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Convergence of radially symmetric solutions for (p,q)-Laplacian elliptic equations with a damping term

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Abstract This study considers the quasilinear elliptic equation with a damping term,

$$\mathrm{div}(D(u)\nabla u) + \frac{k(|\mathbf{x}|)}{|\mathbf{x}|}\,\mathbf{x}\cdot(D(u)\nabla u) + \omega^2(|u|^{p-2}u + |u|^{q-2}u) = 0,$$

where $\mathbf{x}$ is an N-dimensional vector in $\{\mathbf{x} \in \mathbb{R}^N : |\mathbf{x}| \geq \alpha\}$ for some $\alpha > 0$ and $N \in \mathbb{N} \setminus \{1\}$; $D(u) = |\nabla u|^{p-2} + |\nabla u|^{q-2}$ with $1 < q \leq p$; k is a nonnegative and locally integrable function on $[\alpha, \infty)$; and ω is a positive constant. A necessary and sufficient condition is given for all radially symmetric solutions to converge to zero as $|\mathbf{x}| \to \infty$. Our necessary and sufficient condition is expressed by an improper integral related to the damping coefficient k. The case that k is a power function is explained in detail.

Keywords Quasilinear differential equation · (p,q)-Laplacian · Radially symmetric solutions · Convergence of solutions · Damping term

Mathematics Subject Classification (2000) 34D05 · 34D23 · 35J92

1 Introduction

In recent years, quasilinear equations of the (p,q)-Laplacian type have attracted special attention not only because of their mathematical interest to researchers but also because they have rich applications to various sciences. Physical applications have been reported in many studies, for example, see [2, 3, 8, 10, 11, 19, 22] and the references therein. The (p,q)-Laplacian type equations are initially derived from the study of stationary solutions of the reaction-diffusion equation

$$u_t = \mathrm{div}(D(u)\nabla u) + c(u).$$

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Here, $D(u)$ means $|\nabla u|^{p-2} + |\nabla u|^{q-2}$ with $1 < q \le p$. The diffusion term $\mathrm{div}(D(u)\nabla u)$ is usually called the (p,q)-*Laplacian* when $p \ne q$. The second term $c(u)$ is the reaction term. The mathematical interest in quasilinear equations with the (p,q)-Laplacian seems to be primarily concentrated on the existence and multiplicity of positive solutions on a bounded domain Ω of $\mathbb{R}^N$, the existence and nonexistence of nontrivial solutions on $\mathbb{R}^N$, and the nonlinear eigenvalue problem (refer to [4, 7, 8, 10, 11, 15–17, 22, 24, 25]). In this study, we would like to examine the structure of a (p,q)-Laplacian-type quasilinear equation from a different angle.

The stationary solution u of the above-mentioned reaction-diffusion equation satisfies

$$\mathrm{div}(D(u)\nabla u) + c(u) = 0.$$

To relate the reaction term to the diffusion term, we choose $\omega^2(\phi_p(u) + \phi_q(u))$ as $c(u)$, where ω is a positive constant and

$$\phi_r(u) = \begin{cases} |u|^{r-2}u & \text{if } u \ne 0, \\ 0 & \text{if } u = 0, \end{cases}$$

with $r = p$ or $r = q$. It is well known that diffusion causes energy dissipation. Energy loss might occur due to factors other than diffusion in real world applications. In this study, considering another factor that leads to energy loss, we add a damping term and consider the quasilinear elliptic equation

$$\mathrm{div}(D(u)\nabla u) + \frac{k(|\mathbf{x}|)}{|\mathbf{x}|}\mathbf{x}\cdot(D(u)\nabla u) + \omega^2(\phi_p(u) + \phi_q(u)) = 0. \tag{1.1}$$

Here, $\mathbf{x}$ is an N-dimensional vector in an exterior domain $G_\alpha \overset{\text{def}}{=} \{\mathbf{x} \in \mathbb{R}^N : |\mathbf{x}| \ge \alpha\}$ for some $\alpha > 0$; N is an integer that is larger than 1; k is a nonnegative and locally integrable function on $[\alpha, \infty)$; and ω is a positive constant. Let $\beta \ge \alpha$. By a *solution* of (1.1) we mean a function u: $G_\beta \to \mathbb{R}$ that is continuously differentiable together with $|\nabla u|^{q-2}\nabla u$ and satisfies Eq. (1.1) on G_β. Our attention will be focused on the global convergence of radially symmetric solutions of (1.1); that is, those solutions that depend only on $|x|$.

In Eq. (1.1), both the diffusion term and the damping term play a role in energy dissipation. For this reason, many researchers might think that all radially symmetric solutions always converge to zero. However, if the damping term is too strong, those solutions do not decay to zero, and the so-called overdamping phenomenon occurs. The phenomenon of overdamping is that a solution converging to a non-zero value exists. Then, we must consider the limit of the damping term where the overdamping phenomenon does not occur. The following theorem answers this question.

Theorem 1.1 *Suppose that there exists an $\varepsilon_0 > 0$ and a $\delta_0 > 0$ such that $|k(t) - k(s)| < \varepsilon_0$ for all $t \ge \alpha$ and $s \ge \alpha$ with $|t - s| < \delta_0$. Then, all radially symmetric solutions u of* (1.1) *satisfy the property that $u(\mathbf{x})$ and $|\nabla u(\mathbf{x})|$ tend to zero as $|\mathbf{x}| \to \infty$ if and only if condition*

$$\int_\alpha^\infty \varphi^{-1}\left(\frac{\int_\alpha^t e^{K(s)}ds}{e^{K(t)}}\right)dt = \infty, \tag{1.2}$$

where φ^{-1} is the inverse function of $\phi_p + \phi_q$ and $K(t) = \displaystyle\int_\alpha^t \left(k(s) + \frac{N-1}{s}\right)ds$.

To describe our second theorem, we introduce the following family of functions. A function $h\colon [\alpha,\infty) \to [0,\infty)$ is said to belong to $\mathcal{F}_{[\mathrm{IP}]}$ if

$$\sum_{n=1}^{\infty}\int_{\tau_n}^{\sigma_n} h(t)dt = \infty$$

for every pair of sequences $\{\tau_n\}$ and $\{\sigma_n\}$ satisfying $\tau_n < \sigma_n < \tau_{n+1}$,

$$\liminf_{n\to\infty}(\sigma_n - \tau_n) > 0.$$

The concept of integral positivity [IP] was introduced by Matrosov [18]. As typical examples, we can cite any function with a positive lower bound, and any nonnegative periodic function such as $\sin^2 t$.

Theorem 1.2 *Suppose that k belongs to $\mathcal{F}_{[\mathrm{IP}]}$. Then, all radially symmetric solutions u of* (1.1) *satisfy the property that $u(\mathbf{x})$ and $|\nabla u(\mathbf{x})|$ tend to zero as $|\mathbf{x}| \to \infty$ if and only if condition* (1.2) *holds.*

Remark 1.1 When there is no dampimg term, namely, $k(t) = 0$ for $t \geq \alpha$, the growth condition (1.2) is satisfied (for details, see Section 5). Hence, from Theorem 1.1 and 1.2, all radially symmetric solutions of (1.1) converge to zero. This means that the overdamping phenomenon does not occur without the damping factor (which is consistent with common knowledge).

Remark 1.2 The assumption that k is nonnegative is not essential in Theorems 1.1 and 1.2. The value $k(t) + (N-1/t)$ only has to be nonnegative for $t \geq \alpha$. Hence, Theorems 1.1 and 1.2 can be applied even if equation (1.1) has negative damping. To be precise, we need to add the assumption that the function $k + (N-1/t)$ belongs to $\mathcal{F}_{[\mathrm{WIP}]}$ in Theorem 1.1 (for the definition of $\mathcal{F}_{[\mathrm{WIP}]}$, see Section 2). We need to further change the assumption that k belongs to $\mathcal{F}_{[\mathrm{IP}]}$ to the assumption that $k + (N-1/t)$ belongs to $\mathcal{F}_{[\mathrm{IP}]}$ in Theorem 1.2.

2 Quasilinear ordinary differential equations

Consider the quasilinear equation with damped term,

$$(\varphi(\xi'))' + h(t)\varphi(\xi') + \omega^2\varphi(\xi) = 0, \tag{2.1}$$

where $' = d/dt$, h is a nonnegative and integrable function on $[\alpha,\infty)$, and ω is a positive constant. Here, φ is a continuous and strictly increasing function on $\mathbb{R}$ satisfying $\eta\varphi(\eta) > 0$ if $\eta \neq 0$, and $\varphi(\eta)$ tends to $\pm\infty$ as $\eta \to \pm\infty$. Hence, $\varphi(0) = 0$, and the inverse function φ^{-1} exists on $\mathbb{R}$. The origin $(\xi,\xi') = (0,0)$ is clearly the only equilibrium of (2.1).

Let $\chi(t) = (\xi(t),\xi'(t))$ and $\chi_0 \in \mathbb{R}^2$, and let $\|\cdot\|$ be any suitable norm. We denote the solution of (2.1) through (t_0,χ_0) by $\chi(t;t_0,\chi_0)$. The equilibrium is said to be *stable* if, for any $\varepsilon > 0$ and any $t_0 \geq \alpha$, there exists a $\delta(\varepsilon,t_0) > 0$ such that $\|\chi_0\| < \delta$ implies $\|\chi(t;t_0,\chi_0)\| < \varepsilon$ for all $t \geq t_0$. The equilibrium is said to be *attractive* if, for any $t_0 \geq \alpha$, there exists a $\delta_0(t_0) > 0$ such that $\|\chi_0\| < \delta_0$ implies $\|\chi(t;t_0,\chi_0)\| \to 0$ as $t \to \infty$. The equilibrium is said to be *globally attractive* if, for any $t_0 \geq \alpha$ and any $\chi_0 \in \mathbb{R}^2$, there is a $T(t_0,\chi_0,\eta) > 0$ such that $\|\chi(t;t_0,\chi_0)\| < \eta$ for all $t \geq t_0 + T$. The equilibrium is *asymptotically stable* if it is stable and attractive. The equilibrium is *globally asymptotically stable* if it is stable and globally attractive. Refer to the books [1, 5, 6, 9, 12, 13, 20, 21, 26] as to these definitions.

