Integral condition for oscillation of half-linear differential equations with damping

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Abstract

The purpose of this paper is to provide an oscillation theorem that can be applied to half-linear differential equations with time-varying coefficients. A parametric curve by the coefficients is focused in order to obtain our theorem. This parametric curve is a generalization of the curve given by the characteristic equation of the second-order linear differential equation with constant coefficients. The obtained theorem is proved by transforming the half-linear differential equation to a standard polar coordinates system and using phase plane analysis carefully.

Key words: Oscillation; Half-linear differential equation; Phase plane analysis
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1. Introduction

This paper is concerned with an oscillation theorem for the second-order nonlinear differential equation with a damping term,

$$\left(\Phi_p(x')\right)' + a(t)\Phi_p(x') + b(t)\Phi_p(x) = 0, \tag{1}$$

where a and b are locally integrable functions on $[0,\infty)$ and Φ_p is a real-valued function defined by

$$\Phi_p(z) = \begin{cases} |z|^{p-2}z & \text{if } z \neq 0, \\ 0 & \text{if } z = 0. \end{cases}$$

with a real number $p > 1$. Equation (1) has the trivial solution $(x, x') \equiv (0,0)$. When $p = 2$, equation (1) becomes the linear homogeneous differential equation with variable coefficients,

$$x'' + a(t)x' + b(t)x = 0. \tag{2}$$

It is well-known that all solutions of (1) are unique for given initial conditions and continuable in the future as well as those of (2) are (see, for example, [2, 5]). In addition to this property, many commonalities are seen in the asymptotic behavior of solutions of (1) and (2), such as oscillation and stability. For example, see [1, 3, 7, 8, 10, 13, 14, 15, 16, 19, 20, 21]. Equation (1) is one of half-linear differential equations. About half-linear differential equations, refer to the monograph [4] and the references therein.

Since all solutions of (1) are continuable in the future, they can be classified into two groups as follows: a nontrivial solution x of (1) is said to be *oscillatory* if there exists a sequence $\{t_n\}$ tending to ∞ such that $x(t_n) = 0$; otherwise, it is said to be *nonoscillatory*.

Let $u = a(t)$ and $v = b(t)$. Then, the point $(a(t), b(t))$ is considered to move in the (u, v)-plane. Let us call that trajectory a *parametric curve*. We divide the first quadrant of the (u, v)-plane into two regions by the curve $v = (u/p)^p$:

$$R_1 = \{(u,v) \colon u \geq 0 \ \text{ and } \ 0 \leq v \leq (u/p)^p\};$$

$$R_2 = \{(u,v) \colon u \geq 0 \ \text{ and } \ v > (u/p)^p\}.$$

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Using phase plane analysis for a system equivalent to the half-linear differential equation (1), the present author [18] gave two oscillation theorems and one nonoscillation theorem which can be determined by the position of the parametric curve drawn by a given pair of coefficients a and b.

Theorem A. *Let S be a bounded, closed and convex set in R_1. If*

$$(a(t), b(t)) \in S \quad \textit{for all sufficiently large } t,$$

then all nontrivial solutions of (1) *are nonoscillatory.*

Theorem B. *Let S be a bounded and closed set in R_2. If*

$$(a(t), b(t)) \in S \quad \textit{for all sufficiently large } t,$$

then all nontrivial solutions of (1) *are oscillatory.*

Theorem C. *Let S be a bounded set in the closure $\overline{R}_2$ of R_2. If a is periodic and non-constant, and*

$$(a(t), b(t)) \in S \quad \textit{for all sufficiently large } t,$$

then all nontrivial solutions of (1) *are oscillatory.*

Theorems A, B and C are quite different from conventional oscillation theorems and non-oscillation theorems. It is easy to draw a parametric curve by a simple numerical simulation. The advantage of these theorems is to be able to judge whether all nontrivial solutions of (1) oscillate or not by only drawing one figure of the parametric curve. However, Theorems A, B and C cannot be applied the case that the parametric curve does not stay in the first quadrant or it crosses the curve $v = (u/p)^p$. The following result overcomes this weakness.

