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Nonoscillation of Mathieu's equation whose coefficient is a finite Fourier series approximating a square wave

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Abstract Parametric nonoscillation region is given for the Mathieu-type differential equation

$$x'' + (-\alpha + \beta c(t))x = 0,$$

where α and β are real parameters. Oscillation problem about a kind of Meissner's equation is also discussed. The obtained result is proved by using Sturm's comparison theorem and phase plane analysis of the second-order differential equation

$$y'' + a(t)y' + b(t)y = 0,$$

where $a, b\colon [0,\infty) \to \mathbb{R}$ are continuous functions. The feature of the result is the ease of chequing whether the obtained condition is satisfied or not. Parametric nonoscillation region about (α,β) and some solution orbits are drawn to help understand the result.

Keywords Parametric nonoscillation region · Damped linear differential equations · Mathieu's equation · Meissner's equation · Phase plane analysis

Mathematics Subject Classification (2010) 34C10 · 34A36 · 34B30

1 Introduction

As known well, the function

$$f(t) = \begin{cases} \pi/4 & \text{if } 0 \le t < 1, \\ -\pi/4 & \text{if } 1 \le t < 2 \end{cases}$$

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with $f(t) = f(t+2)$ can be expanded to the Fourier series

$$\sum_{n=1}^{\infty} \frac{\sin((2n-1)\pi t)}{2n-1}.$$

The function f expresses a square wave with period 2. Since f is a discontinuous function, the Nth partial sum of the Fourier series

$$S_N(t) = \sum_{n=1}^{N} \frac{\sin((2n-1)\pi t)}{2n-1}$$

converges to the square wave function f for each fixed t but not uniformly in t. Also, the Gibbs phenomenon occurs in the neighbourhood of the points of discontinuities. Since

$$S_N(t) = \pi \int_0^t \sum_{n=1}^{N} \cos((2n-1)\pi s) ds = \frac{\pi}{2} \int_0^t \frac{\sin(2N\pi s)}{\sin(\pi s)} ds,$$

it has local extreme values at $t = k/(2N)$ with $k \in \mathbb{Z}$ but k is not a multiple of $2N$. The maximum value of S_N is

$$S_N(1/(2N)) = \frac{\pi}{2} \int_0^{t_1} \frac{\sin(2N\pi s)}{\sin(\pi s)} ds = \frac{1}{2} \int_0^{\pi} \frac{\tau/(2N)}{\sin(\tau/(2N))} \frac{\sin\tau}{\tau} d\tau.$$

Hence, the peak value of the Gibbs phenomenon is

$$\lim_{N\to\infty} S_N(1/(2N)) = \frac{1}{2} \int_0^{\pi} \frac{\sin\tau}{\tau} d\tau = 0.925968526\dots.$$

Since $\lim_{t\to 0^+} f(t) = \pi/4$, the overshoot is

$$\left| \frac{\pi}{4} - \frac{1}{2} \int_0^{\pi} \frac{\sin\tau}{\tau} d\tau \right| = 0.140570362\dots.$$

The undershoot is the same (see Figure 1). The Gibbs phenomenon never disappears even if the number N of terms of the finite Fourier series S_N is very large. In an actual simulation, we cannot make N infinite. Hence, the upper and lower bounds of S_N are not sharp.

The Gibbs phenomenon has been recognised as a kind of noise in the field of digital signal processing. Hence, this is an undesirable phenomenon. For this reason, various ideas are carried out to avoid this phenomenon. For example, the Gibbs phenomenon is known to be improved by using a smooth method of Cesàro summation of Fourier series. Define

$$T_N(t) = \frac{1}{N} \sum_{n=1}^{N} S_n(t) = \frac{1}{N} \sum_{n=1}^{N} \sum_{k=1}^{n} \frac{\sin((2k-1)\pi t)}{2k-1}.$$

Then, $\lim_{N\to\infty} T_N(t) = f(t)$. The Gibbs phenomenon does not happen for the Cesàro summation T_N. This means that the upper (resp., lower) bound of T_N approaches $\pi/4$

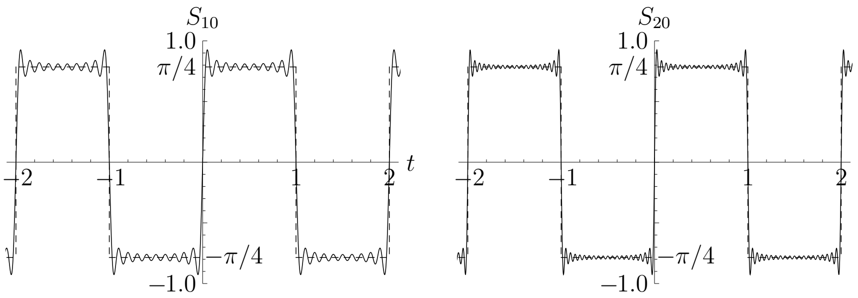

Fig. 1 The graph of the finite Fourier series $S_N(t)$ when $N = 10$ and $N = 20$

(resp., $-\pi/4$) from above (resp., below) as $N \to \infty$ (see Figure 2). Hence, we see that for any sufficiently small $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that

$$|T_N(t)| < \pi/4 + \varepsilon < 1$$

for $t \in \mathbb{R}$.

Let p be a periodic function on $[0, \infty)$. The function p is said to be periodic of *mean value zero* if p is not identically zero and

$$\int_0^{\omega} p(t)dt = 0,$$

where ω is the period of p. Note that any indefinite integral of p is also a periodic function with the same period as that of p.

