# Oscillation problems for Hill's equation with periodic damping

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**Abstract**

This paper deals with the second-order linear differential equation $x'' + a(t)x' + b(t)x = 0$, where $a$ and $b$ are periodic coefficients. The main purpose is to present new criteria which guarantee that all nontrivial solutions are nonoscillatory and that those are oscillatory. Our nonoscillation theorem and oscillation theorem are proved by using the Riccati technique. In our theorem, the composite function of an indefinite integral of $b$ and a suitable multiple-valued continuously differentiable function are focused, and the composite function of them plays an important role. The results obtained here include a result by Kwong and Wong (2003) and a result by Sugie and Matsumura (2008). An application to a equation of Whittaker–Hill type is given to show the usefulness of our results. Finally, simulations are also attached to illustrate that our oscillation criterion is sharp.

*Key words:* Hill's equation; Oscillation problem; Damped linear differential equations; Riccati inequality; Oscillation constant
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## 1. Introduction

We consider the second-order linear differential equation

$$x'' + a(t)x' + b(t)x = 0, \tag{1.1}$$

where $a$, $b\colon [0,\infty) \to \mathbb{R}$ are continuous and periodic functions with period $T > 0$. Equation (1.1) has obviously the trivial solution $x \equiv 0$. Since all other solutions also exist in the future, they are divided into two groups as follows. A nontrivial solution $x$ of (1.1) is said to be *oscillatory* if it has an infinite number of zeros on the interval $[0,\infty)$. Otherwise, the nontrivial solution is said to be *nonoscillatory*. In other words, if $x$ is a nonoscillatory

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solution of (1.1), then there exists a $t^* \geq 0$ such that $x(t) > 0$ for $t \geq t^*$ or $x(t) < 0$ for $t \geq t^*$.

We can consider equation (1.1) to be a motion equation having a very simple form. For this reason, equation (1.1) has been widely studied as models which appear not only in pure mathematics but also in various fields. We can easily find the literatures related to oscillation theory for equation (1.1) and more generalized equations including (1.1) (refer to [1, 2, 3, 6, 7, 18, 22, 24, 27, 28, 30, 31, 32, 33, 34]).

In the special case when $a$ is equal to zero identically, equation (1.1) becomes the differential equation

$$x'' + b(t)x = 0 \tag{1.2}$$

which is famous in astrophysics. This equation was proposed by George William Hill to analyze the orbit of the moon originally (see [9]). Equation (1.2) is called Hill's equation named after him.

Let $p$ be any continuous and periodic function with period $T > 0$. By taking into account the definite integral of $p$ from 0 to $T$, we can define the following family of functions. The periodic function $p$ is said to be periodic of *mean value zero* if $p$ is not identically zero and

$$\int_0^T p(t)dt = 0.$$

Let us denote by $\mathscr{F}_{[\mathrm{MVZ}]}$ the family of functions which are periodic of mean value zero. About other applications of Hill's equation, refer to [19, 20]. We can find various results about the oscillation problem of (1.2) in many literatures (for example, see [3, 15, 16, 17, 27]).

It is well known that if $b$ belongs to $\mathscr{F}_{[\mathrm{MVZ}]}$, then all nontrivial solutions of (1.2) are oscillatory (for the proof, see [3, p. 25]). For example, if $b(t) = \sin t$ (or $b(t) = \cos t$), then all nontrivial solutions of (1.2) are oscillatory. However, even if $a$ and $b$ belong to $\mathscr{F}_{[\mathrm{MVZ}]}$, all solutions of (1.1) are not always oscillatory. By focusing on this fact, Kwong and Wong [15] have studied oscillation and nonoscillation of equation (1.1). Let $B$ be an indefinite integral of $b$. Then, their nonoscillation criterion [15, Theorem 1] can be stated as follows.

**Theorem A.** *Suppose that $b$ belongs to $\mathscr{F}_{[\mathrm{MVZ}]}$. If*

$$(B(t) - a(t))B(t) \leq 0 \quad \textit{for} \;\; 0 \leq t \leq T,$$

*then all nontrivial solutions of* (1.1) *are nonoscillatory.*

**Remark 1.1.** It is clear that the difference of two indefinite integrals of $b$ is constant. The periodic coefficient $b$ belongs to $\mathscr{F}_{[\mathrm{MVZ}]}$ if and only if all indefinite integrals of $b$ are periodic. The condition that $b$ belongs to $\mathscr{F}_{[\mathrm{MVZ}]}$ is used only to show that $B$ is a periodic function with period $T$. Hence, we can rewritten the statement of Theorem A as "If $B$ is periodic and

$$(B(t) - a(t))B(t) \leq 0 \quad \text{for} \;\; 0 \leq t \leq T,$$

then all nontrivial solutions of (1.1) are nonoscillatory."