Theorem 1. *Suppose that the coefficient a is bounded from above. If*

$$\int_0^\infty \left\{ b(t) - \left(\frac{|a(t)|}{p} \right)^p \right\} \mathrm{d}t = \infty, \tag{3}$$

then all nontrivial solutions of (1) *are oscillatory.*

Theorem 1 is new even in the special case that $p = 2$. If the parametric curve remains in a bounded and closed set $S \subset R_2$, then there exists a $\mu > 0$ such that

$$b(t) - \left(\frac{|a(t)|}{p} \right)^p > \mu \quad \text{for} \quad t \geq 0.$$

Hence, condition (3) inevitably holds. This means that Theorem 1 improves Theorem B.

2. Proof of the main theorem

Let $y = \Phi_p(x')$ as a new variable. Then we can rewrite equation (1) as the planar system

$$\begin{aligned} x' &= \Phi_{p^*}(y), \\ y' &= -a(t)y - b(t)\Phi_p(x), \end{aligned} \tag{4}$$

where p^* is the number satisfying

$$\frac{1}{p} + \frac{1}{p^*} = 1.$$

Since $(p-1)(p^*-1) = 1$, the number p^* is also greater than 1. Let (ξ, η) be a vector in $\mathbb{R}^2$ and t_0 be a nonnegative number. Because of the uniqueness of solutions to the initial conditions, we can find only one solution x of (1) satisfying that $(x(t_0), x'(t_0)) = (\xi, \eta)$. Let (x, y) be the solution of (4) corresponding to the solution x of (1). Then the initial condition of the solution (x, y) is that $(x(t_0), y(t_0)) = (\xi, \Phi_p(\eta)) \in \mathbb{R}^2$. The projection of the solution (x, y) of (4) onto the phase plane $\mathbb{R}^2$ becomes a curve starting at the point

$(\xi, \Phi_p(\eta))$. We call this curve a *solution curve*. We may consider that the point $(x(t), y(t))$ moves on the solution curve as t increases.

Using generalized polar coordinates by

$$x = r\cos\theta \quad \text{and} \quad y = \Phi_p(r\sin\theta),$$

where $\theta \neq n\pi/2$ ($n \in \mathbb{N}$), we can transform system (4) into

$$\begin{aligned} r' &= r\,\frac{\sin\theta\cos\theta}{p-1}\left[p-1-a(t)\tan\theta-\frac{b(t)}{|\tan\theta|^{p-2}}\right],\\ \theta' &= -\frac{\sin^2\theta}{p-1}\left[p-1+\frac{a(t)}{\tan\theta}+\frac{b(t)}{|\tan\theta|^{p}}\right]. \end{aligned} \tag{5}$$

This change of variables is often called the generalized *Prüfer transformation* (see, for example, [2, 4, 6, 9]). By the Prüfer transformation, a pair of the functions r and θ satisfying that $r(t_0)\cos\theta(t_0) = \xi$ and $r(t_0)\sin\theta(t_0) = \eta$ becomes a solution of (5) except for the time t when the solution curve corresponding to the solution (x, y) of (4) crosses the x-axis or the y-axis. If the solution curve does not cross the x-axis and the y-axis, the pair of the functions r and θ satisfies system (5) on the interval $[t_0, \infty)$.

Suppose that the solution curve crosses the negative y-axis. Then there exists a $\tau \geq t_0$ such that $x(\tau) < 0 = y(\tau)$. In other words, the point $(x(t), y(t))$ is on the negative x-axis at $t = \tau$. Let us consider about how the point $(x(t), y(t))$ moves after that. Judging from the vector field of (4), there are three possibilities. Since $x'(\tau) = \Phi_p(y(\tau)) = 0$, the point $(x(t), y(t))$ moves up or down vertically (parallel to the y-axis), or it stays in the same place on the negative x-axis. Note that if $b(t) \geq 0$ for $t \geq 0$, then the point $(x(t), y(t))$ does not move down vertically. From the vector field of (4), we also see that the coefficient b has to be zero on the time interval when the point $(x(t), y(t))$ stays on the negative x-axis. Of course, the solution curve does not necessarily cross the negative x-axis at the time when the coefficient b is zero.