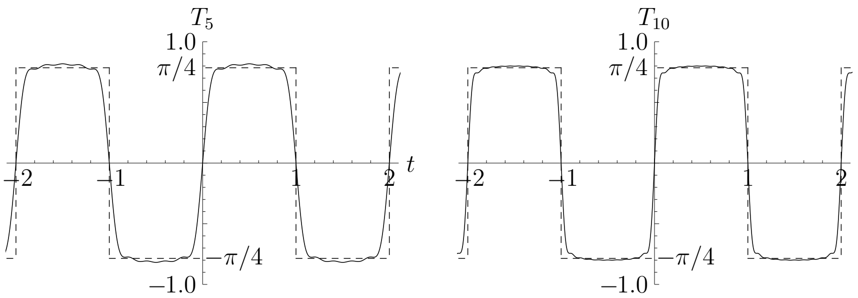

Fig. 2 The graph of the Cesàro summation T_N when $N = 5$ and $N = 10$

In this paper, we consider the second-order differential equation

$$x'' + (-\alpha + \beta c(t))x = 0, \tag{1.1}$$

where the prime denotes d/dt, the parameters α and β are real numbers, and the function c is continuous on $[0, \infty)$ and periodic of mean value zero. Since c is periodic, it is bounded. Let c^* be an upper bound of $|c|$; that is,

$$|c(t)| \leq c^* \quad \text{for } t \geq 0. \tag{1.2}$$

Since c is periodic with mean value zero, the integral function $\int_0^t c(s)ds$ has a lower bound $\underline{C}$ and an upper bound $\overline{C}$. Let $C^* = (\overline{C} - \underline{C})/2$ and define

$$C(t) = \int_0^t c(s)ds - \frac{1}{2}\left(\underline{C} + \overline{C}\right).$$

Then, we have

$$|C(t)| \le C^* \quad \text{for } t \ge 0. \tag{1.3}$$

The trigonometric functions $\sin \pi t$ and $\cos \pi t$ are periodic with period 2. We may regard 1 and $1/\pi$ as c^* and C^*, respectively. The finite Fourier series S_N and T_N also satisfy the assumptions of c and C. As was shown above, if $c = S_N$, then $c^* = 0.9260$ and $C^* = 0.3927$ for N sufficiently large; if $c = T_N$, then $c^* = 0.7854$ and $C^* = 0.3927$ for N sufficiently large.

Equation (1.1) may be considered as a generalised Mathieu equation. Mathieu's equation often describes parametric excitation. Parametric excitation is a famous vibration phenomenon that appears in mechanical engineering, electrical engineering, acoustical engineering, and so on. This vibration phenomenon is caused by the periodic change of the parameters which is inherent in the mechanical system. For example, the position of a pivot point, the arm length of a pendulum, the inductance of an electrical circuit, and the tension of a string are cited as those parameters. We can find various concrete examples of Mathieu's equation in [2–4, 6, 10, 22, 23, 25, 26].

Let us consider an inverted pendulum whose pivot point vibrates periodically in the vertical direction. In the case that the motion speed of the pivot point is the function C, we can write the motion equation of the inverted pendulum as

$$x'' + (-\alpha + \beta c(t))\sin x = 0.$$

Here, we ignore the friction at the pivot point. Equation (1.1) is the linear approximation of this motion equation.

The purpose of this paper is to present parametric conditions on (α, β) which guarantees that all nontrivial solutions of (1.1) are nonoscillatory (see Section 2 for the definitions).

Since c is a periodic function, equation (1.1) belongs to Hill's equation

$$x'' + g(t)x = 0, \tag{1.4}$$

where g is a periodic function. About applications of Hill's equation, refer to [13, 21, 23]. We can find various results about the oscillation problem of (1.4) in many literatures (for example, see [7, 18–20, 31]). It is well-known that if g is periodic of mean value zero, then all nontrivial solutions of (1.4) are oscillatory (for the proof, see [7, p. 25]). Hence, if $\alpha = 0$ and $\beta \ne 0$, then all nontrivial solutions of (1.1) are oscillatory. It follows from Sturm's comparison theorem that if $\alpha < 0$ and $\beta \ne 0$, then all nontrivial solutions of (1.1) are oscillatory. It is clear that

(a) if $\alpha < 0$ and $\beta = 0$, then all nontrivial solutions of (1.1) are oscillatory;

(b) if $\alpha = 0$ and $\beta = 0$, then all nontrivial solutions of (1.1) are nonoscillatory.

Thus, a necessary condition for all nontrivial solutions of (1.1) to be nonoscillatoy is that $\alpha \geq 0$. Our result is as follows.

Theorem 1.1 *If*

$$|\beta| \leq \begin{cases} \dfrac{1}{C^*}\sqrt{\alpha} & \text{if } 0 \leq \alpha < \left(\dfrac{c^*}{2C^*}\right)^2, \\ \dfrac{\alpha}{c^*} + \dfrac{c^*}{4(C^*)^2} & \text{if } \alpha \geq \left(\dfrac{c^*}{2C^*}\right)^2, \end{cases} \tag{1.5}$$

then all nontrivial solutions of (1.1) *are nonoscillatory*.

Theorem 1.1 has a feature that it is very easy to cheque. For a given value $\alpha \geq 0$, we can seek a value of β satisfying condition (1.5) immediately. It is also manageable to obtain a positive value of α satisfying condition (1.5) for a given value $\beta \in \mathbb{R}$.