The purpose of this section is to present criteria for judging whether the equilibrium of (2.1) is globally asymptotically stable. To accomplish this, we set the following conditions on the nonlinear function φ:

$$\text{there exists a positive function } f \text{ such that } \varphi^{-1}(-ab) \geq -f(a)\varphi^{-1}(b) \text{ or } \varphi^{-1}(ab) \leq f(a)\varphi^{-1}(b) \text{ for all } a > 0 \text{ and } b > 0; \tag{2.2}$$

$$\text{there exists a positive function } g \text{ such that } \varphi^{-1}(-ab) \leq -g(a)\varphi^{-1}(b) \text{ and } \varphi^{-1}(ab) \geq g(a)\varphi^{-1}(b) \text{ for all } a > 0 \text{ and } b > 0 \tag{2.3}$$

In the next section, we will give examples in which conditions (2.2) and (2.3) are satisfied.

The following result gives a necessary condition for the global asymptotic stability of the equilibrium of (2.1).

Theorem 2.1 *Let condition* (2.2) *hold. If the equilibrium of* (2.1) *is globally asymptotically stable, then*

$$\int_\alpha^\infty \varphi^{-1}\left(\frac{\int_\alpha^t e^{H(s)}ds}{e^{H(t)}}\right)dt = \infty, \tag{2.4}$$

where $H(t) = \int_\alpha^t h(s)ds$.

To show a sufficient condition that guarantees the equilibrium of (2.1) is globally asymptotically stable, we define another family of functions in addition to $\mathcal{F}_{[\mathrm{IP}]}$. A function $h: [\alpha,\infty) \to [0,\infty)$ is said to belong to $\mathcal{F}_{[\mathrm{WIP}]}$ if

$$\sum_{n=1}^\infty \int_{\tau_n}^{\sigma_n} h(t)dt = \infty$$

for every pair of sequences $\{\tau_n\}$ and $\{\sigma_n\}$ satisfying $\tau_n < \sigma_n < \tau_{n+1}$,

$$\liminf_{n\to\infty}(\sigma_n - \tau_n) > 0 \quad \text{and} \quad \limsup_{n\to\infty}(\tau_{n+1} - \sigma_n) < \infty.$$

The concept of the weak integral positivity [WIP] was first published in Hatvani [14]. This concept is much broader than that of integral positivity. If h decreases to zero monotonically, then it does not belong to $\mathcal{F}_{[\mathrm{IP}]}$. However, there is a possibility that h belongs to $\mathcal{F}_{[\mathrm{WIP}]}$ even if $\liminf_{t\to\infty} h(t) = 0$. For example, the functions $1/t$ and $\sin^2 t/t$ belong to $\mathcal{F}_{[\mathrm{WIP}]}$ (for the proof, see [23, Proposition 2.1]).

Using the concept of the weak integral positivity [WIP], we can state the following result.

Theorem 2.2 *Let condition* (2.3) *hold. Suppose that there exist an* $\varepsilon_0 > 0$ *and a* $\delta_0 > 0$ *such that* $|h(t) - h(s)| < \varepsilon_0$ *for all* $t \geq \alpha$ *and* $s \geq \alpha$ *with* $|t - s| < \delta_0$ *and suppose that* h *belongs to* $\mathcal{F}_{[\mathrm{WIP}]}$. *If condition* (2.4) *is satisfied, then the equilibrium of* (2.1) *is globally asymptotically stable.*

If h is uniformly continuous on $[\alpha,\infty)$, then the first assumption of h in Theorem 2.2 is inevitably satisfied. Even if h belongs to $\mathcal{F}_{[\mathrm{WIP}]}$, Theorem 2.2 does not hold without the first assumption of h. However, the first assumption of h becomes unnecessary if h belongs to $\mathcal{F}_{[\mathrm{IP}]}$.

Theorem 2.3 *Let condition* (2.3) *hold*. *Suppose that h belongs to* $\mathcal{F}_{[\mathrm{IP}]}$. *If condition* (2.4) *is satisfied*, *then the equilibrium of* (2.1) *is globally asymptotically stable*.

To prove Theorems 2.1–2.3, we use phase plane analysis. Let $x = v$ and $y = \varphi(v')/\omega$ be new variables. Then, we can transform Eq. (2.1) into the planar equivalent system

$$\begin{aligned} x' &= \varphi^{-1}(\omega y), \\ y' &= -\omega\varphi(x) - h(t)y. \end{aligned} \tag{2.5}$$

System (2.5) has the zero solution $(x,y) \equiv (0,0)$, which corresponds to the equilibrium of (2.1).

Proof of Theorem 2.1 We show that there exists a solution of (2.5) that does not approach the origin provided that condition (2.4) is satisfied.

By way of contradiction, we suppose that the equilibrium of (2.1) is globally asymptotically stable and

$$\int_\alpha^\infty \varphi^{-1}\left(\frac{\int_\alpha^t e^{H(s)}ds}{e^{H(t)}}\right)dt < \infty.$$

Because of (2.2), we need to consider two cases: (i) $\varphi^{-1}(-ab) \ge -f(a)\varphi^{-1}(b)$ for all $a > 0$ and $b > 0$, and (ii) $\varphi^{-1}(ab) \le f(a)\varphi^{-1}(b)$ for all $a > 0$ and $b > 0$. We will mention only the former case, because the latter case is proved in the same way.

We can choose a $T \ge \alpha$ so large that

$$\int_T^\infty \varphi^{-1}\left(\frac{\int_\alpha^t e^{H(s)}ds}{e^{H(t)}}\right)dt < \frac{1}{2f(\omega^2\varphi(1))}. \tag{2.6}$$

Consider the solution $(\tilde{x},\tilde{y})$ of (2.5) passing through $(1,0)$ at $t = T$. Since

$$\tilde{x}'(T) = \varphi^{-1}(\omega\tilde{y}(T)) = 0 \quad \text{and} \quad \tilde{y}'(T) = -\omega\varphi(\tilde{x}(T)) - h(t)\tilde{y}(T) = -\omega\varphi(1) < 0,$$

the solution curve of $(\tilde{x},\tilde{y})$ enters the fourth quadrant

$$Q_4 \stackrel{\text{def}}{=} \{(x,y) : x > 0 \text{ and } y < 0\}$$

in a right-hand neighborhood of $t = T$. Considering the vector field in Q_4, we see that the solution curve does not directly move to the first quadrant

$$Q_1 \stackrel{\text{def}}{=} \{(x,y) : x > 0 \text{ and } y > 0\}$$

from Q_4 as t increases. We also see that $0 \le \tilde{x}(t) < 1$ as long as the solution curve is in Q_4.

Suppose that the solution curve of $(\tilde{x},\tilde{y})$ crosses the straight line $x = 1/2$ in Q_4. Then, we can find a $T^* > T$ such that $\tilde{x}(T^*) = 1/2$ and $\tilde{x}(t) > 1/2$ for $T \le t < T^*$. Since

$$\tilde{y}'(t) + h(t)\tilde{y}(t) = -\omega\varphi(\tilde{x}(t)) \ge -\omega\varphi(1)$$

for $T \le t < T^*$, it follows that

$$(e^{H(t)}\tilde{y}(t))' \ge -\omega\varphi(1)e^{H(t)} \quad \text{for } T \le t < T^*.$$

We integrate both sides of this inequality from T to $t < T^*$ to obtain

$$e^{H(t)}\tilde{y}(t) \geq e^{H(T)}\tilde{y}(T) - \omega\varphi(1)\int_T^t e^{H(s)}ds = -\omega\varphi(1)\int_T^t e^{H(s)}ds.$$

Hence, by (2.5), we have

$$\begin{aligned}\tilde{x}'(t) = \varphi^{-1}(\omega\tilde{y}(t)) &\geq \varphi^{-1}\left(-\omega^2\varphi(1)\frac{\int_T^t e^{H(s)}ds}{e^{H(t)}}\right)\\ &\geq -f(\omega^2\varphi(1))\varphi^{-1}\left(\frac{\int_T^t e^{H(s)}ds}{e^{H(t)}}\right)\end{aligned}$$

for $T \leq t < T^*$. From this estimation and (2.6), we see that

$$\begin{aligned}\tilde{x}(T^*) &\geq \tilde{x}(T) - f(\omega^2\varphi(1))\int_T^{T^*}\varphi^{-1}\left(\frac{\int_T^t e^{H(s)}ds}{e^{H(t)}}\right)dt\\ &\geq 1 - f(\omega^2\varphi(1))\int_T^{\infty}\varphi^{-1}\left(\frac{\int_\alpha^t e^{H(s)}ds}{e^{H(t)}}\right)dt > \frac{1}{2}.\end{aligned}$$

This contradicts the assumption that $\tilde{x}(T^*) = 1/2$. Hence, the solution curve does not intersect the straight line $x = 1/2$.