Proof of Theorem 1. We proceed by contradiction. Suppose that there exists a nonoscillatory solution x of (1). Then we can find a $T \geq 0$ such that $x(t) > 0$ or $x(t) < 0$ for $t \geq T$. We consider only the latter, because the former is carried out in the same way. Let y be the functions on $[T, \infty)$ satisfying $y(t) = \Phi_p(x'(t))$ for $t \geq T$. Recall that the pair of the functions x and y is the solution of (4) starting from T. Consider the solution curve corresponding to the solution (x, y) of (4). Since the solution curve is the locus of the point $(x(t), y(t))$, the point $(x(T), y(T))$ is on the solution curve. From the above assumption of the solution x it turns out that the solution curve ultimately remains in the left half-plane. We divide the discussion into two cases: (i) the solution curve crosses the negative x-axis and (ii) it does not cross the negative x-axis.

Case (i): Note that the intersection between the solution curve and the negative x-axis is not necessarily only one. The solution curve can even intersect the negative x-axis infinitely many times. Let m be the number of the intersections. Then $m \in \mathbb{N} \cup \{\infty\}$. For convenience, we write the set $\{1, 2, \dots, m\}$ or $\{1, 2, 3, \dots\}$ as S. Let x_n be the x-coordinate of the n-th intersection with $n \in S$. The point $(x(t), y(t))$ either leaves instantaneously from the intersection $(x_n, 0)$, or it stays for a while at the intersection $(x_n, 0)$. We can find two sequences $\{t_n\}$ and $\{s_n\}$ with $T \leq t_n \leq s_n < t_{n+1}$ ($n \in S$) satisfying

$$x(t)\begin{cases} = x_n & \text{if } t \in [t_n, s_n] \\ \neq x_n & \text{otherwise} \end{cases} \quad \text{and} \quad y(t)\begin{cases} = 0 & \text{if } t \in [t_n, s_n] \\ \neq 0 & \text{otherwise.} \end{cases}$$

Here, we regard t_{m+1} as ∞. We divide the set S into two subsets as follows.

$$S_1 = \{n \in S : t_n < s_n\} \quad \text{and} \quad S_2 = \{n \in S : t_n = s_n\}.$$

As mentioned in the paragraph just before entering the proof, if $n \in S_1$, then $b(t) = 0$ for $t \in [t_n, s_n]$. If $n \in S_2$, then the value of $b(t_n)$ is not necessarily zero. Its value may be positive or negative.

Let r and θ be the functions on $[T, \infty)$ satisfying

$$x(t) = r(t)\cos\theta(t) \quad \text{and} \quad y(t) = \Phi_p(r(t)\sin\theta(t)).$$

Since $x(t) < 0$ for $t \geq T$, we may assume that

$$\frac{1}{2}\pi < \theta(t) < \frac{3}{2}\pi \quad \text{for} \quad t \geq T. \tag{6}$$

Note that $\theta(t) = \pi$ for $t \in [t_n, s_n]$. The pair of the functions r and θ is a solution of (5) as long as $t \notin [t_n, s_n]$. Hence, from the second equation of (5) it follows that

$$\begin{aligned}\theta'(t) &= -\frac{\sin^2\theta(t)}{p-1}\left[p-1+\frac{a(t)}{\tan\theta(t)}+\frac{b(t)}{|\tan\theta(t)|^p}\right]\\ &= -\frac{\sin^2\theta(t)}{p-1}\left[p-1+\frac{a(t)}{\tan\theta(t)}+\left(\frac{|a(t)|}{p|\tan\theta(t)|}\right)^p-\left(\frac{|a(t)|}{p|\tan\theta(t)|}\right)^p+\frac{b(t)}{|\tan\theta(t)|^p}\right] \qquad (7)\end{aligned}$$

for $t \notin [t_n, s_n]$. Define $h(x) = (|x|/p)^p + x + p - 1$ for $x \in \mathbb{R}$. Since