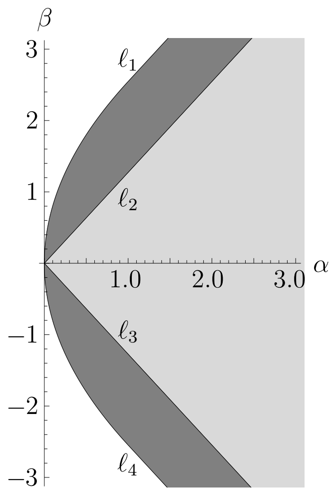

Fig. 3 Parametric nonoscillation region about (α,β) given by (1.5) when $c^* = 0.7854$ and $C^* = 0.3927$

Figure 3 shows a nonoscillation region for equation (1.1) with $c = T_N$ for any $N \in \mathbb{N}$. In Figure 3, the equations of the curved lines ℓ_1 and ℓ_4 are

$$\beta = \begin{cases} \dfrac{2}{c^*}\sqrt{\alpha} & \text{if } 0 \leq \alpha < 1, \\ \dfrac{1}{c^*}(\alpha+1) & \text{if } \alpha \geq 1, \end{cases}$$

and

$$\beta = \begin{cases} -\dfrac{2}{c^*}\sqrt{\alpha} & \text{if } 0 \leq \alpha < 1, \\ -\dfrac{1}{c^*}(\alpha+1) & \text{if } \alpha \geq 1, \end{cases}$$

respectively; and the equations of the straight lines ℓ_2 and ℓ_3 are $\beta = \alpha/c^*$ and $\beta = -\alpha/c^*$, respectively, where $c^* = 0.7854$. It is easy to prove that if (α,β) is in the area surrounded by two straight lines ℓ_2 and ℓ_3, namely, the light grey part, then all nontrivial solutions of (1.1) are nonoscillatory (see Section 3 for details). Theorem 1.1 guarantees that the nonoscillation region for equation (1.1) is more wider than this area.

2 Nonoscillation theorem by using phase plane analysis

We consider the second-order differential equation

$$y'' + a(t)y' + b(t)y = 0, \tag{2.1}$$

where the prime means d/dt and $a,\, b\colon [0,\infty) \to \mathbb{R}$ are continuous functions. Equation (2.1) has naturally the trivial solution $y \equiv 0$. We can divide the other solutions into two groups as follows. A nontrivial solution y of (2.1) is said to be *oscillatory* if it has an infinite number of zeros. Otherwise, the nontrivial solution is said to be *nonoscillatory*. Hence, a nonoscillatory solution y of (2.1) is eventually positive or eventually negative.

Equation (2.1) is a typical object of research in the qualitative theory of ordinary differential equations, because it often appears as an important model in natural science, applied science and technology. To judge whether a solution is oscillatory or nonoscillatory is an important theme in the qualitative theory of (2.1). Since Sturm's separation theorem holds in equation (2.1), nonoscillatory solutions do not coexist with oscillatory solutions.

A lot of effort has been made to find sufficient conditions which guarantee that all nontrivial solutions of (2.1) (and more general nonlinear equations) are nonoscillatory (resp., oscillatory). For example, see [1, 5, 11, 28, 33–37] and the references cited therein. Such conditions are expressed by several kinds of integration that are written by using the coefficients a and b of (2.1). However, in general, we cannot seek the concrete integration for given a and b. In this section, we give a sufficient condition for nonoscillation of (2.1) which can be chequed without using the integration. We will pay attention to the parameter curve $(a(t), b(t))$ instead of the integration.

Let d and h be any real numbers satisfying $0 < d \le h$. Define

$$T = T(h,d) = \{(u,v)\colon\ 2h - d \le u \le 2h + d \ \text{ and } \ 0 \le v \le hu - h^2\}.$$

The trapezoid T is contained in the domain

$$U = \{(u,v)\colon\ u \ge 0 \ \text{ and } \ 0 \le v \le u^2/4\}.$$

In fact, if $(u,v) \in T$, then $u \ge 2h - d \ge h > 0$ and

$$\begin{aligned} 0 \le v \le hu - h^2 &= \frac{u^2}{4} - \left(\frac{u^2}{4} - hu + h^2\right) \\ &= \frac{u^2}{4} - \left(\frac{u}{2} - h\right)^2 \le \frac{u^2}{4}. \end{aligned}$$

Hence, $(u,v) \in U$. By taking the trapezoidal domain T into account, we obtain the following result.

Theorem 2.1 *Suppose that there exist numbers γ and δ with $\gamma \geq \delta > 0$ such that*

$$(a(t), b(t)) \in T(\gamma,\delta) \tag{2.2}$$

for t sufficiently large. Then all nontrivial solutions of (2.1) *are nonoscillatory.*

Proof of Theorem 2.1 By way of contradiction, we suppose that equation (2.1) has an oscillatory solution. Then, from Sturm's separation theorem, we see that all nontrivial solutions of (2.1) are oscillatory. Let $z = y'$. Then, equation (2.1) becomes the planar linear system

$$\begin{aligned} y' &= z, \\ z' &= -b(t)y - a(t)z. \end{aligned} \tag{2.3}$$

Since all nontrivial solutions of (2.1) are oscillatory, these derivatives are also oscillatory. Hence, judging from the vector field of (2.1), we conclude that all positive orbits of (2.3) rotate in a clockwise direction about the origin infinitely many times.