Therefore, we can conclude that the solution curve of $(\tilde{x}, \tilde{y})$ stays in the region

$$\{(x, y) : 1/2 < x \leq 1 \text{ and } y \leq 0\}$$

for $t \geq T$. In other words, the solution $(\tilde{x}, \tilde{y})$ of (2.5) does not approach the origin. Hence, the equilibrium of (2.1) is not globally asymptotically stable. This is a contradiction. Thus, the proof of Theorem 2.1 is complete. □

It is convenient to introduce some notation to prove Theorem 2.2. We denote functions Φ and Ψ by

$$\Phi(x) = \int_0^x \omega\varphi(\xi)d\xi \quad \text{and} \quad \Psi(y) = \int_0^y \varphi^{-1}(\omega\eta)d\eta,$$

respectively. Since φ is strictly increasing and $\eta\varphi(\eta) > 0$ if $\eta \neq 0$, we see that $\Phi(0) = 0$; $\Phi(x)$ is increasing for $x \geq 0$ and decreasing for $x \leq 0$, and $\Phi(x)$ diverges to ∞ as $x \to \pm\infty$. Let

$$\widetilde{\Phi}(x) = \Phi(x)\,\mathrm{sgn}\,x.$$

Then, $\widetilde{\Phi}$ is an increasing function on $\mathbb{R}$, and $\widetilde{\Phi}(x)$ tends to $\pm\infty$ as $x \to \pm\infty$. Hence, the inverse function $\widetilde{\Phi}^{-1}$ exists on $\mathbb{R}$. The function Ψ has the same property. Let

$$\widetilde{\Psi}(y) = \Psi(y)\,\mathrm{sgn}\,y.$$

Then, $\widetilde{\Psi}$ also has an inverse function $\widetilde{\Psi}^{-1}$ that is defined on $\mathbb{R}$. Moreover, we define

$$\Delta(y) = y\varphi^{-1}(\omega y).$$

From the property of φ^{-1}, it follows that $\Delta(0) = 0$; $\Delta(y)$ is increasing for $y \geq 0$ and decreasing for $y \leq 0$.

To prove Theorems 2.2 and 2.3, we need to show two facts: (i) stability of the equilibrium of (2.1), and (ii) global attractivity. It is relatively easy to prove fact (i). However, detailed mathematical analysis and considerable patience are necessary to demonstrate fact (ii). For this reason, we first show that the equilibrium of (2.1) is stable.

Let (x,y) be any solution of (2.5) with the initial time $t_0 \geq \alpha$ and define

$$v(t) = \Phi(x(t)) + \Psi(y(t)). \tag{2.7}$$

Then, we obtain

$$\begin{aligned} v'(t) &= \omega\varphi(x(t))x'(t) + \varphi^{-1}(\omega y(t))y'(t) \\ &= \omega\varphi(x(t))\varphi^{-1}(\omega y(t)) + \varphi^{-1}(\omega y(t))(-\,\omega\varphi(x(t)) - h(t)y) \\ &= -\,h(t)y(t)\varphi^{-1}(\omega y(t)) = -\,h(t)\varDelta(y(t)) \end{aligned}$$

for $t \geq t_0$. Since $h(t) \geq 0$ for $t \geq 0$, we see that

$$v(t) \leq v(t_0) \quad \text{for } \ t \geq t_0.$$

Hence, we obtain the following result.

Proposition 2.4 *The equilibrium of* (2.1) *is stable*.

We are now ready to prove Theorems 2.2 and 2.3.

Proof of Theorem 2.2 By virtue of Proposition 2.4, we only have to show that the equilibrium of (2.1) is globally attractive. As shown already,

$$v'(t) = -\,h(t)\varDelta(y(t)) \leq 0$$

for $t \geq t_0$; namely, v is a decreasing function on $[t_0,\infty)$. It also follows from (2.7) that $v(t) \geq 0$ for $t \geq t_0$. Hence, the function v has a limiting value $v^* \geq 0$. If $v^* = 0$, the solution (x,y) of (2.5) clearly tends to the origin as $t \to \infty$. This is our desired conclusion. To complete the proof, we will show that v^* is not positive.

By way of contradiction, we suppose that v^* is positive. Then, we can find a $T_1 \geq t_0$ satisfying

$$0 < v^* \leq v(t) \leq 2v^* \quad \text{for } \ t \geq T_1. \tag{2.8}$$

We advance the argument by dividing it into two steps. In the first step, we consider the asymptotic behavior of the second component y of the solution and show that y approaches zero. Hence, by (2.7), we conclude that $\lim_{t\to\infty} x(t) = \widetilde{\Phi}^{-1}(v^*) > 0$ or $\lim_{t\to\infty} x(t) = \widetilde{\Phi}^{-1}(-v^*)$. In the second step, we show that the solution curve of (x,y) does not approache the points $(\widetilde{\Phi}^{-1}(v^*),0)$ and $(\widetilde{\Phi}^{-1}(-v^*),0)$. This is a contradiction.

Step (1): If $\liminf_{t\to\infty}|y(t)| > 0$, then we can choose a $\lambda > 0$ and a $T_2 \geq T_1$ such that $|y(t)| > \lambda$ for $t \geq T_2$. Let $\varGamma = \min\{\varDelta(\lambda),\varDelta(-\lambda)\}$. Then, we have

$$v'(t) = -\,h(t)\varDelta(y(t)) \leq -\,\varGamma h(t)$$

for $t \geq T_2$. Integrating this inequality from T_2 to t, we obtain

$$v(t) - v(T_2) = \int_{T_2}^{t} v'(s)ds \leq -\varGamma\int_{T_2}^{t} h(s)ds.$$

Since h belongs to $\mathcal{F}_{[\mathrm{WIP}]}$, we see that

$$\lim_{t\to\infty} \Gamma \int_{T_2}^{t} h(s)ds = \infty.$$

On the other hand, from (2.8), we get

$$v(t) - v(T_2) \geq v^* - 2v^* = -v^*$$

for $t \geq T_2$. This is a contradiction. Thus, we conclude that $\liminf_{t\to\infty} |y(t)| = 0$.

Suppose that $\limsup_{t\to\infty} |y(t)| > 0$, and let

$$\mu = \limsup_{t\to\infty} |y(t)|.$$

Then, we can choose an ε so small that

$$0 < \varepsilon < \min\left\{\frac{\mu}{2}, \frac{\widetilde{\Psi}^{-1}(v^*)}{2}, -\frac{\widetilde{\Psi}^{-1}(-v^*)}{2}\right\} \tag{2.9}$$

and

$$\frac{4\varepsilon}{\delta_0} + 2(1+2\varepsilon_0)\varepsilon < \omega \min\left\{\varphi\left(\widetilde{\Phi}^{-1}(M)\right), -\varphi\left(\widetilde{\Phi}^{-1}(-M)\right)\right\}, \tag{2.10}$$

where $M = M(\varepsilon) = v^* - \max\left\{\Psi(2\varepsilon), \Psi(-2\varepsilon)\right\}$. In fact, the left-hand side of (2.10) approaches 0 as $\varepsilon \to 0$. On the other hand, since $M(\varepsilon)$ tends to v^* as $\varepsilon \to 0$, the right-hand side of (2.10) approaches $\omega \min\left\{\varphi\left(\widetilde{\Phi}^{-1}(v^*)\right), -\varphi\left(\widetilde{\Phi}^{-1}(-v^*)\right)\right\}$. Note that $M(\varepsilon)$ is positive for any $\varepsilon > 0$ that satisfies (2.9).

It follows from (2.9) that

$$\liminf_{t\to\infty} |y(t)| = 0 < 2\varepsilon < \mu = \limsup_{t\to\infty} |y(t)|.$$

Since the inferior limit of $|y(t)|$ is zero, we can find a $t_* > T_1$ so that $|y(t_*)| < \varepsilon$. Moreover, since the superior limit of $|y(t)|$ is larger than 2ε, we can choose numbers s_1, τ_1, and σ_1 such that $s_1 = \inf\{t > t_* : |y(t)| > 2\varepsilon\}$, $\tau_1 = \sup\{t < s_1 : |y(t)| < \varepsilon\}$, and $\sigma_1 = \inf\{t > s_1 : |y(t)| < \varepsilon\}$. It is easy to verify that $|y(s_1)| = 2\varepsilon$, $|y(\tau_1)| = |y(\sigma_1)| = \varepsilon$, and $|y(t)| \geq \varepsilon$ for $\tau_1 < t < \sigma_1$. Using σ_1 instead of t_*, we define τ_2 and σ_2 similarly to τ_1 and σ_1, and so on. Then, we obtain numbers s_n, τ_n, and σ_n with $n \in \mathbb{N}$ such that $s_n = \inf\{t > \sigma_{n-1} : |y(t)| > 2\varepsilon\}$, $\tau_n = \sup\{t < s_n : |y(t)| < \varepsilon\}$, and $\sigma_n = \inf\{t > s_n : |y(t)| < \varepsilon\}$. It is clear that $T_1 < \tau_n < s_n < \sigma_n < \tau_{n+1}$ and $\tau_n \to \infty$ as $n \to \infty$. The three sequences $\{s_n\}$, $\{\tau_n\}$, and $\{\sigma_n\}$ satisfy $|y(s_n)| = 2\varepsilon$, $|y(\tau_n)| = |y(\sigma_n)| = \varepsilon$, and

$$|y(t)| \geq \varepsilon \quad \text{for } \tau_n < t < \sigma_n, \tag{2.11}$$

$$|y(t)| \leq 2\varepsilon \quad \text{for } \sigma_n < t < \tau_{n+1}, \tag{2.12}$$

$$\varepsilon < |y(t)| < 2\varepsilon \quad \text{for } \tau_n < t < s_n. \tag{2.13}$$