A typical example that can be applied to Theorem A is

$$x'' + (\sin t)x' + (\cos t)x = 0. \tag{1.3}$$

Since $b(t) = \cos t$ in equation (1.3), we can choose $B(t) = \sin t$. Hence, we have

$$(B(t) - a(t))B(t) = (\sin t - \sin t)\sin t = 0 \quad \text{for } 0 \le t \le 2\pi.$$

In fact, equation (1.3) has a nonoscillatory solution $x(t) = \exp(\cos t)$. Kwong and Wong [15, Theorem 2] also presented a criterion for all solutions of (1.1) to be oscillatory. Their result was extended by Sugie and Matsumura [26, Theorem 3.1] as follows.

**Theorem B.** *Suppose that $a$ and $B$ belong to $\mathscr{F}_{[\mathrm{MVZ}]}$. If*

$$\int_0^T \left\{ (B(t) - a(t))B(t) \exp \int_0^t (a(s) - 2B(s))\, ds \right\} dt > 0,$$

*then all nontrivial solutions of* (1.1) *are oscillatory.*

**Remark 1.2.** If $B$ belongs to $\mathscr{F}_{[\mathrm{MVZ}]}$, then $b$ also belongs to $\mathscr{F}_{[\mathrm{MVZ}]}$. Hence, it is inevitably assumed that $b$ belongs to $\mathscr{F}_{[\mathrm{MVZ}]}$ in Theorem B.

It is known that all nontrivial solutions of

$$x'' + (\cos t)x' + (\sin t)x = 0$$

are oscillatory (for example, see [31, pp. 255–256] and [34, p. 427]). This fact can also be verified by using Theorem B. To apply Theorem A or Theorem B to equation (1.1), the periodic coefficient $b$ needs to belong to $\mathscr{F}_{[\mathrm{MVZ}]}$. For example, we consider the damped linear equation

$$x'' + (\sin t)x' + (\varepsilon + \cos t)x = 0, \tag{1.4}$$

where $\varepsilon$ is a real number. Comparing equation (1.4) with equation (1.1), we see that $a(t) = \sin t$ and $b(t) = \varepsilon + \cos t$. Hence, there exists no indefinite integral of $b$ belonging to $\mathscr{F}_{[\mathrm{MVZ}]}$ if $\varepsilon \ne 0$, and therefore, we cannot apply Theorems A and B to equation (1.4).

Then, what is the condition that guarantees that all nontrivial solutions of (1.1) with a periodic coefficient $b$ which does not belong to $\mathscr{F}_{[\mathrm{MVZ}]}$ oscillate or not? In this paper, we deal with this problem. For this purpose, we focus on the composite function of an indefinite integral $B$ and a continuously differentiable function $F$ defined on the range of $B$. Let $f$ be a derivative function of $F$. Our results are as follows.

**Theorem 1.1.** *If the composite function $F(B)$ is periodic and*

$$\big(F(B(t)) - a(t)\big)F(B(t)) + \big(1 - f(B(t))\big)b(t) \le 0 \tag{1.5}$$

*holds for $0 \le t \le T$, then all nontrivial solutions of* (1.1) *are nonoscillatory.*

**Theorem 1.2.** *Suppose that $a$ and $F(B)$ belong to $\mathscr{F}_{[\mathrm{MVZ}]}$. Let*

$$E(t) = \exp \int_0^t \big(a(s) - 2F(B(s))\big)\, ds.$$

*If*

$$\int_0^T \Big\{\big(F(B(t)) - a(t)\big)F(B(t)) + \big(1 - f(B(t))\big)b(t)\Big\} E(t)\, dt > 0, \tag{1.6}$$

*then all nontrivial solutions of* (1.1) *are oscillatory.*

**Remark 1.3.** In Theorems 1.1 and 1.2, the function $F$ is permitted to be multiple-valued. However, the composite function $F(B)$ should be a continuous and single-valued function even if $F$ is a multiple-valued function. When $F$ is a multiple-valued function, its derivative function $f$ is also multiple-valued. In this case, a combination of suitable branches of $f$ has only to satisfy inequality (1.6) (see the concrete example given in Section 3 on how to decide suitable branches).

**Remark 1.4.** Let $u = B(t)$. Since $B$ and $F$ in Theorems 1.1 and 1.2 are differentiable functions, their composite function $F(B)$ is also differentiable, and its derivative with respect to $t$ is

$$\big(F(B(t))\big)' = \frac{du}{dt}\frac{d}{du}F(u)\Big|_{u=B(t)} = b(t)\, f(u)\Big|_{u=B(t)} = f(B(t))b(t).$$

In the equality above, $F$ and $f$ mean a suitable branch when $F$ is a multiple-valued function.

**Remark 1.5.** In Theorem 1.2, the periodic coefficient $a$ and the composite function $F(B)$ belong to $\mathscr{F}_{[\mathrm{MVZ}]}$. Hence, the function $E$ is a periodic function with period $T$.

## 2. Proof of Theorems 1.1 and 1.2

The Riccati inequality corresponding to equation (1.1) is

$$r' \geq r^2 - a(t)r + b(t). \tag{2.1}$$

The following relation between equation (1.1) and inequality (2.1) is known well (for example, refer to the book [7, pp. 362–363]).

**Lemma 2.1.** *All nontrivial solutions of* (1.1) *are nonoscillatory if and only if there exist a $t_0 \geq 0$ and a continuously differentiable function $r : [t_0, \infty) \to \mathbb{R}$ satisfying inequality* (2.1).