Condition (2.2) means that there exists a sufficiently large t_0 such that

$$0 < \gamma \leq 2\gamma - \delta \leq a(t) \leq 2\gamma + \delta$$

and

$$0 \leq b(t) \leq \gamma a(t) - \gamma^2 \tag{2.4}$$

for $t \geq t_0$. Since $a(t)$ is bounded, we can define

$$u_0 = \sup_{t \geq t_0} a(t) \quad \text{and} \quad v_0 = \gamma u_0 - \gamma^2 \tag{2.5}$$

We choose a $t_1 \geq t_0$ so that $a(t_1) = u_0$. Note that t_1 may be ∞. Since the trapezoidal domain $T(\gamma,\delta)$ is closed, we see that $(u_0, b(t_1)) \in T(\gamma,\delta)$. From (2.4) and (2.5) it follows that

$$0 \leq b(t_1) \leq \gamma u_0 - \gamma^2 = v_0.$$

We here consider the autonomous system

$$\begin{aligned} y' &= z, \\ z' &= -v_0 y - u_0 z. \end{aligned} \tag{2.6}$$

From (2.5), we see that system (2.6) has a solution

$$(y(t), z(t)) = (-e^{-\gamma t}, \gamma e^{-\gamma t}).$$

Hence, the solution curve is given by $z = -\gamma y$ for $y < 0$. This curve is in the second quadrant

$$Q_2 = \{(y,z) \colon y < 0 < z\}.$$

Since

$$\gamma = \frac{\gamma^2}{u_0} + \frac{v_0}{u_0} > \frac{v_0}{u_0},$$

we can define the sectorial domain

$$D = \{(y,z):\ y < 0 \ \text{ and } \ -(v_0/u_0)y < z < -\gamma y\} \subset Q_2.$$

Let P be any point in D. We denote by $\Gamma^+_{(2.6)}(P)$ the positive orbit of (2.6) starting at the point P. Note that system (2.6) is autonomous. By the uniqueness of solutions of initial value problems, $\Gamma^+_{(2.6)}(P)$ does not cross the solution curve $z = -\gamma y$ in D. Taking account of the vector field of (2.6) in R, we see that $\Gamma^+_{(2.6)}(P)$ does not intersect the straight line $z = -(v_0/u_0)y$ and approaches the origin through D.

Consider the positive orbit of (2.3) starting from the point P at $t = \tau \ge t_0$. We express this positive orbit by $\Gamma^+_{(2.3)}(P)$. Let us compare $\Gamma^+_{(2.3)}(P)$ with $\Gamma^+_{(2.6)}(P)$. It follows from (2.5) that $a(t) \le u_0$ for $t \ge t_0$. Hence, by (2.4) we have

$$\begin{aligned} v_0 - b(t) &= \gamma u_0 - \gamma^2 - b(t) \\ &\ge \gamma a(t) - \gamma^2 - b(t) \ge 0 \end{aligned}$$

for $t \ge t_0$. Using (2.4) and (2.5) again, we see that if $(y,z) \in R$, then

$$\begin{aligned} b(t)y + a(t)z &= v_0 y + u_0 z - (v_0 - b(t))y - (u_0 - a(t))z \\ &> v_0 y + u_0 z + \frac{v_0 - b(t)}{\gamma} z - (u_0 - a(t))z \\ &= v_0 y + u_0 z + \left(a(t) - \gamma - \frac{b(t)}{\gamma}\right) z \\ &\ge v_0 y + u_0 z \end{aligned}$$

for $t \ge t_0$. Hence, we obtain

$$-\frac{b(t)y + a(t)z}{z} < -\frac{v_0 y + u_0 z}{z} < 0 \quad \text{for } \ t \ge t_0.$$

From this inequality it turns out that

(1) the slope of $\Gamma^+_{(2.3)}(P)$ is steeper than the slope of $\Gamma^+_{(2.6)}(P)$ at the point P;

(2) $\Gamma^+_{(2.3)}(P)$ and $\Gamma^+_{(2.6)}(P)$ do not have a common point in R.

Hence, $\Gamma^+_{(2.3)}(P)$ runs under $\Gamma^+_{(2.6)}(P)$, and therefore, $\Gamma^+_{(2.3)}(P)$ does not intersect the solution curve $z = -\gamma y$ in R. However, this contradicts the above-mentioned conclusion that $\Gamma^+_{(2.3)}(P)$ goes around the origin clockwise. Thus, all nontrivial solutions of (2.1) are nonoscillatory. The proof of Theorem 2.1 is complete. □

3 Proof of the main theorem

We denote by R the nonoscillation region defined by inequality (1.5). Let k be an arbitrary number larger than 2 and let R_k be the region defined by

$$|\beta| \le \begin{cases} \dfrac{\alpha}{c^*} - \dfrac{c^*(k-2)}{(C^*k)^2} + \dfrac{1}{C^*k}\sqrt{8\alpha + \left(\dfrac{c^*(k-2)}{C^*k}\right)^2} & \text{if } \ 0 \le \alpha < \left(\dfrac{c^*(k-2)}{C^*k}\right)^2, \\ \dfrac{\alpha}{c^*} + \dfrac{2\,c^*(k-2)}{(C^*k)^2} & \text{if } \ \alpha \ge \left(\dfrac{c^*(k-2)}{C^*k}\right)^2. \end{cases}$$

Before proving Theorem 1.1, we will show that

$$R = \bigcup_{k>2} R_k. \tag{3.1}$$

For this purpose, we consider the curve

$$\beta = \begin{cases} \dfrac{\alpha}{c^*} - \dfrac{c^*(k-2)}{(C^*k)^2} + \dfrac{1}{C^*k}\sqrt{8\alpha + \left(\dfrac{c^*(k-2)}{C^*k}\right)^2} & \text{if } 0 \le \alpha < \left(\dfrac{c^*(k-2)}{C^*k}\right)^2, \\ \dfrac{\alpha}{c^*} + \dfrac{2c^*(k-2)}{(C^*k)^2} & \text{if } \alpha \ge \left(\dfrac{c^*(k-2)}{C^*k}\right)^2. \end{cases} \tag{3.2}$$