$$\frac{\mathrm{d}}{\mathrm{d}x}h(x) = \frac{\Phi_p(x)}{p^{p-1}} + 1 \quad \text{and} \quad h(-p) = 0,$$

we see that $h(x) \ge 0$ for $x \in \mathbb{R}$. We therefore conclude that

$$\theta'(t) \le -\frac{\sin^2\theta(t)}{(p-1)|\tan\theta(t)|^p}\left\{b(t)-\left(\frac{|a(t)|}{p}\right)^p\right\}$$

for $t \notin [t_n, s_n]$. This inequality can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_p(\tan\theta(t)) \le -b(t) + \left(\frac{|a(t)|}{p}\right)^p.$$

Using the inequality above, we can estimate that

$$\begin{aligned}\Phi_p(\tan\theta(t)) - \Phi_p(\tan\theta(T)) &= \Phi_p(\tan\theta(t_1)) - \Phi_p(\tan\theta(T)) + \Phi_p(\tan\theta(t_2)) - \Phi_p(\tan\theta(s_1))\\ &\quad + \Phi_p(\tan\theta(t_3)) - \Phi_p(\tan\theta(s_2)) + \cdots + \Phi_p(\tan\theta(t)) - \Phi_p(\tan\theta(s_n))\\ &= -\int_T^{t_1}\left\{b(t)-\left(\frac{|a(t)|}{p}\right)^p\right\}\mathrm{d}t - \sum_{i=1}^{n-1}\int_{s_i}^{t_{i+1}}\left\{b(t)-\left(\frac{|a(t)|}{p}\right)^p\right\}\mathrm{d}t\\ &\quad - \int_{s_n}^{t}\left\{b(s)-\left(\frac{|a(s)|}{p}\right)^p\right\}\mathrm{d}s \qquad (8)\end{aligned}$$

for $t \in (s_n, t_{n+1})$. Taking into account that $n \in S_1$ implies $b(t) = 0$ for $t \in [t_n, s_n]$, we obtain

$$\begin{aligned}\int_T^{\infty}\left\{b(t)-\left(\frac{|a(t)|}{p}\right)^p\right\}\mathrm{d}t &= \int_T^{t_1}\left\{b(t)-\left(\frac{|a(t)|}{p}\right)^p\right\}\mathrm{d}t + \sum_{i=1}^{m-1}\int_{s_i}^{t_{i+1}}\left\{b(t)-\left(\frac{|a(t)|}{p}\right)^p\right\}\mathrm{d}t\\ &\quad + \int_{s_n}^{\infty}\left\{b(t)-\left(\frac{|a(t)|}{p}\right)^p\right\}\mathrm{d}t + \sum_{i=1}^{m}\int_{t_i}^{s_i}\left\{b(t)-\left(\frac{|a(t)|}{p}\right)^p\right\}\mathrm{d}t\\ &= \int_T^{t_1}\left\{b(t)-\left(\frac{|a(t)|}{p}\right)^p\right\}\mathrm{d}t + \sum_{i=1}^{m-1}\int_{s_i}^{t_{i+1}}\left\{b(t)-\left(\frac{|a(t)|}{p}\right)^p\right\}\mathrm{d}t\\ &\quad + \int_{s_n}^{\infty}\left\{b(t)-\left(\frac{|a(t)|}{p}\right)^p\right\}\mathrm{d}t + \sum_{i\in S_1}\int_{t_i}^{s_i}\left\{b(t)-\left(\frac{|a(t)|}{p}\right)^p\right\}\mathrm{d}t\\ &\quad + \sum_{i\in S_2}\int_{t_i}^{s_i}\left\{b(t)-\left(\frac{|a(t)|}{p}\right)^p\right\}\mathrm{d}t\\ &= \int_T^{t_1}\left\{b(t)-\left(\frac{|a(t)|}{p}\right)^p\right\}\mathrm{d}t + \sum_{i=1}^{m-1}\int_{s_i}^{t_{i+1}}\left\{b(t)-\left(\frac{|a(t)|}{p}\right)^p\right\}\mathrm{d}t\\ &\quad + \int_{s_n}^{\infty}\left\{b(t)-\left(\frac{|a(t)|}{p}\right)^p\right\}\mathrm{d}t - \sum_{i\in S_1}\int_{t_i}^{s_i}\left\{\left(\frac{|a(t)|}{p}\right)^p\right\}\mathrm{d}t\\ &\quad + \sum_{i\in S_2}\int_{t_i}^{t_i}\left\{b(t)-\left(\frac{|a(t)|}{p}\right)^p\right\}\mathrm{d}t\\ &\le \int_T^{t_1}\left\{b(t)-\left(\frac{|a(t)|}{p}\right)^p\right\}\mathrm{d}t + \sum_{i=1}^{m-1}\int_{s_i}^{t_{i+1}}\left\{b(t)-\left(\frac{|a(t)|}{p}\right)^p\right\}\mathrm{d}t\\ &\quad + \int_{s_n}^{\infty}\left\{b(t)-\left(\frac{|a(t)|}{p}\right)^p\right\}\mathrm{d}t.\end{aligned}$$