Using Lemma 2.1, we can easily prove Theorem 1.1. For the convenience of the reader, we give the proof briefly.

**Proof of Theorem 1.1.** By assumption, the coefficients $a$ and $b$, the composite function $F(B)$ and its derivative $f(B)b$ are periodic functions with period $T$. Since the derivative of $F(B)$ is $f(B)b$, it follows from (1.5) that

$$\big(F(B(t))\big)' \geq F^2(B(t)) - a(t)F(B(t)) + b(t)$$

holds for $t \geq 0$. This means that $F(B)$ satisfies the Riccati inequality (2.1) for $t \geq t_0 = 0$. Hence, by Lemma 2.1, all nontrivial solutions of (1.1) are nonoscillatory. □

Define

$$r(t) = \exp\left(\int_0^t a(s)ds\right) \quad \text{for } t \geq 0.$$

Then we can rewrite equation (1.1) as the Sturm–Liouville differential equation

$$(r(t)x')' + c(t)x = 0, \tag{2.2}$$

where $c(t) = b(t)r(t)$ for $t \geq 0$. The following lemma is the so-called Leighton–Wintner oscillation criterion (for the proof, see [27, pp. 70–71, Theorem 2.24]).

**Lemma 2.2.** *All nontrivial solutions of* (2.2) *are oscillatory if*

$$\lim_{t\to\infty}\int^t \frac{1}{r(s)}ds = \lim_{t\to\infty}\int^t c(s)ds = \infty.$$

Needless to say, all nontrivial solutions of (2.2) are oscillatory if and only if those of (1.1) are oscillatory. We prove Theorem 1.2 by means of Lemmas 2.1 and 2.2.

**Proof of Theorem 1.2.** The proof is by contradiction. Suppose that equation (1.1) has a nonoscillatory solution $x$. Then we may assume without loss of generality that there exists a $t_0 \geq 0$ such that $x(t) > 0$ for $t \geq t_0$. Let

$$r(t) = -\frac{x'(t)}{x(t)} \quad \text{for } t \geq t_0.$$

Then, the function $r$ is continuously differentiable for $t \geq t_0$ and

$$r'(t) = \left(\frac{x'(t)}{x(t)}\right)^2 - \frac{x''(t)}{x(t)} = r^2(t) - a(t)r(t) + b(t).$$

Let $R(t) = r(t) - F(B(t))$ for $t \geq t_0$. Then we have

$$\begin{aligned}
R'(t) &= r^2(t) - a(t)r(t) + b(t) - f(B(t))b(t) \\
&= \big(R(t) + F(B(t))\big)^2 - \big(R(t) + F(B(t))\big)a(t) + b(t) - f(B(t))b(t) \\
&= R^2(t) - \big(a(t) - 2F(B(t))\big)R(t) \\
&\qquad + \left\{\big(F(B(t)) - a(t)\big)F(B(t)) + \big(1 - f(B(t))\big)b(t)\right\}.
\end{aligned}$$

Since $R$ is a continuously differentiable function on $[t_0, \infty)$, we can regard the function $R$ as a solution of the Riccati-type equation

$$r' = r^2 - \big(a(t) - 2F(B(t))\big)r + \big\{\big(F(B(t)) - a(t)\big)F(B(t)) + \big(1 - f(B(t))\big)b(t)\big\}.$$

The second-order linear differential equation corresponding to the above equation is

$$y'' + \big(a(t) - 2F(B(t))\big)y' + \big\{\big(F(B(t)) - a(t)\big)F(B(t)) + \big(1 - f(B(t))\big)b(t)\big\}y = 0. \tag{2.3}$$

Hence, Lemma 2.1 leads to the conclusion that all nontrivial solutions of (2.3) are non-oscillatory.

We can transform equation (2.3) into the differential equation of Sturm–Liouville type,

$$\big(E(t)y'\big)' + \big\{\big(F(B(t)) - a(t)\big)F(B(t)) + \big(1 - f(B(t))\big)b(t)\big\}E(t)y = 0. \tag{2.4}$$

From the assumption that both $a$ and $F(B)$ belong to $\mathscr{F}_{[\mathrm{MVZ}]}$, it follows that $E$ is a periodic function with period $T$. Hence, we can find an $m > 0$ such that $0 < E(t) \le m$ for $t \ge 0$. We therefore conclude that

$$\lim_{t\to\infty}\int_0^t \frac{1}{E(s)}ds \ge \lim_{t\to\infty}\int_0^t \frac{1}{m}ds = \infty. \tag{2.5}$$