This curve passes through the point

$$(\alpha_0(k), \beta_0(k)) = \left(\left(\frac{c^*(k-2)}{C^*k}\right)^2, \frac{c^*(k-2)}{(C^*)^2k}\right), \tag{3.3}$$

and it is a concave curve on the interval $[0, (c^*(k-2)/(C^*k))^2]$ and a straight line on the interval $[(c^*(k-2)/(C^*k))^2, \infty)$. Since

$$\lim_{\alpha\to\alpha_0+0} \frac{\beta(\alpha)-\beta(\alpha_0)}{\alpha-\alpha_0} = \frac{1}{c^*} < \frac{1}{c^*}\left(1 + \frac{4}{3(k-2)}\right) = \lim_{\alpha\to\alpha_0-0} \frac{\beta(\alpha)-\beta(\alpha_0)}{\alpha-\alpha_0},$$

this curve has a sharp corner at the point (α_0, β_0). From (3.3), we see that the point (α_0, β_0) moves on the curve $\beta = \sqrt{\alpha}/C^*$ with the change of $k > 2$. Note that

$$0 < \frac{2(k-2)}{k^2} \le \frac{1}{4} \quad \text{for } k > 2$$

and the equality holds only when $k = 4$. Hence, the straight line $\beta = \alpha/c^* + c^*/(2C^*)^2$ is located above the other straight lines $\beta = \alpha/c^* + 2c^*(k-2)/(C^*k)^2$ with $2 < k < 4$ and $k > 4$. The curve $\beta = \sqrt{\alpha}/C^*$ is connected with the straight line $\beta = \alpha/c^* + c^*/(2C^*)^2$ smoothly at the point

$$(\alpha_0(4), \beta_0(4)) = \left((c^*/(2C^*))^2, c^*/(2(C^*)^2)\right).$$

In the case when $\beta < 0$, we can use the same argument as in the case when $\beta \ge 0$. We therefore conclude that the region R is the union for all $k > 4$ of R_k (see Figure 4).

The relation (3.1) means that if $(\alpha, \beta) \in R$, then there exists a $k_0 > 2$ such that $(\alpha, \beta) \in R_{k_0}$. Let

$$\gamma = \frac{1}{2}\left\{C^*k_0\left(\beta - \frac{\alpha}{c^*}\right) + \sqrt{\left(C^*k_0\left(\beta - \frac{\alpha}{c^*}\right)\right)^2 + 4\left|\frac{k_0}{2}\left(\beta - \frac{\alpha}{c^*}\right) - \beta\right| + 4\alpha}\right\} \tag{3.4}$$

and $\delta = C^*k_0\left(\beta - \dfrac{\alpha}{c^*}\right)$. Consider equation (2.1) with

$$\begin{aligned} a(t) &= 2\gamma - k_0\left(\beta - \frac{\alpha}{c^*}\right)C(t), \\ b(t) &= \left(\gamma - \frac{k_0}{2}\left(\beta - \frac{\alpha}{c^*}\right)C(t)\right)^2 - \frac{k_0}{2}\left(\beta - \frac{\alpha}{c^*}\right)c(t) - \alpha + \beta c(t). \end{aligned} \tag{3.5}$$

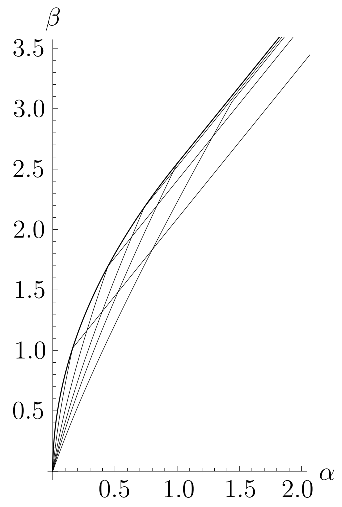

Fig. 4 The curves defined by (3.2) when $c^* = 0.7854$, $C^* = 0.3927$ and $k = 2.5, 3, 3.5, 4, 5$

It turns out that

$$\begin{aligned}\frac{1}{4}a^2(t)+\frac{1}{2}a'(t)-\alpha+\beta c(t) &= \gamma^2-k_0\left(\beta-\frac{\alpha}{c^*}\right)\gamma C(t)+\frac{k_0^2}{4}\left(\beta-\frac{\alpha}{c^*}\right)^2C^2(t)\\ &\quad -\frac{k_0}{2}\left(\beta-\frac{\alpha}{c^*}\right)c(t)-\alpha+\beta c(t)\\ &= \left(\gamma-\frac{k_0}{2}\left(\beta-\frac{\alpha}{c^*}\right)C(t)\right)^2-\frac{k_0}{2}\left(\beta-\frac{\alpha}{c^*}\right)c(t)\\ &\quad -\alpha+\beta c(t)\\ &= b(t).\end{aligned}$$

Define

$$x = y\exp\left(\frac{1}{2}\int_0^t a(\tau)d\tau\right).$$

Then, we have

$$\begin{aligned}x''+(-\alpha+\beta c(t))x &= \left(y''+a(t)y'+\left(\frac{1}{4}a^2(t)+\frac{1}{2}a'(t)-\alpha+\beta c(t)\right)y\right)\\ &\quad\times\exp\left(\frac{1}{2}\int_0^t a(\tau)d\tau\right)\\ &= (y''+a(t)y'+b(t)y)\exp\left(\frac{1}{2}\int_0^t a(\tau)d\tau\right).\end{aligned}$$

Hence, all nontrivial solutions of (1.1) are nonoscillatory if and only if those of (2.1) are nonoscillatory under the assumption (3.5).

By using Theorem 2.1, we can prove our main theorem which was presented in Section 1.