From (2.8), we see that $\Phi(x(t)) \leq v(t) \leq 2v^*$ for $t \geq T_1$. Hence, we have

$$\widetilde{\Phi}^{-1}(-2v^*) \leq x(t) \leq \widetilde{\Phi}^{-1}(2v^*) \quad \text{for } t \geq T_1,$$

and therefore,

$$|\varphi(x(t))| \leq \max\left\{\varphi\left(\widetilde{\Phi}^{-1}(2v^*)\right), -\varphi\left(\widetilde{\Phi}^{-1}(-2v^*)\right)\right\}$$

for $t \geq T_1$. For simplicity, let $L = \max\left\{\varphi\left(\widetilde{\Phi}^{-1}(2v^*)\right), -\varphi\left(\widetilde{\Phi}^{-1}(-2v^*)\right)\right\} > 0$. Using (2.5) and (2.13), we can estimate that

$$\begin{aligned}3\varepsilon^2 &= y^2(s_n) - y^2(\tau_n) = 2\int_{\tau_n}^{s_n} y(t)y'(t)dt \\ &= -2\omega\int_{\tau_n}^{s_n}\varphi(x(t))y(t)dt - 2\int_{\tau_n}^{s_n}h(t)y^2(t)dt \\ &\leq 2\omega\int_{\tau_n}^{s_n}|\varphi(x(t))||y(t)|dt \leq 4\varepsilon\omega\int_{\tau_n}^{s_n}|\varphi(x(t))|dt \\ &\leq 4L\varepsilon\omega(s_n - \tau_n).\end{aligned}$$

Thus, we can estimate that

$$s_n - \tau_n \geq \frac{3\varepsilon}{4L\omega} \stackrel{\text{def}}{=} m > 0$$

for each $n \in \mathbb{N}$. It is clear that the positive number m is independent of $n \in \mathbb{N}$. Since $[\tau_n, s_n] \subsetneq [\tau_n, \sigma_n]$, we conclude that $\liminf_{n\to\infty}(\sigma_n - \tau_n) \geq m > 0$.

Define

$$S = \{n \in \mathbb{N} : h(\sigma_n) \geq 1 + \varepsilon_0\},$$

and let $\operatorname{card} S$ denote the number of elements of the set S. We show that $\operatorname{card} S$ is finite. By way of contradiction, we suppose that $\operatorname{card} S$ is infinite. From (2.11), we see that

$$\Delta(y(t)) \geq \min\{\Delta(\varepsilon), \Delta(-\varepsilon)\} \stackrel{\text{def}}{=} \rho$$

for $\tau_n \leq t \leq \sigma_n$. As shown above, $\tau_n + m \leq \sigma_n$ for each $n \in \mathbb{N}$. Hence,

$$\Delta(y(t)) \geq \rho \quad \text{for } \sigma_n - m \leq t \leq \sigma_n. \tag{2.14}$$

From the assumption of $h(t)$, it follows that

$$|h(t) - h(\sigma_n)| < \varepsilon_0 \quad \text{for } \sigma_n - \delta_0 < t < \sigma_n + \delta_0.$$

Hence, $n \in S$ implies that

$$1 + \varepsilon_0 - h(t) \leq h(\sigma_n) - h(t) \leq |h(t) - h(\sigma_n)| < \varepsilon_0$$

for $\sigma_n - \delta_0 < t < \sigma_n + \delta_0$; namely,

$$h(t) > 1 \quad \text{for } \sigma_n - \delta_0 < t < \sigma_n + \delta_0. \tag{2.15}$$

Let $2d = \min\{\delta_0, m\}$. Then, by (2.14) and (2.15), we have

$$\int_{\sigma_n - d}^{\sigma_n} h(t)\Delta(y(t))dt > \ell\rho \quad \text{if } n \in S.$$

Using this inequality, we get

$$\begin{aligned}v^* - v(t_0) \leq v(t) - v(t_0) &= \int_{t_0}^{t} v'(s)ds = -\int_{t_0}^{t} h(s)y^2(s)ds \\ &\leq -\sum_{n\in S}\int_{\sigma_n - d}^{\sigma_n} h(t)\Delta(y(t))dt = -d\rho\operatorname{card} S = -\infty,\end{aligned}$$

which is a contradiction.

Since card S is finite, we can find an $N \in \mathbb{N}$ such that

$$h(\sigma_n) < 1 + \varepsilon_0 \quad \text{for } n \geq N. \tag{2.16}$$

We next show that $\tau_{n+1} - \sigma_n \leq \delta_0$ for $n \geq N$. Suppose there exists an $n_0 \geq N$ such that

$$\sigma_{n_0} + \delta_0 < \tau_{n_0+1}. \tag{2.17}$$

From (2.7), (2.8), and (2.12), we obtain

$$\Phi(x(t)) = v(t) - \Psi(y(t)) \geq M > 0$$

for $\sigma_{n_0} \leq t \leq \tau_{n_0+1}$. Hence, it is necessary to address two cases: (a) $x(t) \geq \widetilde{\Phi}^{-1}(M) > 0$ for $\sigma_{n_0} \leq t \leq \tau_{n_0+1}$, and (b) $x(t) \leq \widetilde{\Phi}^{-1}(-M) < 0$ for $\sigma_{n_0} \leq t \leq \tau_{n_0+1}$. Note that

$$h(t) < \varepsilon_0 + h(\sigma_{n_0}) < 1 + 2\varepsilon_0 \quad \text{for } \sigma_{n_0} \leq t \leq \sigma_{n_0} + \delta_0$$

because of (2.14) and (2.16). In the former case, using (2.5), (2.10), (2.12), and (2.17), we get

$$\begin{aligned} y'(t) &= -\omega\varphi(x(t)) - h(t)y(t) \leq -\omega\varphi\left(\widetilde{\Phi}^{-1}(M)\right) + h(t)|y(t)| \\ &< -\frac{4\varepsilon}{\delta_0} - 2(1+2\varepsilon_0)\varepsilon + 2(1+2\varepsilon_0)\varepsilon = -\frac{4\varepsilon}{\delta_0} \end{aligned}$$

for $\sigma_{n_0} \leq t \leq \sigma_{n_0} + \delta_0$. In the latter case, we get

$$\begin{aligned} y'(t) &= -\omega\varphi(x(t)) - h(t)y(t) \geq -\omega\varphi\left(\widetilde{\Phi}^{-1}(-M)\right) - h(t)|y(t)| \\ &> \frac{4\varepsilon}{\delta_0} + 2(1+2\varepsilon_0)\varepsilon - 2(1+2\varepsilon_0)\varepsilon = \frac{4\varepsilon}{\delta_0} \end{aligned}$$

for $\sigma_{n_0} \leq t \leq \sigma_{n_0} + \delta_0$. Thus, in either case, we have

$$|y'(t)| > \frac{4\varepsilon}{\delta_0} \quad \text{for } \sigma_{n_0} \leq t \leq \sigma_{n_0} + \delta_0.$$

Integrating this inequality from σ_{n_0} to $\sigma_{n_0} + \delta_0$, we obtain

$$|y(\sigma_{n_0} + \delta_0)| + |y(\sigma_{n_0})| \geq \left| \int_{\sigma_{n_0}}^{\sigma_{n_0}+\delta_0} y'(t)dt \right| = \int_{\sigma_{n_0}}^{\sigma_{n_0}+\delta_0} |y'(t)|dt > 4\varepsilon.$$

However, it follows from (2.12) and (2.17) that

$$|y(\sigma_{n_0} + \delta_0)| + |y(\sigma_{n_0})| \leq 4\varepsilon.$$

This is a contradiction. We therefore conclude that $\limsup_{n\to\infty}(\tau_{n+1} - \sigma_n) \leq \delta_0 < \infty$.

Recall that $\tau_n < \sigma_n < \tau_{n+1}$ and $\liminf_{n\to\infty}(\sigma_n - \tau_n) \geq m > 0$. Since h belongs to $\mathcal{F}_{[\mathrm{WIP}]}$, we see that

$$\sum_{n=1}^{\infty} \int_{\tau_n}^{\sigma_n} h(t)dt = \infty. \tag{2.18}$$

On the other hand, it follows from (2.11) that

$$\int_{t_0}^{\infty} v'(t)dt = -\int_{t_0}^{\infty} h(t)\Delta(y(t))dt \leq -\rho \sum_{n=1}^{\infty} \int_{\tau_n}^{\sigma_n} h(t)dt.$$

Since

$$\int_{t_0}^{\infty} v'(t)dt = \lim_{t\to\infty} v(t) - v(t_0) = v^* - v(t_0) < 0,$$

we obtain

$$\sum_{n=1}^{\infty}\int_{\tau_n}^{\sigma_n} h(t)dt \le \frac{v(t_0)-v^*}{\rho} < \infty.$$

This contradicts (2.18). Thus, we conclude that $\limsup_{t\to\infty}|y(t)| = \mu = 0$. The proof of Step (1) is now complete.