For convenience, let us denote by $G(t)$ the coefficient $\{(F(B(t)) - a(t))F(B(t)) + (1 - f(B(t)))b(t)\}E(t)$ and let

$$\rho = \int_0^T G(t)dt > 0.$$

Since $G$ is a periodic function with period $T$, there exists a $C \ge \rho$ such that

$$\int_0^t |G(s)|\,ds \le C \quad \text{for } 0 \le t \le T. \tag{2.6}$$

For an arbitrary $t \ge 0$, there is an $n \in \mathbb{N}$ such that $nT \le t < (n+1)T$. Hence, by (2.6) we have

$$\begin{aligned}\int_0^t G(s)ds &= \int_0^T G(s)ds + \int_T^{2T} G(s)ds + \cdots + \int_{(n-1)T}^{nT} G(s)ds + \int_{nT}^t G(s)ds \\ &\ge n\rho - \int_{nT}^t |G(s)|ds = n\rho - \int_0^{t-nT}|G(s)|ds \ge n\rho - C.\end{aligned}$$

Since the integer $n$ tends to infinity as $t \to \infty$, we obtain

$$\lim_{t\to\infty}\int_0^t G(s)ds \ge \lim_{n\to\infty} n\rho - C = \infty. \tag{2.7}$$

From the estimations (2.5) and (2.7) it turns out that all conditions of Lemma 2.2 are satisfied. Hence, all nontrivial solutions of (2.4) are oscillatory. Since equation (2.3) is equivalent to equation (2.4), all nontrivial solutions of (2.3) are also oscillatory.

The conclusion of the second paragraph contradicts that of the first paragraph. Thus, the proof is complete. □

## 3. Application to the Whittaker–Hill equation

Whittaker–Hill equation has the form

$$\frac{d^2y}{ds^2} + \big(\lambda + 4mq\cos(2s) + 2q^2\cos(4s)\big)y = 0,$$

where $\lambda$ and $q$ are real numbers and $m$ is a natural number (for example, see [19, pp. 106–107]). It has been reported that this equation has a deep relationship with various equations such as Mathieu's equation, Ince's equation and the confluent hypergeometric differential equation. For this reason, it has been extensively studied in various research fields. For example, see [4, 5, 8, 10, 12, 13, 14, 21, 25, 29]) and the references therein. Whittaker–Hill equation appears in various fields of natural science and engineering. For example, we can cite a theory of internal rotation in the hydrogen peroxide molecule (see [11, 23]).

Let

$$t = 2s \quad \text{and} \quad x(t) = y(s)e^{-q(1-\cos t)}.$$

Then, by a straightforward calculation, Whittaker–Hill equation is transformed into

$$x'' + 2q(\sin t)x' + \left(\frac{\lambda}{4} + \frac{q^2}{2} + q(1+m)\cos t\right)x = 0. \tag{3.1}$$

From this transformation, we see that all nontrivial solutions of Whittaker–Hill equation are oscillatory if and only if those of (3.1) are oscillatory.

Consider the special case that $m = 1$ and $q = 1/2$. Then, by letting $\varepsilon = \lambda/4 + 1/8$, this special case becomes the damped linear equation (1.4). As mentioned in Section 1, all nontrivial solutions of (1.4) are nonoscillatory when $\varepsilon = 0$. Those are also nonoscillatory when $\varepsilon < 0$. In fact, by multiplying both sides by $\exp(1-\cos t)$, equation (1.4) is rewritten as the differential equation of Sturm–Liouville type,

$$\big(e^{1-\cos t}x'\big)' + e^{1-\cos t}(\varepsilon + \cos t)x = 0. \tag{3.2}$$

Hence, the classical Sturm comparison theorem is valid (for example, see [27, pp. 1–2] about Sturm's comparison theorem).

Then, how about the case when $\varepsilon > 0$? The classical Leighton–Wintner oscillation criterion, namely, Lemma 2.2 gives a partial answer. Comparing (3.2) with (2.2), we see that

$$r(t) = e^{1-\cos t} \quad \text{and} \quad c(t) = e^{1-\cos t}(\varepsilon + \cos t).$$

Since $1 \le r(t) \le e^2$ for $t \ge 0$, it follows that

$$\lim_{t\to\infty} \int^t \frac{1}{r(s)}ds = \infty.$$

Numerical computation shows that

$$21.6237 < \int_0^{2\pi} e^{1-\cos t}dt < 21.6238 \quad \text{and} \quad -9.65263 < \int_0^{2\pi} e^{1-\cos t}\cos t\,dt < -9.65262.$$

Hence, we have

$$\begin{aligned}\int_0^{2\pi} c(t)dt &= \int_0^{2\pi} e^{1-\cos t}(\varepsilon + \cos t)dt \\ &= \varepsilon \int_0^{2\pi} e^{1-\cos t} dt + \int_0^{2\pi} e^{1-\cos t} \cos t \, dt \\ &< 21.6238\,\varepsilon - 9.65262\end{aligned}$$

and

$$\int_0^{2\pi} c(t)dt > 21.6237\,\varepsilon - 9.65263.$$

Form these estimations it turns out that

$$\lim_{t\to\infty} \int^t c(s)ds < \infty \quad \text{if } 0 < \varepsilon \leq 0.4463887$$

and

$$\lim_{t\to\infty} \int^t c(s)ds = \infty \quad \text{if } \varepsilon \geq 0.4463912.$$

We therefore conclude that if $\varepsilon \geq 0.4463912$, then all nontrivial solutions of (1.4) are oscillatory. However, since the Leighton–Wintner oscillation criterion is not satisfied if $0 < \varepsilon \leq 0.4463887$, we cannot judge that all nontrivial solutions of (1.4) are oscillatory.