Proof of Theorem 1.1 If $\alpha = 0$, then it follows from (1.5) that $\beta = 0$. In this case, it is clear that all nontrivial solutions of (1.1) are nonoscillatory. Hence, we only need to consider the case when $\alpha > 0$. Let

$$s = t - 1 \quad \text{and} \quad z(s) = x(t).$$

Then, we can transform equation (1.1) into

$$\frac{d^2 z}{ds^2} + (-\alpha - \beta c(s))z = 0$$

which has the same form as equation (1.1). Hence, we only deal with the case when $\beta \geq 0$. If $\alpha \geq c^*\beta$, then

$$-\alpha + \beta c(t) \leq -\alpha + c^*\beta \leq 0$$

for t sufficiently large. Hence, by virtue of Sturm's comparison theorem, all nontrivial solutions of (1.1) are nonoscillatory (see the light grey part in Figure 3). Thus, the only remaining case is $0 < \alpha < c^*\beta$.

Since $0 < \alpha < c^*\beta$, we see that

$$\begin{aligned} 0 < \delta &= C^* k_0 \left(\beta - \frac{\alpha}{c^*}\right) \\ &< \frac{1}{2}\left\{C^* k_0 \left(\beta - \frac{\alpha}{c^*}\right) + \sqrt{\left(C^* k_0 \left(\beta - \frac{\alpha}{c^*}\right)\right)^2 + 4\left|\frac{k_0}{2}\left(\beta - \frac{\alpha}{c^*}\right) - \beta\right| + 4\alpha}\right\} = \gamma. \end{aligned}$$

We will cheque whether that $(a(t), b(t))$ given by (3.5) satisfies condition (2.2). Put $u = a(t)$ and $v = b(t)$. Then it is clear that

$$2\gamma - \delta \leq u \leq 2\gamma + \delta.$$

By (3.4), we have

$$\begin{aligned} v = {} & \gamma^2 - C^* k_0 \left(\beta - \frac{\alpha}{c^*}\right)\gamma - c^* \left|\frac{k_0}{2}\left(\beta - \frac{\alpha}{c^*}\right) - \beta\right| - \alpha \\ & + k_0 \left(\beta - \frac{\alpha}{c^*}\right)\gamma\,(C^* - C(t)) + \frac{k_0^2}{4}\left(\beta - \frac{\alpha}{c^*}\right)^2 C^2(t) \\ & + c^* \left|\frac{k_0}{2}\left(\beta - \frac{\alpha}{c^*}\right) - \beta\right| + \alpha - \frac{k_0}{2}\left(\beta - \frac{\alpha}{c^*}\right)c(t) - \alpha + \beta c(t) \\ = {} & k_0 \left(\beta - \frac{\alpha}{c^*}\right)\gamma\,(C^* - C(t)) + \frac{k_0^2}{4}\left(\beta - \frac{\alpha}{c^*}\right)^2 C^2(t) \\ & + c^* \left|\frac{k_0}{2}\left(\beta - \frac{\alpha}{c^*}\right) - \beta\right| - \left(\frac{k_0}{2}\left(\beta - \frac{\alpha}{c^*}\right) - \beta\right)c(t). \end{aligned}$$

From the assumption (1.3), we see that $v \geq 0$. It also follows from (3.5) that

$$\begin{aligned} 0 \leq v &= \frac{u^2}{4} - \left(\frac{k_0}{2}\left(\beta - \frac{\alpha}{c^*}\right) - \beta\right)c(t) - \alpha \\ &\leq \frac{u^2}{4} - \left(\alpha - c^*\left|\frac{k_0}{2}\left(\beta - \frac{\alpha}{c^*}\right) - \beta\right|\right). \end{aligned}$$

Hence, we conclude that the parameter curve $(a(t), b(t))$ is included in the the domain

$$S = \left\{(u,v) : 2\gamma - \delta \leq u \leq 2\gamma + \delta \ \text{ and } \ 0 < v \leq \frac{u^2}{4} - \left(\alpha - c^*\left|\frac{k_0}{2}\left(\beta - \frac{\alpha}{c^*}\right) - \beta\right|\right)\right\}.$$

Let $A(2\gamma - \delta, \gamma^2 - \gamma\delta)$ and $B(2\gamma + \delta, \gamma^2 + \gamma\delta)$ be two points in the (u, v)-plane. Note that the points A and B are on the straight line $v = \gamma u - \gamma^2$. Since the line $v = \gamma u - \gamma^2$ is the tangent to the quadratic curve $v = u^2/4$ at the point $(2\gamma, \gamma^2)$, the vertical distance between the line segment $\overline{AB}$ and the curve $v = u^2/4$ is less than or equal to $\delta^2/4$.

We will show that the line segment $\overline{AB}$ is located above the quadratic curve

$$v = \frac{u^2}{4} - \left(\alpha - c^*\left|\frac{k_0}{2}\left(\beta - \frac{\alpha}{c^*}\right) - \beta\right|\right).$$

To this end, we have only to verify that

$$\frac{\delta^2}{4} = \frac{1}{4}\left(C^* k_0\left(\beta - \frac{\alpha}{c^*}\right)\right)^2 \leq \alpha - c^*\left|\frac{k_0}{2}\left(\beta - \frac{\alpha}{c^*}\right) - \beta\right|, \tag{3.6}$$

because this quadratic curve is convex. Since

$$\frac{k_0}{2}\left(\beta - \frac{\alpha}{c^*}\right) - \beta = \frac{1}{2c^*}\left(c^*(k_0 - 2)\beta - k_0\alpha\right),$$

there are two cases to be considered:

(i) $\alpha < c^*\beta < k_0\alpha/(k_0 - 2)$;

(ii) $c^*\beta \geq k_0\alpha/(k_0 - 2)$.