Step (2): From the conclusion of Step (1), it follows that $\lim_{t\to\infty} x(t) = \widetilde{\Phi}^{-1}(v^*) > 0$ or $\lim_{t\to\infty} x(t) = \widetilde{\Phi}^{-1}(-v^*) < 0$. Considering the vector field of (2.5), we see that the solution (x,y) must approach the point $\left(\widetilde{\Phi}^{-1}(v^*),0\right)$ or the point $\left(\widetilde{\Phi}^{-1}(-v^*),0\right)$ by ultimately passing through the region

$$\left\{(x,y) : x > \widetilde{\Phi}^{-1}(v^*) \text{ and } y < 0\right\}$$

or the region

$$\left\{(x,y) : x < \widetilde{\Phi}^{-1}(-v^*) \text{ and } y > 0\right\}.$$

Hence, we can find a $T_3 \ge t_0$ such that

$$x(t) > \widetilde{\Phi}^{-1}(v^*) \quad \text{and} \quad y(t) < 0 \quad \text{for } t \ge T_3 \tag{2.19}$$

or

$$x(t) < \widetilde{\Phi}^{-1}(-v^*) \quad \text{and} \quad y(t) > 0 \quad \text{for } t \ge T_3. \tag{2.20}$$

We consider only the former case, because the latter case is proved in the same way by using (2.20) instead of (2.19). By (2.5) and (2.19), we have

$$y'(t) + h(t)y(t) = -\omega\varphi(x(t)) < -\omega\varphi\left(\widetilde{\Phi}^{-1}(v^*)\right)$$

for $t \ge T_3$. Multiplying both sides of this inequality by $e^{H(t)}$ and integrating from T_3 to t, we obtain

$$y(t) < y(t) - e^{(H(T_3)-H(t))}y(T_3) < -\omega\varphi\left(\widetilde{\Phi}^{-1}(v^*)\right)\frac{\int_{T_3}^{t} e^{H(s)}ds}{e^{H(t)}}$$

for $t \ge T_3$. Since $h(t) \ge 0$ for $t \ge \alpha$, it is clear that

$$\int_{\alpha}^{\infty} e^{H(t)}dt = \infty.$$

Hence, we can choose a $T_4 \ge T_3$ such that

$$\int_{T_3}^{t} e^{H(s)}ds > \frac{1}{2}\int_{\alpha}^{t} e^{H(s)}ds \quad \text{for } t \ge T_4.$$

Using this inequality, we can estimate that

$$y(t) < -\frac{\omega}{2}\varphi\left(\widetilde{\Phi}^{-1}(v^*)\right)\frac{\int_{\alpha}^{t} e^{H(s)}ds}{e^{H(t)}} \quad \text{for } t \ge T_4.$$

From this estimation and condition (2.3), we see that

$$\begin{aligned} x'(t) = \varphi^{-1}(\omega y(t)) &< \varphi^{-1}\left(-\frac{\omega^2}{2}\varphi\left(\widetilde{\Phi}^{-1}(v^*)\right)\frac{\int_{\alpha}^{t} e^{H(s)}ds}{e^{H(t)}}\right) \\ &\le -g\left(\frac{\omega^2}{2}\varphi\left(\widetilde{\Phi}^{-1}(v^*)\right)\right)\varphi^{-1}\left(\frac{\int_{\alpha}^{t} e^{H(s)}ds}{e^{H(t)}}\right) \end{aligned}$$

for $t \geq T_4$. We therefore conclude that

$$\begin{aligned}\widetilde{\Phi}^{-1}(v^*) < x(t) &< -g\left(\frac{\omega^2}{2}\varphi\left(\widetilde{\Phi}^{-1}(v^*)\right)\right)\int_{T_4}^{t}\varphi^{-1}\left(\frac{\int_\alpha^t e^{H(s)}ds}{e^{H(t)}}\right)dt + x(T_4)\\ &= -g\left(\frac{\omega^2}{2}\varphi\left(\widetilde{\Phi}^{-1}(v^*)\right)\right)\int_{\alpha}^{t}\varphi^{-1}\left(\frac{\int_\alpha^t e^{H(s)}ds}{e^{H(t)}}\right)dt\\ &\quad + g\left(\frac{\omega^2}{2}\varphi\left(\widetilde{\Phi}^{-1}(v^*)\right)\right)\int_{\alpha}^{T_4}\varphi^{-1}\left(\frac{\int_\alpha^t e^{H(s)}ds}{e^{H(t)}}\right)dt + x(T_4).\end{aligned}$$

This contradicts condition (2.4). The proof of Step (2) is now complete.

Theorem 2.2 is thus proved. □

Proof of Theorem 2.3 Let (x, y) be any solution of (2.5) with the initial time $t_0 \geq \alpha$, and let v be the function defined by (2.7). Since the damping coefficient h is nonnegative, the function v has a limiting value $v^* \geq 0$. Recall that we showed that $v^* = 0$ using two steps in the proof of Theorem 2.2. Theorem 2.3 can be proved in the same manner as Theorem 2.2.

Recall that $\mathcal{F}_{[\mathrm{IP}]} \subsetneq \mathcal{F}_{[\mathrm{WIP}]}$. In the first step, we can conclude that $\liminf_{t\to\infty} |y(t)|$ by using $h \in \mathcal{F}_{[\mathrm{IP}]}$ instead of $h \in \mathcal{F}_{[\mathrm{WIP}]}$. Suppose that $\mu = \limsup_{t\to\infty} |y(t)| > 0$. Then, as in the proof of Theorem 2.2, we can define three sequences $\{s_n\}$, $\{\tau_n\}$, and $\{\sigma_n\}$ that satisfy $|y(s_n)| = 2\varepsilon$, $|y(\tau_n)| = |y(\sigma_n)| = \varepsilon$ and inequalities (2.11)–(2.13), where ε is a small number satisfying (2.9) and (2.10). We can also show that $\liminf_{n\to\infty}(\sigma_n - \tau_n) \geq m$ for some positive number m. Hence, from the assumption that h belongs to $\mathcal{F}_{[\mathrm{IP}]}$, we obtain the estimation (2.18). However, from (2.11) we have

$$-\infty < v^* - v(t_0) = \int_{t_0}^{\infty} v'(t)dt = -\int_{t_0}^{\infty} h(t)\Delta(y(t))dt \leq -\rho\sum_{n=1}^{\infty}\int_{\tau_n}^{\sigma_n} h(t)dt,$$

where $\rho = \min\{\Delta(\varepsilon), \Delta(-\varepsilon)\}$. This contradicts (2.18). Hence, we conclude that $\mu = 0$.

The proof of second step is the same as that of Theorem 2.2. □

The following results are direct conclusions of Theorems 2.1–2.3.

Theorem 2.5 *Let conditions* (2.2) *and* (2.3) *hold. Suppose that there exists an* $\varepsilon_0 > 0$ *and a* $\delta_0 > 0$ *such that* $|h(t) - h(s)| < \varepsilon_0$ *for all* $t \geq \alpha$ *and* $s \geq \alpha$ *with* $|t - s| < \delta_0$ *and suppose that* h *belongs to* $\mathcal{F}_{[\mathrm{WIP}]}$. *Then, the equilibrium of* (2.1) *is globally asymptotically stable if and only if condition* (2.4) *holds.*

Theorem 2.6 *Let conditions* (2.2) *and* (2.3) *hold. Suppose that* h *belongs to* $\mathcal{F}_{[\mathrm{IP}]}$. *Then, the equilibrium of* (2.1) *is globally asymptotically stable if and only if condition* (2.4) *holds.*

3 Functions which satisfy conditions (2.2) and (2.3)

Eq. (2.1) expressed using the nonlinear function φ satisfies (2.2) and/or (2.3). Of course, when φ is linear, that is, $z = \varphi(w) = w$ for $w \in \mathbb{R}$, the inverse function φ^{-1} satisfies $w = \varphi^{-1}(z) = z$. Therefore, conditions (2.2) and (2.3) are satisfied with $f = \varphi^{-1}$ and $g = \varphi^{-1}$, respectively. For this reason, Eq. (2.1) is a natural generalization of the damped linear oscillator

$$x'' + h(t)x' + \omega^2 x = 0.$$

In this sense, we may call Eq. (2.1) the *damped quasilinear oscillator*.

We give several examples of functions that satisfy conditions (2.2) and (2.3) in this section. First, we define

$$z = \phi_p(w) = \begin{cases} |w|^{p-2}w & \text{if } w \neq 0, \\ 0 & \text{if } w = 0, \end{cases}$$

with $w \in \mathbb{R}$ and $p > 1$. Then, ϕ_p is a continuous and strictly increasing function on $\mathbb{R}$ satisfying $w\phi_p(w) > 0$ if $w \neq 0$ and $\varphi(w)$ tends to $\pm\infty$ as $w \to \pm\infty$. Let

$$\frac{1}{p} + \frac{1}{p^*} = 1.$$

Then, p^* is also greater than 1. Since $(p-1)(p^*-1) = 1$, we see that

$$w = \phi_{p^*}(z) = \begin{cases} |z|^{p-2}z & \text{if } z \neq 0, \\ 0 & \text{if } z = 0, \end{cases}$$

with $z \in \mathbb{R}$; namely, ϕ_{p^*} is the inverse function of ϕ_p. It is clear that $\phi_{p^*}(ab) = \phi_{p^*}(a)\phi_{p^*}(b)$ and $\phi_{p^*}(-ab) = -\phi_{p^*}(a)\phi_{p^*}(b)$ for all $a > 0$ and $b > 0$. Hence, when $\varphi(w) = \phi_p(w)$ for $w \in \mathbb{R}$, conditions (2.2) and (2.3) are satisfied with $f = \varphi^{-1} = \phi_{p^*}$ and $g = \varphi^{-1} = \phi_{p^*}$.