Then, how about the case when $\varepsilon$ is a sufficiently small positive value? Theorem 1.2 answers this question as follows.

**Theorem 3.1.** *All nontrivial solutions of* (1.4) *are oscillatory when* $\varepsilon > 0$.

**Proof.** By Sturm's comparison theorem, we only need to prove the case when $\varepsilon$ is small sufficiently. Note that $a(t) = \sin t$ and $b(t) = \varepsilon + \cos t$. It is clear that $a$ belongs to $\mathscr{F}_{[\mathrm{MVZ}]}$. We choose the function $\varepsilon t + \sin t$ as an indefinite integral $B$ of $b$. Define

$$u = H(v) = v + \varepsilon \sin^{-1} v$$

for $v \in [-1,1]$, where $\sin^{-1} v$ means the set of numbers $\xi$ satisfying $v = \sin \xi$. Then, $H$ is a multi-valued function (see Figure 1). Note that

$$H(\sin t) = \sin t + \varepsilon t = B(t) \quad \text{for } t \geq 0.$$

Of course, $H$ has no inverse function. However, each branch of $H$ has an inverse function because it is strictly increasing on $[-1,1]$. There are an infinite number of branches of $H$. Define a sequence $\{I_n\}$ of intervals by

$$I_n = \left[-1 + \left(2n - \frac{1}{2}\right)\varepsilon\pi, 1 + \left(2n + \frac{1}{2}\right)\varepsilon\pi\right] \stackrel{\text{def}}{=} [\alpha_n, \beta_n]$$

for each $n \in \mathbb{Z}$. Note that the intersection of $I_n$ and $I_{n+1}$ exists for each $n \in \mathbb{Z}$. Let $F_n : I_n \to [-1,1]$ be the inverse function of a branch of $H$, and let $F$ be a combination of

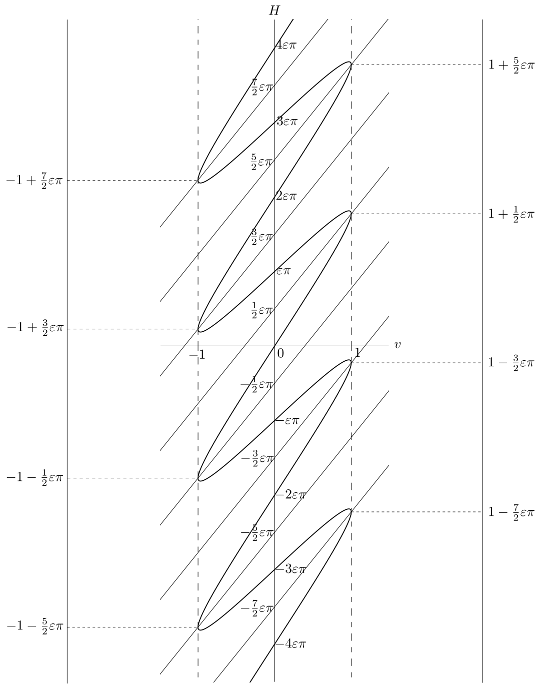

Figure 1: The graph of the multi-valued function $H$ when $\varepsilon = 0.25$

all $F_n$ ($n \in \mathbb{Z}$). In other words, each function $F_n$ is a branch of $F$ which is a multi-valued function (see Figure 2). We see that $F$ is a continuously differentiable function on $\mathbb{R}$. Since

$$F(B(t)) = F(\sin t + \varepsilon t) = \sin t \quad \text{for } t \geq 0,$$

$F(B)$ belongs to $\mathscr{F}_{[\mathrm{MVZ}]}$.

For each $n \in \mathbb{Z}$, the branch $F_n$ is a single-valued function on $I_n$. The principal branch of $F$ is the single-valued function $F_0$. The branch $F_n$ is differentiable on the open interval $(\alpha_n, \beta_n)$ for each $n \in \mathbb{Z}$. It is clear that $2n\varepsilon\pi \in I_n$, $F_n(2n\varepsilon\pi) = 0$, $F_n(\alpha_n) = -1$ and $F_n(\beta_n) = 1$. When the branch $F_n$ moves $2\varepsilon\pi$ parallel to the right, it coincides with the next branch $F_{n+1}$, namely,

$$F_{n+1}(u) = F_n(u - 2\varepsilon\pi) \quad \text{for } u \in I_n. \tag{3.3}$$

The branch $F_n$ is an odd function around $u = 2n\varepsilon\pi$. Hence, we have

$$F_n(u) = -F_n(4n\varepsilon\pi - u) \quad \text{for } u \in I_n. \tag{3.4}$$

Since $\varepsilon$ is small enough, it follows that

$$-1 + \left(2n + \frac{3}{2}\right)\varepsilon\pi < (2n+1)\varepsilon\pi < 1 + \left(2n + \frac{1}{2}\right)\varepsilon\pi.$$