Hence, by condition (3), we have

$$\int_T^{t_1}\left\{b(t)-\left(\frac{|a(t)|}{p}\right)^p\right\}\mathrm{d}t+\sum_{i=1}^{m-1}\int_{s_i}^{t_{i+1}}\left\{b(t)-\left(\frac{|a(t)|}{p}\right)^p\right\}\mathrm{d}t+\int_{s_n}^{\infty}\left\{b(t)-\left(\frac{|a(t)|}{p}\right)^p\right\}\mathrm{d}t=\infty.$$

From this and (8) it turns out that

$$\tan\theta(t)\to-\infty\quad\text{as}\quad t\to\infty;$$

that is, $\theta(t)\searrow\pi/2$ as $t\to\infty$, and therefore, $\sin^2\theta(t)\to 1$ as $t\to\infty$. Since a is bounded from above, we can find a $\tau_1\geq T$ satisfying

$$\frac{\sin^2\theta(t)}{p-1}\,h\left(\frac{a(t)}{\tan\theta(t)}\right)>\frac{1}{2}\quad\text{for}\quad t\geq\tau_1.$$

Using (7) again, we get

$$\begin{aligned}\theta'(t)&=-\frac{\sin^2\theta(t)}{p-1}\left[h\left(\frac{a(t)}{\tan\theta(t)}\right)-\left(\frac{|a(t)|}{p|\tan\theta(t)|}\right)^p+\frac{b(t)}{|\tan\theta(t)|^p}\right]\\&\leq-\frac{1}{2}-\frac{\sin^2\theta(t)}{(p-1)|\tan\theta(t)|^p}\left[b(t)-\left(\frac{|a(t)|}{p}\right)^p\right]\end{aligned}\tag{9}$$

for $t\geq\tau_1$. From (3) it follows that there exists a $\tau_2\geq\tau_1$ such that

$$\int_{\tau_1}^{t}\left\{b(s)-\left(\frac{|a(s)|}{p}\right)^p\right\}\mathrm{d}s>0\quad\text{for}\quad t\geq\tau_2.\tag{10}$$

Let $\tau_3=\tau_2+2\pi$ and

$$\nu=\min_{\tau_1\leq t\leq\tau_3}\frac{\sin^2\theta(t)}{(p-1)|\tan\theta(t)|^p}>0.$$

Then, by (9) and (10), we have

$$\theta(\tau_3)-\theta(\tau_1)\leq-\frac{1}{2}(\tau_3-\tau_1)-\nu\int_{\tau_1}^{\tau_3}\left\{b(t)-\left(\frac{|a(t)|}{p}\right)^p\right\}\mathrm{d}t<-\pi.$$

This contradicts (6). Thus, the proof of case (i) is complete.