Let $\varphi = \phi_p + \phi_q$ with $1 < q \leq p$. If $0 \leq w < 1$, then $\phi_q(w) \geq \phi_p(w) \geq 0$. Hence, we have

$$2\phi_p(w) \leq \varphi(w) \leq 2\phi_q(w)$$

and

$$\phi_q(w) \leq \varphi(w)$$

for $0 \leq w < 1$. From these inequalities, it follows that

$$\phi_{q^*}\left(\frac{z}{2}\right) \leq \varphi^{-1}(z) \leq \phi_{p^*}\left(\frac{z}{2}\right) \tag{3.1}$$

and

$$\varphi^{-1}(z) \leq \phi_{q^*}(z) \tag{3.2}$$

for $0 \leq z \leq \varphi(1) = \phi_p(1) + \phi_q(1) = 2$. On the other hand, if $w \geq 1$, then $1 \leq \phi_q(w) \leq \phi_p(w)$. Hence, we have

$$2\phi_q(w) \leq \varphi(w) \leq 2\phi_p(w)$$

and

$$\phi_p(w) \leq \varphi(w)$$

for $w \geq 1$. From these inequalities, it follows that

$$\phi_{p^*}\left(\frac{z}{2}\right) \leq \varphi^{-1}(z) \leq \phi_{q^*}\left(\frac{z}{2}\right) \tag{3.3}$$

and

$$\varphi^{-1}(z) \leq \phi_{p^*}(z) \tag{3.4}$$

for $z \ge 2$. Let us divide the region $R = \{(a,b) : a > 0 \text{ and } b > 0\}$ into four parts:

$$R_1 = \{(a,b) : ab \ge 2 \text{ and } b \ge 2\};$$

$$R_2 = \{(a,b) : ab \ge 2 \text{ and } 0 < b < 2\};$$

$$R_3 = \{(a,b) : ab < 2 \text{ and } 0 < b < 2\};$$

$$R_4 = \{(a,b) : ab < 2 \text{ and } b \ge 2\}.$$

If $(a,b) \in R_1$, then by (3.3) and (3.4), we have

$$\varphi^{-1}(ab) \le \phi_{p^*}(ab) = \phi_{p^*}(2a)\,\phi_{p^*}\left(\frac{b}{2}\right) \le \phi_{p^*}(2a)\,\varphi^{-1}(b),$$

$$\varphi^{-1}(ab) \ge \phi_{p^*}\left(\frac{ab}{2}\right) = \phi_{p^*}\left(\frac{a}{2}\right)\phi_{p^*}(b) \ge \phi_{p^*}\left(\frac{a}{2}\right)\varphi^{-1}(b).$$

If $(a,b) \in R_2$, then by (3.1) and (3.3), we have

$$\varphi^{-1}(ab) \le \phi_{q^*}\left(\frac{ab}{2}\right) = \phi_{q^*}(a)\,\phi_{q^*}\left(\frac{b}{2}\right) \le \phi_{q^*}(a)\,\varphi^{-1}(b),$$

$$\varphi^{-1}(ab) \ge \phi_{p^*}\left(\frac{ab}{2}\right) = \phi_{p^*}(a)\,\phi_{p^*}\left(\frac{b}{2}\right) \ge \phi_{p^*}(a)\,\varphi^{-1}(b).$$

If $(a,b) \in R_3$, then by (3.1) and (3.2), we have

$$\varphi^{-1}(ab) \le \phi_{q^*}(ab) = \phi_{q^*}(2a)\,\phi_{q^*}\left(\frac{b}{2}\right) \le \phi_{q^*}(2a)\,\varphi^{-1}(b),$$

$$\varphi^{-1}(ab) \ge \phi_{q^*}\left(\frac{ab}{2}\right) = \phi_{q^*}\left(\frac{a}{2}\right)\phi_{q^*}(b) \ge \phi_{q^*}\left(\frac{a}{2}\right)\varphi^{-1}(b).$$

If $(a,b) \in R_4$, then by (3.1) and (3.3), we have

$$\varphi^{-1}(ab) \le \phi_{p^*}\left(\frac{ab}{2}\right) = \phi_{p^*}(a)\,\phi_{p^*}\left(\frac{b}{2}\right) \le \phi_{p^*}(a)\,\varphi^{-1}(b),$$

$$\varphi^{-1}(ab) \ge \phi_{q^*}\left(\frac{ab}{2}\right) = \phi_{q^*}(a)\,\phi_{q^*}\left(\frac{b}{2}\right) \ge \phi_{q^*}(a)\,\varphi^{-1}(b).$$

Note that $\phi_{p^*}\left(\frac{a}{2}\right) \le \phi_{p^*}(a) \le \phi_{p^*}(2a)$ and $\phi_{q^*}\left(\frac{a}{2}\right) \le \phi_{q^*}(a) \le \phi_{q^*}(2a)$ for all $a > 0$ and $b > 0$. We therefore conclude that

$$\varphi^{-1}(ab) \le f(a)\,\varphi^{-1}(b),$$

$$\varphi^{-1}(ab) \ge g(a)\,\varphi^{-1}(b)$$

for all $a > 0$ and $b > 0$, where

$$\begin{aligned} f(a) &= \max\{\phi_{p^*}(2a),\, \phi_{q^*}(2a)\} \\ &= \begin{cases} \phi_{q^*}(2a) & \text{if } 0 \le a < 1/2, \\ \phi_{p^*}(2a) & \text{if } a \ge 1/2, \end{cases} \end{aligned}$$

$$
\begin{aligned}
g(a) &= \min\left\{\phi_{p^*}\left(\frac{a}{2}\right), \phi_{q^*}\left(\frac{a}{2}\right)\right\} \\
&= \begin{cases} \phi_{q^*}\left(\dfrac{a}{2}\right) & \text{if } 0 \le a < 2, \\ \phi_{p^*}\left(\dfrac{a}{2}\right) & \text{if } a \ge 2. \end{cases}
\end{aligned}
$$

Since $\phi_p + \phi_q$ is an odd function, we see that

$$
\varphi^{-1}(ab) \ge -f(a)\varphi^{-1}(b),
$$

$$
\varphi^{-1}(ab) \le -g(a)\varphi^{-1}(b)
$$

for all $a > 0$ and $b > 0$, where f and g are the functions given above. Thus, conditions (2.2) and (2.3) are satisfied.

Let us present a different type of function satisfying conditions (2.2) and (2.3). Define

$$
z = \varphi(w) = \begin{cases} e^w - 1 & \text{if } w \ge 0, \\ 1 - e^{-w} & \text{if } w < 0. \end{cases}
$$

Then, φ is a continuous and strictly increasing function on $\mathbb{R}$ satisfying $w\varphi(w) > 0$ if $w \ne 0$ and $\varphi(w)$ tends to $\pm\infty$ as $w \to \pm\infty$. The inverse function of φ is

$$
w = \varphi^{-1}(z) = \begin{cases} \ln(1+z) & \text{if } z \ge 0, \\ -\ln(1-z) & \text{if } z < 0. \end{cases}
$$

For $a > 0$ and $b > 0$, let

$$
\mathcal{G}(a,b) \overset{\text{def}}{=} \frac{\varphi^{-1}(ab)}{\varphi^{-1}(b)} = \frac{\ln(1+ab)}{\ln(1+b)}.
$$

Then, using l'Hôpital's rule, we obtain

$$
\lim_{b\to 0} \mathcal{G}(a,b) = \lim_{b\to 0} \frac{a(1+b)}{1+ab} = a
$$

and

$$
\lim_{b\to\infty} \mathcal{G}(a,b) = \lim_{b\to\infty} \frac{a/b+a}{1/b+a} = 1.
$$

We have

$$
\frac{\partial}{\partial b}\mathcal{G}(a,b) = \frac{a(1+b)\ln(1+b) - (1+ab)\ln(1+ab)}{(1+ab)(1+b)(\ln(1+b))^2}.
$$

Let $\mathcal{K}(a,b) = a(1+b)\ln(1+b) - (1+ab)\ln(1+ab)$, and let a^* be any fixed positive number. Then, $\mathcal{K}(a^*,0) = 0$, and

$$
\frac{d}{db}\mathcal{K}(a^*,b) = a^* \ln \frac{1+b}{1+a^*b}.
$$

Since $b > 0$, the function $\mathcal{K}$ is strictly decreasing with respect to b if $a^* > 1$ and strictly increasing with respect to b if $0 < a^* < 1$. Hence, we have

$$
\mathcal{K}(a^*,b) < 0 \quad \text{if } a^* > 1,
$$

$$
\mathcal{K}(a^*,b) = 0 \quad \text{if } a^* = 1,
$$

$$
\mathcal{K}(a^*,b) > 0 \quad \text{if } 0 < a^* < 1.
$$

This means that $\mathcal{G}(1,b) = 1$ for all $b > 0$; $\mathcal{G}(a,b)$ is strictly decreasing with respect to b in the region $\{(a,b) : a > 1 \text{ and } b > 0\}$, and $\mathcal{G}(a,b)$ is strictly increasing with respect to b in the region $\{(a,b) : 0 < a < 1 \text{ and } b > 0\}$. Hence, we see that

$$\min\{1, a\} \le \mathcal{G}(a,b) = \frac{\varphi^{-1}(ab)}{\varphi^{-1}(b)} \le \max\{1, a\}$$

for all $a > 0$ and $b > 0$. Since φ is an odd function, we see that

$$-\min\{1, a\} \ge \frac{\varphi^{-1}(-ab)}{\varphi^{-1}(b)} = -\frac{\varphi^{-1}(ab)}{\varphi^{-1}(b)} \ge -\max\{1, a\}$$

for all $a > 0$ and $b > 0$. We therefore conclude that conditions (2.2) and (2.3) are satisfied with f and g satisfying

$$f(a) = \max\{1, a\} \quad \text{and} \quad g(a) = \min\{1, a\}.$$

4 Proof of Theorems 1.1 and 1.2

Let $u(\mathbf{x})$ be any radially symmetric solution of (1.1), and let $\xi(t)$ be the function defined by $\xi(t) = u(\mathbf{x})$ and $t = |\mathbf{x}| \ge \alpha$. Then, we have $\nabla u(\mathbf{x}) = \dfrac{\xi'(t)}{t}\mathbf{x}$, and therefore,

$$\begin{aligned}\Delta_r u(\mathbf{x}) &= \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(|\nabla u(\mathbf{x})|^{r-2}\frac{\partial u}{\partial x_i}\right)\\ &= \left(|\xi'(t)|^{r-2}\xi'(t)\right)' + \frac{N-1}{t}|\xi'(t)|^{r-2}\xi'(t)\end{aligned}$$

and

$$\begin{aligned}\mathbf{x}\cdot|\nabla u(\mathbf{x})|^{r-2}\nabla u(\mathbf{x}) &= \sum_{i=1}^{N} x_i|\nabla u(\mathbf{x})|^{r-2}\frac{\partial u}{\partial x_i}\\ &= t|\xi'(t)|^{r-2}\xi'(t).\end{aligned}$$

Hence, the function $\xi(t)$ is a solution of the second-order differential equation

$$\left(\phi_p(\xi') + \phi_q(\xi')\right)' + \left(k(t) + \frac{N-1}{t}\right)\left(\phi_p(\xi') + \phi_q(\xi')\right) + \omega^2\left(\phi_p(\xi) + \phi_q(\xi)\right) = 0, \tag{4.1}$$

where ϕ_p and ϕ_q are functions given in Section 3. Since $\phi_p + \phi_q$ satisfies conditions (2.2) and (2.3) as shown in Section 3, we may regard Eq. (4.1) as a special case of (2.1) with h satisfying $h(t) = k(t) + (N-1)/t$. If the equilibrium of (4.1) is globally asymptotically stable, then $(\xi(t), \xi'(t))$ tends to the origin as $t \to \infty$.