This means that $(2n+1)\varepsilon\pi \in I_n \cap I_{n+1}$. Hence, by (3.3) and (3.4) we obtain

$$\begin{aligned} F_{n+1}\big((2n+1)\varepsilon\pi\big) &= F_n\big((2n+1)\varepsilon\pi - 2\varepsilon\pi\big) \\ &= F_n\big(4n\varepsilon\pi - (2n+1)\varepsilon\pi\big) = -F_n\big((2n+1)\varepsilon\pi\big) \end{aligned} \tag{3.5}$$

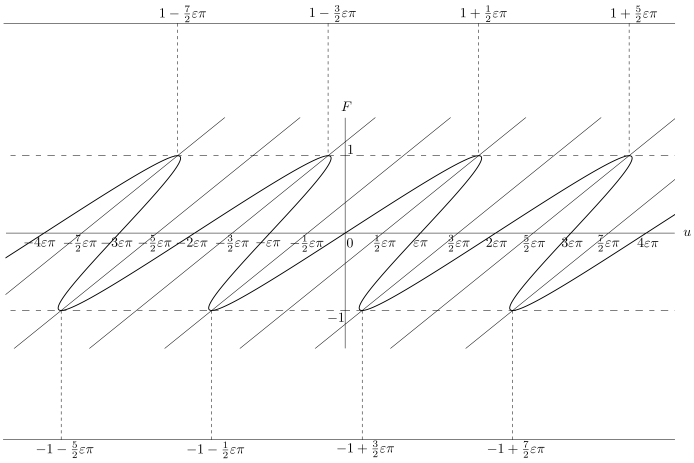


Figure 2: The graph of the multi-valued function $F$ when $\varepsilon = 0.25$

for each $n \in \mathbb{Z}$.

Since $F_n$ is a single-valued and differentiable function on the open interval $(\alpha_n, \beta_n)$ for each $n \in \mathbb{Z}$, we can define the derivative of $F_n$. We have

$$f_n(u) = \frac{d}{du}F_n(u) = \left.\frac{1}{\frac{d}{dv}H(v)}\right|_{v=F_n(u)} = \frac{\sqrt{1-F_n^2(u)}}{\sqrt{1-F_n^2(u)}+\varepsilon} < 1 \tag{3.6}$$

for $\alpha_n < u < \beta_n$. In fact, since the branch of $\sin^{-1}v$ corresponding to $F_n$ is an increasing function on $[-1,1]$ and

$$-1+\left(2n-\frac{1}{2}\right)\varepsilon\pi = \alpha_n < u = H(v) = v+\varepsilon\,\sin^{-1}v < \beta_n = 1+\left(2n+\frac{1}{2}\right)\varepsilon\pi,$$

we see that $(2n-1/2)\pi < \sin^{-1}v < (2n+1/2)\pi$. Let $w = \sin^{-1}v$. Then it follows that $\cos w > 0$ and

$$\frac{dv}{dw} = \cos w = \sqrt{1-\sin^2 w} = \sqrt{1-v^2}.$$

Hence, we obtain

$$\frac{d}{dv}H(v) = 1+\frac{\varepsilon}{\sqrt{1-v^2}} \quad \text{for } -1 < v < 1.$$

Taking into account that $(2n+1)\varepsilon\pi \in I_n \cap I_{n+1}$ for each $n \in \mathbb{Z}$ and using the equalities (3.5) and (3.6), we see that

$$\begin{aligned} f_n\big((2n+1)\varepsilon\pi\big) &= \frac{\sqrt{1-F_n^2((2n+1)\varepsilon\pi)}}{\sqrt{1-F_n^2((2n+1)\varepsilon\pi)}+\varepsilon} \\ &= \frac{\sqrt{1-F_{n+1}^2((2n+1)\varepsilon\pi)}}{\sqrt{1-F_{n+1}^2((2n+1)\varepsilon\pi)}+\varepsilon} = f_{n+1}\big((2n+1)\varepsilon\pi\big). \end{aligned} \tag{3.7}$$

Now, consider the function $B$. Since $-1+\varepsilon t \le B(t) = \sin t + \varepsilon t \le 1+\varepsilon t$ for $t \ge 0$, the graph of $B$ rises while oscillating up and down. The curve $u = B(t)$ crosses the horizontal line $u = \varepsilon\pi$ at three points within the range where $0 \le t \le 2\pi$ (see Figure 3). One of the intersection points is $(t,u) = (\pi, \varepsilon\pi)$. Let $(t_1, \varepsilon\pi)$ and $(t_2, \varepsilon\pi)$ be the other intersection points. From the form of this curve, we see that $0 < t_1 < \pi/2$, $3\pi/2 < t_2 < 2\pi$.