Recall that the curve defined by (3.2) is concave for $0 \leq \alpha \leq (c^*(k-2)/(C^*k))^2$. Hence, the region $R_{k_0} \cap \{(\alpha, \beta) : 0 < \alpha < c^*\beta\}$ is divided into two parts. One is the region defined by

$$\frac{\alpha}{c^*} < \beta \leq \begin{cases} \dfrac{k_0\alpha}{c^*(k_0 - 2)} & \text{if } \ 0 \leq \alpha < \left(\dfrac{c^*(k_0 - 2)}{C^* k_0}\right)^2, \\[2ex] \dfrac{\alpha}{c^*} + \dfrac{2c^*(k_0 - 2)}{(C^* k_0)^2} & \text{if } \ \alpha \geq \left(\dfrac{c^*(k_0 - 2)}{C^* k_0}\right)^2. \end{cases} \tag{3.7}$$

The other is the region defined by

$$\frac{k_0\alpha}{c^*(k_0 - 2)} \leq \beta \leq \frac{\alpha}{c^*} - \frac{c^*(k_0 - 2)}{(C^* k_0)^2} + \frac{1}{C^* k_0}\sqrt{8\alpha + \left(\frac{c^*(k_0 - 2)}{C^* k_0}\right)^2} \tag{3.8}$$

for $0 \leq \alpha \leq (c^*(k_0 - 2)/(C^* k_0))^2$ (see Figure 5).

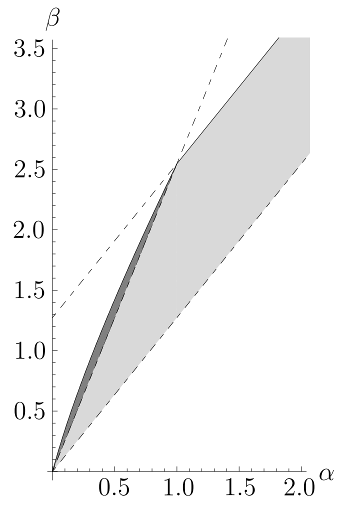

Fig. 5 The region defined by (3.7) (light grey part) and the region defined by (3.8) (dark grey part) when $c^* = 0.7854$, $C^* = 0.3927$ and $k_0 = 4$

Case (i): $\alpha < c^*\beta < k_0\alpha/(k_0-2)$. In this case, (α,β) is in the region given by (3.7). As shown in Figure 5, the parameters α and β satisfy that

$$0 < \beta - \frac{\alpha}{c^*} \le \frac{2\,c^*(k_0-2)}{(C^*k_0)^2}.$$

Hence, we obtain

$$\begin{aligned}\frac{1}{4}\left(C^*k_0\left(\beta-\frac{\alpha}{c^*}\right)\right)^2 &= \frac{1}{4}\,(C^*k_0)^2\left(\beta-\frac{\alpha}{c^*}\right)\left(\beta-\frac{\alpha}{c^*}\right)\\ &\le \frac{c^*(k_0-2)}{2}\left(\beta-\frac{\alpha}{c^*}\right)\\ &= \alpha - c^*\left(\beta-\frac{k_0}{2}\left(\beta-\frac{\alpha}{c^*}\right)\right)\\ &= \alpha - c^*\left|\frac{k_0}{2}\left(\beta-\frac{\alpha}{c^*}\right)-\beta\right|,\end{aligned}$$

namely, the inequality (3.6).

Case (ii): $c^*\beta \ge k_0\alpha/(k_0-2)$. From (3.8) it follows that

$$C^*k_0\left(\beta-\frac{\alpha}{c^*}\right)+\frac{c^*(k_0-2)}{C^*k_0} \le \sqrt{8\alpha+\left(\frac{c^*(k_0-2)}{C^*k_0}\right)^2}.$$

Hence, we obtain

$$\begin{aligned}\frac{1}{4}\left(C^*k_0\left(\beta-\frac{\alpha}{c^*}\right)\right)^2 &= 2\alpha-\frac{c^*(k_0-2)}{2}\left(\beta-\frac{\alpha}{c^*}\right)\\ &= \alpha-c^*\left(\beta-\frac{\alpha}{c^*}\right)+c^*\beta-\frac{c^*(k_0-2)}{2}\left(\beta-\frac{\alpha}{c^*}\right)\\ &= \alpha-c^*\left(\frac{k_0}{2}\left(\beta-\frac{\alpha}{c^*}\right)-\beta\right)\\ &= \alpha-c^*\left|\frac{k_0}{2}\left(\beta-\frac{\alpha}{c^*}\right)-\beta\right|,\end{aligned}$$

namely, the inequality (3.6).

By the above-mentioned argument, it turns out that

$$S\subset T(\gamma,\delta).$$

We therefore conclude that

$$(a(t),b(t))\in T(\gamma,\delta)$$

for t sufficiently large; that is, condition (2.2) holds. Thus, by means of Theorem 2.1, all nontrivial solutions of (2.1) are nonoscillatory under the assumption (3.5), and therefore, those of (1.1) are nonoscillatory. □

4 Simulation and discussion

Consider the discontinuous differential equation

$$x''+(-\alpha+\beta f(t))x=0, \tag{4.1}$$

where f is the piece-wise constant function given in Section 1. Equation (4.1) is a kind of Meissner's equation. There are many studies on the stability theory regarding more general Meissner equations. For example, refer to [8, 12, 14, 21, 24, 27].