Since $h(t) = k(t) + (N-1)/t$, condition (1.2) coincides with condition (2.4). If there exist an $\varepsilon_0 > 0$ and a $\delta_0 > 0$ such that $|k(t) - k(s)| < \varepsilon_0$ for all $t \ge \alpha$ and $s \ge \alpha$ with $|t - s| < \delta_0$, then

$$\begin{aligned}|h(t) - h(s)| &\le |k(t) - k(s)| + \left|\frac{N-1}{t} - \frac{N-1}{s}\right|\\ &< \varepsilon_0 + \frac{N-1}{\alpha^2}\delta_0\end{aligned}$$

for $t \ge \alpha$ and $s \ge \alpha$ with $|t - s| < \delta_0$. Because $h(t)$ is larger than $(N-1)/t$ for $t \ge \alpha$, it naturally belongs to $\mathcal{F}_{[\mathrm{WIP}]}$. Hence, all conditions of Theorems 2.1 and 2.2 are satisfied, and thus, we see that the equilibrium of (4.1) is globally asymptotically stable if and only if condition (1.2) holds. We therefore conclude that condition (1.2) is a necessary and sufficient condition for $u(\mathbf{x})$ and $|\nabla u(\mathbf{x})|$ to tend to zero as $|\mathbf{x}| \to \infty$. Theorem 1.1 is thus proved.

If k belongs to $\mathcal{F}_{[\mathrm{IP}]}$, the function h also belongs to $\mathcal{F}_{[\mathrm{IP}]}$. Hence, Theorem 1.2 can be deduced from Theorems 2.1 and 2.3. □

5 Power dissipation

To show the usefulness of Theorems 1.1 and 1.2, we consider the first-order linear differential equation

$$x' = 1 - \left(k(t) + \frac{N-1}{t}\right)x, \tag{5.1}$$

where k is a power function t^{ℓ} with $\ell \in \mathbb{R}$. Then, it is obvious that

$$z(t) = \frac{\int_{\alpha}^{t} e^{K(s)} ds}{e^{K(t)}}$$

is the particular solution of (5.1) satisfying the initial condition $z(\alpha) = 0$. For this reason, we first examine the asymptotic behaviour of the particular solution. Let

$$h(t) = k(t) + \frac{N-1}{t} \quad \text{for } t \geq \alpha.$$

The particular solution has a close relationship with the curve defined by $1/h(t)$ for $t \geq \alpha$. If this curve and the trajectory of the particular solution have a point of intersection, then the trajectory moves horizontally at this point.

We classify our argument into two cases: (i) $\ell \leq 0$, and (ii) $\ell > 0$.

Case (i) $\ell \leq 0$: The function h satisfies

$$\lim_{t\to\infty} \frac{1}{h(t)} = \begin{cases} \infty & \text{if } \ell < 0, \\ 1 & \text{if } \ell = 0, \end{cases}$$

where $1/h(t)$ is identically equal to 1 when $\ell = 0$ and $N = 1$. Since

$$\left(\frac{1}{h(t)}\right)' = -\frac{\ell t^{\ell-1} - (N-1)/t^2}{(t^{\ell} + (N-1)/t)^2},$$

we see that it is zero for $t > \alpha$ if $\ell = 0$ and $N = 1$; otherwise, it is positive for $t > \alpha$. Hence, there are three subcases. The curve defined by $1/h(t)$ is strictly increasing and tends to ∞ as $t \to \infty$ if $\ell < 0$; it is strictly increasing and approaches 1 as $t \to \infty$ if $\ell = 0$ and $N > 1$; and it is a horizontal line whose value is 1 if $\ell = 0$ and $N = 1$.

Case (ii) $\ell > 0$: The function h satisfies

$$\lim_{t\to\infty} \frac{1}{h(t)} = 0.$$

We divide this case into two subcases: $N > 1$ and $N = 1$. If $N > 1$, then there exists a $t^* \geq \alpha$ such that

$$\left(\frac{1}{h(t)}\right)' \begin{cases} > 0 & \text{for } \alpha < t < t^*, \\ < 0 & \text{for } t > t^*, \end{cases}$$

which means that, if $t^* = \alpha$, then $(1/h(t))' < 0$ for $t > \alpha$. If $N = 1$, then $(1/h(t))' < 0$ for $t > \alpha$. Hence, the curve defined by $1/h(t)$ is strictly decreasing for all sufficiently large t, and it tends to 0 as $t \to \infty$. The inflection point of this curve is at most one. The slope of this curve is zero at such an inflection point.

Example 5.1 Consider Eq. (1.1) with $k(t) = t^\ell$. If $\ell \le 0$, then all radially symmetric solutions u satisfy the property that $u(\mathbf{x})$ and $|\nabla u(\mathbf{x})|$ tend to zero as $|\mathbf{x}| \to \infty$.

Proof Since the power dissipation k is strictly decreasing or is equal to 1, we see that

$$|k(t) - k(s)| \le k(t) + k(s) \le 2\,k(\alpha)$$

for all $t \ge \alpha$ and $s \ge \alpha$. Hence, the assumption of Theorem 1.1 is satisfied in any $\varepsilon_0 > 2\,k(\alpha)$ and $\delta_0 > 0$. We check that condition (1.2) is also satisfied.

Let us compare the position of the curve defined by $1/h(t)$ and that of the trajectory of the particular solution z of (5.1) satisfying the initial condition that $z(\alpha) = 0$. Since

$$\frac{1}{h(\alpha)} = \frac{1}{\alpha^\ell + (N-1)/\alpha} > 0,$$

the trajectory is located below the curve in a neighborhood of the point $(\alpha, 0)$. To be precise, there exists a $\beta > \alpha$ such that

$$z(t) < \frac{1}{h(t)} \quad \text{for } \alpha \le t \le \beta.$$

From (5.1), we see that $z'(t) > 0$ for $\alpha \le t \le \beta$. This means that the trajectory rises in the neighborhood of the point $(\alpha, 0)$.

Suppose that the trajectory crosses the curve. Then, we can find a $\gamma > \beta$ such that $z(\gamma) = 1/h(\gamma)$,

$$z(t) < \frac{1}{h(t)} \quad \text{for } \alpha \le t < \gamma$$

and

$$z(t) > \frac{1}{h(t)} \quad \text{for } t > \gamma. \tag{5.2}$$

Since $z(\gamma) = 1/h(\gamma)$, it follows that $z'(\gamma) = 0$. Hence, the slope of the trajectory is zero on the curve. On the other hand, the slope of the curve is nonnegative in case (i) above. Thus, the trajectory cannot pass through the curve. This contradicts (5.2).

We therefore conclude that

$$z(t) \le \frac{1}{h(t)} \quad \text{for } t \ge \alpha;$$

that is, the trajectory does not drop. Hence, $z(t) \ge z(\beta) > 0$ for $t \ge \beta$. Since φ^{-1} is a strictly increasing function, we obtain

$$\int_\alpha^\infty \varphi^{-1}\left(\frac{\int_\alpha^t e^{K(s)} ds}{e^{K(t)}}\right) dt \ge \int_\alpha^\beta \varphi^{-1}(z(t))\,dt + \int_\beta^\infty \varphi^{-1}(z(\beta))\,dt = \infty,$$

and therefore, condition (1.2) is satisfied.

Thus, we can apply Theorem 1.1 to this example. □

Example 5.2 Consider Eq. (1.1) with $k(t) = t^\ell$. If $0 < \ell \le q-1$, then all radially symmetric solutions u satisfy the property that $u(\mathbf{x})$ and $|\nabla u(\mathbf{x})|$ tend to zero as $|\mathbf{x}| \to \infty$.

To prove Example 5.2, we need the following lemma and a technique in Zheng and Sugie [27].

Lemma 5.1 *Let $\zeta(t)$ be a nonnegatve continuous function on $[\alpha, \infty)$. If there exist numbers $\kappa > 0$ and $\theta > 0$ such that*

$$\theta = \lim_{t\to\infty} t^\kappa \zeta(t). \tag{5.3}$$

Then,

(a) *if* $\kappa \le 1$, *then* $\displaystyle\int_\alpha^\infty \zeta(t)dt = \infty$;

(b) *if* $\kappa > 1$, *then* $\displaystyle\int_\alpha^\infty \zeta(t)dt < \infty$.