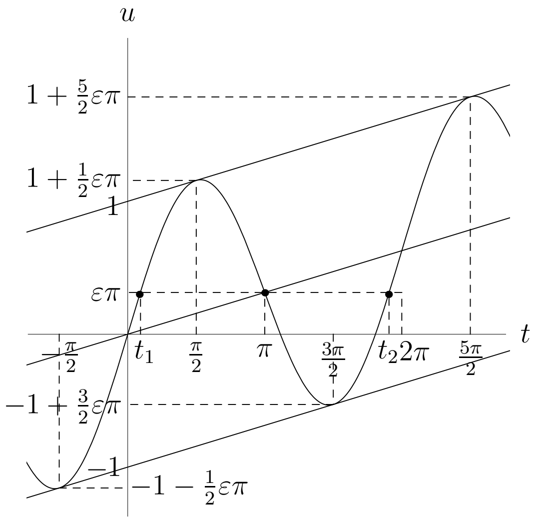


Figure 3: The graph of the function $B$ when $\varepsilon = 0.1$

Since $a(t) = F_0(B(t)) = F_1(B(t)) = \sin t$, it follows that

$$\int_0^{2\pi} \left\{ \big(F(B(t)) - a(t)\big) F(B(t)) + \big(1 - f(B(t))\big) b(t) \right\} E(t)\, dt$$
$$= \int_0^{2\pi} \big(1 - f(B(t))\big) b(t)\, E(t)\, dt.$$

Recall that $f$ is a multiple-valued function. We can estimate that

$$\alpha_0 = -1 - \frac{1}{2}\varepsilon\pi < 0 \le B(t) \le \varepsilon\pi < 1 + \frac{1}{2}\varepsilon\pi = \beta_0 \quad \text{for } 0 \le t \le t_1,$$
$$\alpha_1 = -1 + \frac{3}{2}\varepsilon\pi < \varepsilon\pi < B(t) < 1 + \frac{5}{2}\varepsilon\pi = \beta_1 \quad \text{for } t_1 < t < \pi,$$
$$\alpha_0 = -1 - \frac{1}{2}\varepsilon\pi < -1 < B(t) \le \varepsilon\pi < 1 + \frac{1}{2}\varepsilon\pi = \beta_0 \quad \text{for } \pi \le t \le t_2,$$

and

$$\alpha_1 = -1 + \frac{3}{2}\varepsilon\pi < \varepsilon\pi < B(t) \le 2\varepsilon\pi < 1 + \frac{5}{2}\varepsilon\pi = \beta_1 \quad \text{for } t_2 < t \le 2\pi.$$

Hence, we may choose the multiple-valued function $f$ as follows:

$$f(B(t)) = \begin{cases} f_0(B(t)) & \text{if } 0 \le t \le t_1 \text{ or } \pi \le t \le t_2, \\ f_1(B(t)) & \text{if } t_1 < t < \pi \text{ or } t_2 < t \le 2\pi. \end{cases}$$

Since $B(t)=\varepsilon\pi$ at $t=t_1$, $t=\pi$ and $t=t_2$, it follows from (3.7) that

$$f_0(B(t_1))=f_0(\varepsilon\pi)=f_1(\varepsilon\pi)=f_1(B(t_1)),$$

$$f_0(B(\pi))=f_0(\varepsilon\pi)=f_1(\varepsilon\pi)=f_1(B(\pi)),$$

and

$$f_0(B(t_2))=f_0(\varepsilon\pi)=f_1(\varepsilon\pi)=f_1(B(t_2)).$$

This means that $f(B(t)$ is continuous for $0\le t\le 2\pi$. Hence, we are allowed to calculate as follows:

$$\begin{aligned}&\int_0^{2\pi}\big(1-f(B(t))\big)b(t)E(t)\,dt\\ &=\int_0^{t_1}\big(1-f_0(B(t))\big)b(t)E(t)\,dt+\int_{t_1}^{\pi}E(t)\big(1-f_1(B(t))\big)b(t)E(t)\,dt\\ &\quad+\int_{\pi}^{t_2}\big(1-f_0(B(t))\big)b(t)E(t)\,dt+\int_{t_2}^{2\pi}E(t)\big(1-f_1(B(t))\big)b(t)E(t)\,dt.\end{aligned}$$

From (3.6) it turns out that

$$f_0(B(t))=\frac{\sqrt{1-F_0^2(B(t))}}{\sqrt{1-F_0^2(B(t))}+\varepsilon}=\frac{\sqrt{1-\sin^2 t}}{\sqrt{1-\sin^2 t}+\varepsilon}=\frac{|\cos t|}{|\cos t|+\varepsilon}$$

for $0\le t\le t_1$ or $\pi\le t\le t_2$. Similarly,

$$f_1(B(t))=\frac{|\cos t|}{|\cos t|+\varepsilon}\quad\text{for }\ t_1<t<\pi\ \text{ or }\ t_2<t\le 2\pi.$$

Since $b(t)=\varepsilon+\cos t$ and

$$E(t)=\exp\int_0^t\big(a(s)-2F(B(s))\big)\,ds=\exp\int_0^t(\sin s-2\sin s)ds=e^{\cos t-1}$$

for $t\ge 0$, we obtain

$$\int_0^{2\pi}\big(1-f(B(t))\big)b(t)E(t)\,dt=\varepsilon\int_0^{2\pi}e^{\cos t-1}\frac{\varepsilon+\cos t}{\varepsilon+|\cos t|}\,dt.$$