Case (ii): As in the proof of case (i), we consider the pair of the functions r and θ. Since the solution curve does not cross the negative x-axis, the pair is a solution of (5) on $[T,\infty)$, and therefore,

$$\theta'(t)=-\frac{\sin^2\theta(t)}{p-1}\left[h\left(\frac{a(t)}{\tan\theta(t)}\right)-\left(\frac{|a(t)|}{p|\tan\theta(t)|}\right)^p+\frac{b(t)}{|\tan\theta(t)|^p}\right]$$

for $t\geq T$. Hence, using the same way as in the case (ii), we can obtain

$$\Phi_p(\tan\theta(t))-\Phi_p(\tan\theta(T))=-\int_T^t\left\{b(s)-\left(\frac{|a(s)|}{p}\right)^p\right\}\mathrm{d}s\quad\text{for}\quad t\geq T.$$

By using this and (3), we can choose a $\tau_1\leq T$ such that

$$\theta'(t)\leq-\frac{1}{2}-\frac{\sin^2\theta(t)}{(p-1)|\tan\theta(t)|^p}\left[b(t)-\left(\frac{|a(t)|}{p}\right)^p\right]\quad\text{for}\quad t\geq\tau_1.$$

The rest of the proof is the same as that of case (i). Thus, all nontrivial solutions of (1) are oscillatory. □

3. Example and simulation

Consider the nonlinear differential equation

$$\left((x')^3\right)'+(c+r\cos t)(x')^3+(d+r\sin t)x^3=0.\tag{11}$$

Note that $p=4$, $a(t)=c+r\cos t$ and $b(t)=d+r\sin t$ for $t\geq 0$ in equation (11). It is clear that the parametric curve drawn by the coefficients a and b is a circle with radius r whose center is (c,d). Since

$$a(t)\leq c+r\quad\text{for}\quad t\geq 0,$$

the coefficient a is bounded from above. We obtain

$$b(t) - \left(\frac{|a(t)|}{p}\right)^p = d + r\sin t - \frac{1}{2^8}(c + r\cos t)^4 = d - \frac{1}{256}\left(c^4 + 3c^2r^2 + \frac{3}{8}r^4\right)$$
$$+ r\sin t - \frac{1}{256}\left(cr\left(4c^2 + 3r^2\right)\cos t + r^2\left(3c^2 + \frac{1}{2}r^2\right)\cos 2t + cr^3\cos 3t + \frac{1}{8}r^4\cos 4t\right).$$

Hence, condition (3) holds provided that

$$d > \frac{1}{256}\left(c^4 + 3c^2r^2 + \frac{3}{8}r^4\right). \tag{12}$$

Example 1. Let $p = 4$, $c = 2$, $d = 6$ and $r = 7$. Then all nontrivial solutions of (11) are oscillatory.

It is clear that the numbers $p = 4$, $c = 2$, $d = 6$ and $r = 7$ satisfy the inequality (12).

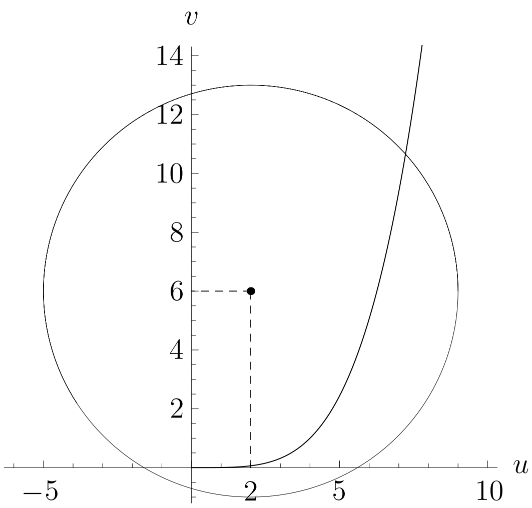

Figure 1: The parametric curve drawn by a pair of $a(t) = 2 + 7\cos t$ and $b(t) = 6 + 7\sin t$