Proof of Example 5.2 As in the proof of Example 5.1, we compare the position of the curve defined by $1/h(t)$ and that of the trajectory of the particular solution z of (5.1) satisfying the initial condition that $z(\alpha) = 0$. The trajectory is located below the curve in a neighborhood of the point $(\alpha, 0)$. The trajectory continues to rise as long as it is located below the curve. As shown above, in case (ii), the curve is strictly decreasing for all sufficiently large t, and it tends to zero as $t \to \infty$. Hence, the trajectory has to cross the curve. Let $(\gamma, 1/h(\gamma))$ be the intersecting point of the trajectory and the curve. Then, $z(t) > 1/h(t)$ for t in a right-hand side neighborhood of γ.

The trajectory does not intersect the curve again afterwards. In fact, the slope of the trajectory is zero on the curve, and the slope of the curve is negative. Hence, we conclude that

$$z(t) \ge \frac{1}{h(t)} \quad \text{for } t \ge \gamma. \tag{5.4}$$

This means that the trajectory is ultimately located above the curve. From (5.1) and (5.4), we see that $z'(t) \le 0$ for $t \ge \gamma$. Since $1/h(t)$ tends to zero as $t \to \infty$, the trajectory also ultimately tends to zero. If the trajectory does not tend to zero, then we find $\delta > 0$ such that $z(t) > \delta$ for $t \ge \gamma$. Since $z'(t) \le 0$ for $t \ge \gamma$, the slope of the trajectory ultimately approaches zero. However, we have

$$z(t) - \frac{1}{h(t)} > \frac{\delta}{2}$$

for all sufficiently large t. Hence, we obtain

$$z'(t) = 1 - h(t)z(t) < -\frac{\delta}{2}h(t) = -\frac{\delta}{2}\left(t^\ell + \frac{N-1}{t}\right),$$

which tends to $-\infty$ as $t \to \infty$. This is a contradiction. Thus, we conclude that $z(t)$ is decreasing and tends to zero as $t \to \infty$.

As proved in Section 3, the estimation (3.1) holds. Hence, there exists a $T > \gamma$ such that

$$\varphi^{-1}(z(t)) \ge \phi_{q^*}(z(t)/2) \quad \text{for } t \ge T.$$

Using this inequality, we get

$$\begin{aligned}\int_\alpha^\infty \varphi^{-1}\left(\frac{\int_\alpha^t e^{K(s)}ds}{e^{K(t)}}\right)dt &= \int_\alpha^T \varphi^{-1}(z(t))\,dt + \int_T^\infty \varphi^{-1}(z(t))\,dt \\ &\ge \int_\alpha^T \varphi^{-1}(z(t))\,dt + \frac{1}{\phi_{q^*}(2)}\int_T^\infty \phi_{q^*}(z(t))\,dt \\ &= \int_\alpha^T \varphi^{-1}(z(t))\,dt - \frac{1}{\phi_{q^*}(2)}\int_\alpha^T \phi_{q^*}(z(t))\,dt \\ &\quad + \frac{1}{\phi_{q^*}(2)}\int_\alpha^\infty \phi_{q^*}(z(t))\,dt.\end{aligned}$$

Hence, if the integral from α to ∞ of $\phi_{q^*}(z(t))$ diverges to ∞, then condition (1.2) is satisfied.

Since

$$\begin{aligned}K(t) &= \int_\alpha^t \left(s^\ell + \frac{N-1}{s}\right)ds \\ &= \frac{1}{\ell+1}\left(t^{\ell+1} - \alpha^{\ell+1}\right) + (N-1)\log\frac{t}{\alpha},\end{aligned}$$

it follows that

$$e^{K(t)} = c\,t^{N-1}e^{t^{\ell+1}/(\ell+1)},$$

where $c = 1\Big/\left(\alpha^{N-1}e^{\alpha^{\ell+1}/(\ell+1)}\right)$. Let $\kappa = \ell(q^*-1)$ and $\zeta(t) = \phi_{q^*}(z(t))$. Then, we have

$$t^\kappa = \left(t^\ell\right)^{q^*-1} = \phi_{q^*}\left(t^\ell\right)$$

and

$$\zeta(t) = \phi_{q^*}\left(\frac{t^\ell \int_\alpha^t s^{N-1}e^{\frac{1}{\ell+1}s^{\ell+1}}ds}{t^{N-1}e^{\frac{1}{\ell+1}t^{\ell+1}}}\right).$$

Using l'Hôpital's rule twice, we obtain

$$\begin{aligned}\lim_{t\to\infty} t^\kappa\zeta(t) &= \lim_{t\to\infty}\phi_{q^*}\left(\frac{t^\ell\int_\alpha^t s^{N-1}e^{\frac{1}{\ell+1}s^{\ell+1}}ds}{t^{N-1}e^{\frac{1}{\ell+1}t^{\ell+1}}}\right) = \phi_{q^*}\left(\lim_{t\to\infty}\frac{t^{\ell-(N-1)}\int_\alpha^t s^{N-1}e^{\frac{1}{\ell+1}s^{\ell+1}}ds}{e^{\frac{1}{\ell+1}t^{\ell+1}}}\right) \\ &= \phi_{q^*}\left(\lim_{t\to\infty}\frac{(\ell-(N-1))\,t^{\ell-N}\int_\alpha^t s^{N-1}e^{\frac{1}{\ell+1}s^{\ell+1}}ds + t^\ell\,e^{\frac{1}{\ell+1}t^{\ell+1}}}{t^\ell\,e^{\frac{1}{\ell+1}t^{\ell+1}}}\right) \\ &= \phi_{q^*}\left(\lim_{t\to\infty}\frac{(\ell-(N-1))\int_\alpha^t s^{N-1}e^{\frac{1}{\ell+1}s^{\ell+1}}ds}{t^N e^{\frac{1}{\ell+1}t^{\ell+1}}} + 1\right) \\ &= \phi_{q^*}\left(\lim_{t\to\infty}\frac{(\ell-(N-1))\,t^{N-1}e^{\frac{1}{\ell+1}t^{\ell+1}}}{N\,t^{N-1}e^{\frac{1}{\ell+1}t^{\ell+1}} + t^{\ell+N}e^{\frac{1}{\ell+1}t^{\ell+1}}} + 1\right) \\ &= \phi_{q^*}\left(\lim_{t\to\infty}\frac{\ell-(N-1)}{N+t^{\ell+1}} + 1\right) = \phi_{q^*}(1) = 1.\end{aligned}$$

Hence, condition (5.3) is satisfied with $\kappa = \ell(q^*-1)$ and $\theta = 1$. Since $(q-1)(q^*-1) = 1$, we see that $\kappa \le 1$ if and only if $\ell \le q-1$. Therefore, by virtue of Lemma 5.1 (a), we can conclude that

$$\int_\alpha^\infty \phi_{q^*}(z(t))\,dt = \int_\alpha^\infty \zeta(t)dt = \infty$$

if $\ell \le q-1$.

It is clear that k belongs to $\mathcal{F}_{[\mathrm{IP}]}$ in case (ii) above. Thus, we can apply Theorem 1.2 to this example. □

Example 5.3 Consider Eq. (1.1) with $k(t) = t^{\ell}$. If all radially symmetric solutions u satisfy the property that $u(\mathbf{x})$ and $|\nabla u(\mathbf{x})|$ tend to zero as $|\mathbf{x}| \to \infty$, then $\ell \le p-1$.

Proof We prove that if $\ell > p-1$, then condition (1.2) does not hold. Using the same method as in the proof of Example 5.2, we can conclude that

$$z(t) = \frac{\int_\alpha^t e^{K(s)} ds}{e^{K(t)}}$$

is decreasing and tends to zero as $t \to \infty$. From the estimation (3.1), we see that there exists a $T > 0$ such that

$$\varphi^{-1}(z(t)) \le \phi_{p^*}(z(t)/2) \quad \text{for } t \ge T.$$

Using this inequality, we get

$$\begin{aligned}
\int_\alpha^\infty \varphi^{-1}\left(\frac{\int_\alpha^t e^{K(s)} ds}{e^{K(t)}}\right) dt &= \int_\alpha^T \varphi^{-1}(z(t))\, dt + \int_T^\infty \varphi^{-1}(z(t))\, dt \\
&\le \int_\alpha^T \varphi^{-1}(z(t))\, dt + \frac{1}{\phi_{p^*}(2)} \int_T^\infty \phi_{p^*}(z(t))\, dt \\
&= \int_\alpha^T \varphi^{-1}(z(t))\, dt - \frac{1}{\phi_{p^*}(2)} \int_\alpha^T \phi_{p^*}(z(t))\, dt \\
&\quad + \frac{1}{\phi_{p^*}(2)} \int_\alpha^\infty \phi_{p^*}(z(t))\, dt.
\end{aligned}$$

Hence, if the integral from α to ∞ of $\phi_{p^*}(z(t))$ converges, then condition (1.2) is not satisfied.

Let $\kappa = \ell(p^* - 1)$ and $\zeta(t) = \phi_{p^*}(z(t))$. Then, as in the proof of Example 5.2, we obtain

$$\lim_{t\to\infty} t^\kappa \zeta(t) = \phi_{p^*}(1) = 1.$$

Hence, condition (5.3) is satisfied with $\kappa = \ell(p^* - 1)$ and $\theta = 1$. Since $(p-1)(p^* - 1) = 1$, we see that $\kappa > 1$ if and only if $\ell > p-1$. Therefore, by virtue of Lemma 5.1 (b), we can conclude that

$$\int_\alpha^\infty \phi_{p^*}(z(t))\, dt = \int_\alpha^\infty \zeta(t) dt < \infty$$

if $\ell > p-1$.

Because k belongs to $\mathcal{F}_{[\mathrm{IP}]}$ in case (ii) above, we can apply Theorem 1.2 to this example. □

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