n+m-1} a_i + \sum_{i=2^n+m}^{2^{n+1}} a_i = \sum_{i=m}^{2^{n+1}} a_i \leq 2^{2(n+1)}K.$$

Hence, it follows from the Cauchy-Bunyakovski-Schwarz inequality that

$$(2^n - m + 1)^2 = \left(\sum_{j=m}^{2^n} \sqrt{b_j}\, \frac{1}{\sqrt{b_j}} \right)^2 \leq \sum_{j=m}^{2^n} b_j \sum_{j=m}^{2^n} \frac{1}{b_j} < 2^{2(n+1)}K \sum_{j=m}^{2^n} \frac{1}{b_j}.$$

Thus, we obtain

$$\sum_{j=m}^{2^n} \frac{1}{b_j} > \frac{(2^n - m + 1)^2}{2^{2(n+1)}K}.$$

We therefore conclude that

$$\sum_{i=m}^{\infty} \frac{1}{a_j} > \sum_{k=0}^{2^m-1} \frac{1}{a_{m+k}} + \sum_{n=m}^{\infty} \left(\sum_{j=m}^{2^n} \frac{1}{a_{2^n+j}} \right) > \sum_{n=m}^{\infty} \left(\sum_{j=m}^{2^n} \frac{1}{b_j} \right)$$
$$> \sum_{n=m}^{\infty} \frac{(2^n-m+1)^2}{2^{2(n+1)}K} = \frac{1}{K} \sum_{n=m}^{\infty} \left(\frac{1-(m-2)/2^n}{2} \right)^2.$$

Since $\dfrac{1-(m-1)/2^n}{2} \nearrow \dfrac{1}{2}$ as $n \to \infty$, we see that

$$\sum_{i=m}^{\infty} \frac{1}{a_i} = \infty.$$

This completes the proof. $\square$

We are ready to prove Proposition 5.

Proof of Proposition 5. From condition (A) it follows that

(17) there exists a $K > 0$ and a $T_1 > 0$ such that $\displaystyle\int_0^t h(s)ds < Kt^2$ for $t \geq T_1$.

Since h satisfies condition (3), we see that

(18) there exists a $\nu > 0$ and a $T_2 > 0$ such that $\displaystyle\int_t^{t+1} h(s)ds \geq \nu$ for $t \geq T_2$.

Let m be an integer satisfying $m \geq \max\{T_1, T_2\}$. Define

$$a_n = \int_{n-1}^{n} h(s)ds.$$

Then, from (17) and (18), it turns out that $a_n \geq \nu > 0$ for $n \geq m$ and

$$\sum_{i=m}^{\ell} a_i = \int_{m-1}^{\ell} h(s)ds \leq \int_0^{\ell} h(s)ds < K\ell^2$$

for $\ell \geq m$. Hence, by Lemma 6, condition (H) holds. The proof is complete. $\square$

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