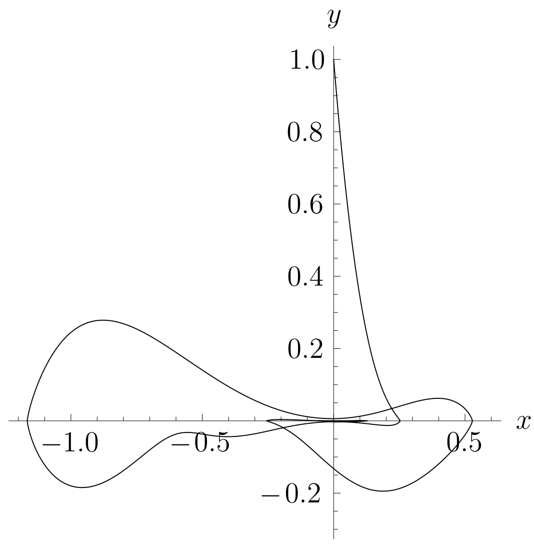

Figure 2: The solution curve of (4) with $p = 4$, $c = 2$, $d = 6$, $r = 7$, $a(t) = 2 + 7\cos t$ and $b(t) = 6 + 7\sin t$ satisfying the initial condition $(x(0), y(0)) = (0, 1)$

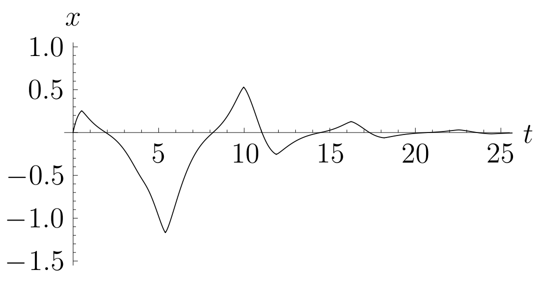

Figure 3: The solution of (11) with $p = 4$, $c = 2$, $d = 6$ and $r = 7$ satisfying the initial condition $(x(0), x'(0)) = (0, 1)$

4. Final comment

As a classical oscillation theorem for the equation

$$y'' + c(t)y = 0, \tag{13}$$

where c is a continuous function, the following Leighton − Wintner criterion is very well-known (see, for example, [12, 17]).

Theorem D. *All nontrivial solutions of* (13) *are oscillatory if*

$$\int_0^\infty c(t)\,\mathrm{d}t = \infty.$$

In the case that a is continuously differentiable on $[0,\infty)$, equation (1) can be transformed into equation (13), where

$$c(t) = b(t) - \frac{1}{4}a^2(t) - \frac{1}{2}a'(t).$$

Hence, it turns out that all nontrivial solutions of (2) are oscillatory if

$$\int_0^\infty \left\{ b(t) - \frac{1}{4}a^2(t) - \frac{1}{2}a'(t) \right\} \mathrm{d}t = \infty. \tag{14}$$

If a is bounded, then condition (14) coincides with condition (3) when $p = 2$. This means that Theorem 1 is a partial generalization of the well-known result above.

Let p be any continuous and periodic function with period $T > 0$. A periodic function p is said to be *periodic with mean value zero* if p is not identically zero and

$$\int_0^T p(t)\,\mathrm{d}t = 0.$$

Recently, Došlý et al. [3] gave the following oscillation theorem for equation (1).

Theorem E. *Suppose that a and b are periodic with mean value zero and B is an indefinite integral of b such that $\Phi_{p^*}(B)$ is also periodic with mean value zero. If*

$$\int_0^T \Big\{ (p-1)\Phi_{p^*}(B(t)) - a(t) \Big\} B(t)E(t)\,\mathrm{d}t > 0,$$

where

$$E(t) = \exp\left\{ \int_0^t \Big(a(s) - p\Phi_{p^*}(s) \Big)\,\mathrm{d}s \right\},$$

then all nontrivial solutions of (1) *are oscillatory.*

Theorem E is a generalization to equation (1) of oscillation theorems for equation (2) by Kwong and Wong [11] and Sugie and Matsumura [20]. However, the coefficients a and b have to be at least periodic with mean value zero. Hence, Theorem E cannot be applied to Example 1 because a and b are periodic but not periodic with mean value zero. On the other hand, Theorem 1 can be applied even when a and b are not periodic.

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