We divide the interval $[0,2\pi]$ into two groups. Let

$$I_+=\big\{t\in[0,2\pi]:\cos t>0\big\}\quad\text{and}\quad I_-=\big\{t\in[0,2\pi]:\cos t\le 0\big\}.$$

Then we have

$$\varepsilon\int_0^{2\pi}e^{\cos t-1}\frac{\varepsilon+\cos t}{\varepsilon+|\cos t|}\,dt>\varepsilon\int_{I_+}e^{\cos t-1}dt-\varepsilon\int_{I_-}e^{\cos t-1}dt.$$

Since $e^{\cos t-1} > 1/e$ for $t \in I_+$ and $e^{\cos t-1} \leq 1/e$ for $t \in I_-$, we see that

$$\begin{aligned}\varepsilon\int_{I_+} e^{\cos t-1}dt - \varepsilon\int_{I_-} e^{\cos t-1}dt &> \varepsilon\int_{I_+}\left(e^{\cos t-1}-\frac{1}{e}\right)dt + \varepsilon\int_{I_+}\frac{1}{e}\,dt - \varepsilon\int_{I_-}\frac{1}{e}\,dt \\ &> \varepsilon\int_{I_+}\left(e^{\cos t-1}-\frac{1}{e}\right)dt > 0.\end{aligned}$$

We therefore conclude that

$$\int_0^{2\pi}\left\{\big(F(B(t))-a(t)\big)F(B(t)) + \big(1-f(B(t))\big)b(t)\right\}E(t)\,dt > 0.$$

Thus, by Theorem 1.2, all nontrivial solutions of (1.4) are oscillatory even when $\varepsilon$ is a sufficiently small positive value. The proof is now complete. □

The lower bound that all nontrivial solutions are oscillatory is often called the *oscillation constant*. Theorem 3.1 indicates that the oscillation constant for equation (1.4) is zero by combining with Sturm's comparison theorem and the fact that all nontrivial solutions of (1.3) are nonoscillatory.

## 4. Simulation

To verify Theorem 3.1, we give an example in the section. Let us consider the case that $\varepsilon = 0.0003$, namely, the equation

$$x'' + (\sin t)x' + (0.0003 + \cos t)x = 0, \tag{4.1}$$

Equation (4.1) is equivalent to the system

$$\begin{aligned} x' &= y, \\ y' &= -(\sin t)y - (0.0003 + \cos t)x. \end{aligned} \tag{4.2}$$

In Figure 4, we draw a positive orbit of (4.2) starting at the point $(0,1)$. The initial time is 0 and the finish time is 500. The positive orbit first runs around in the right half-plane, and then it moves into the left half-plane. The same movement repeats infinitely. As a result, the positive orbit looks like a bivalve shell. Since the positive orbit goes alternately in the right half-plane and the left half-plane, we see that the solution of (4.1) corresponding to this positive orbit is oscillatory. Indeed, the solution $x(t)$ of (4.1) satisfying the initial condition that $(x(0), x'(0)) = (0,1)$ repeats finely up and down, and it changes the sign infinitely while shaking greatly (see Figure 5).

Since equation (4.1) is linear, all nontrivial solutions are oscillatory. This matches the statement of Theorem 3.1 correctly. When $\varepsilon$ is made smaller, the size of this bivalve increases and the amplitude of the solution curve of (4.1) also increases.

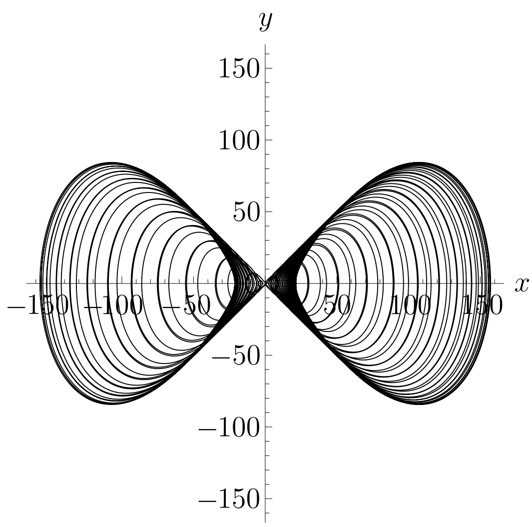


Figure 4: A positive orbit of (4.2) when $\varepsilon = 0.0003$

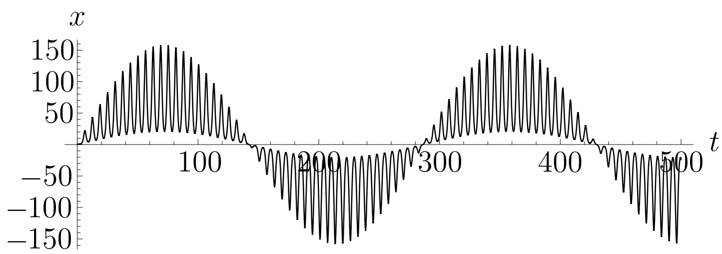


Figure 5: A solution curve of (4.1) when $\varepsilon = 0.0003$

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