Solutions of (4.1) satisfy the second-order linear differential equation with a constant coefficient,

$$x_1''+(-\alpha+\frac{\pi}{4}\beta)x_1=0,\qquad 2(m-1)\le t<2m-1 \tag{4.2}$$

and

$$x_2''+(-\alpha-\frac{\pi}{4}\beta)x_2=0,\qquad 2m-1\le t<2m \tag{4.3}$$

with $m\in\mathbb{N}$. There are four cases to be considered:

(1) $-\alpha+\frac{\pi}{4}\beta>0$ and $-\alpha-\frac{\pi}{4}\beta>0$, namely, $\alpha<0$ and $\frac{4}{\pi}\alpha<\beta<-\frac{4}{\pi}\alpha$;

(2) $-\alpha+\frac{\pi}{4}\beta\le 0$ and $-\alpha-\frac{\pi}{4}\beta>0$, namely, $\beta\le\frac{4}{\pi}\beta$ and $\beta<-\frac{4}{\pi}\beta$;

(3) $-\alpha+\frac{\pi}{4}\beta>0$ and $-\alpha-\frac{\pi}{4}\beta\le 0$, namely, $\beta>\frac{4}{\pi}\beta$ and $\beta\ge-\frac{4}{\pi}\beta$;

(4) $-\alpha+\frac{\pi}{4}\beta\le 0$ and $-\alpha-\frac{\pi}{4}\beta\le 0$, namely, $\alpha\ge 0$ and $-\frac{4}{\pi}\alpha\le\beta\le\frac{4}{\pi}\alpha$.

Case (1): Equations (4.2) and (4.3) have solutions

$$x_1(t) = A\sin\left(t\sqrt{-\alpha+\pi\beta/4}\right) + B\cos\left(t\sqrt{-\alpha+\pi\beta/4}\right)$$

and

$$x_2(t) = C\sin\left(t\sqrt{-\alpha-\pi\beta/4}\right) + D\cos\left(t\sqrt{-\alpha-\pi\beta/4}\right),$$

respectively.

Case (2): Equations (4.2) and (4.3) have solutions

$$x_1(t) = A\exp\left(t\sqrt{\alpha-\pi\beta/4}\right) + B\exp\left(-t\sqrt{\alpha-\pi\beta/4}\right)$$

and

$$x_2(t) = C\sin\left(t\sqrt{-\alpha-\pi\beta/4}\right) + D\cos\left(t\sqrt{-\alpha-\pi\beta/4}\right),$$

respectively.

Case (3): Equations (4.2) and (4.3) have solutions

$$x_1(t) = A\sin\left(t\sqrt{-\alpha+\pi\beta/4}\right) + B\cos\left(t\sqrt{-\alpha+\pi\beta/4}\right)$$

and

$$x_2(t) = C\exp\left(t\sqrt{\alpha+\pi\beta/4}\right) + D\exp\left(-t\sqrt{\alpha+\pi\beta/4}\right),$$

respectively.

Case (4): Equations (4.2) and (4.3) have solutions

$$x_1(t) = A\exp\left(t\sqrt{\alpha-\pi\beta/4}\right) + B\exp\left(-t\sqrt{\alpha-\pi\beta/4}\right)$$

and

$$x_2(t) = C\exp\left(t\sqrt{\alpha+\pi\beta/4}\right) + D\exp\left(-t\sqrt{\alpha+\pi\beta/4}\right),$$

respectively.

Here, A, B, C and D are any real numbers. Since each solution of (4.1) is a combination of solutions of (4.2) and (4.3), we see that

(a) if $\alpha < 0$ and $4\alpha/\pi < \beta < -4\alpha/\pi$, then all nontrivial solutions of (4.1) are oscillatory;

(b) if $\alpha \ge 0$ and $-4\alpha/\pi < \beta < 4\alpha/\pi$, then all nontrivial solutions of (4.1) are nonoscillatory.

However, in the other cases, we cannot immediately decide whether all nontrivial solutions of (4.1) are oscillatory or not.

It is clear that $|f(t)| \le \pi/4$ for $t \ge 0$. Since

$$\int_0^t f(s)ds = \begin{cases} (\pi/4)t - \pi(m-1)/2 & \text{if } 2(m-1) \le t < 2m-1, \\ -(\pi/4)t + \pi m/2 & \text{if } 2m-1 \le t < 2m \end{cases}$$

with $m \in \mathbb{N}$, we see that $\underline{C} = 0$ and $\overline{C} = \pi/4$. Hence, we may choose $\pi/4$ and $\pi/8$ as c^* and C^*, respectively. Theorem 1.1 infers that if

$$|\beta| \le \begin{cases} \dfrac{8}{\pi}\sqrt{\alpha} & \text{if } 0 \le \alpha < 1, \\ \dfrac{4}{\pi}\alpha + \dfrac{4}{\pi} & \text{if } \alpha \ge 1, \end{cases}$$

then all nontrivial solutions of (4.1) are nonoscillatory.

Recall that the Cesàro summation T_N is a good approximation of the step function f. For this reason, we can expect that the asymptotic behaviour of a solution of (4.1) is very near to that of the solution of

$$x'' + (-\alpha + \beta T_N(t))x = 0 \tag{4.4}$$

satisfying the same initial condition. We confirm this expectation by simulation. In the left part of Figure 6, we draw two solution orbits of (4.1) and (4.4) with $N = 5$; in the right part of Figure 6, we draw two solution orbits of (4.1) and (4.4) with $N = 20$. In either part, the lower curve represents the solution of (4.1) satisfying the initial condition $(x(0), x'(0)) = (0, 1)$, and the upper curve represents the solution of (4.4) satisfying the same initial condition. As shown in Figure 6, the upper curve approaches the lower curve as N increases. If we draw the solution orbit of (4.4) with $N = 30$, we cannot distinguish between the upper curve and the lower curve.

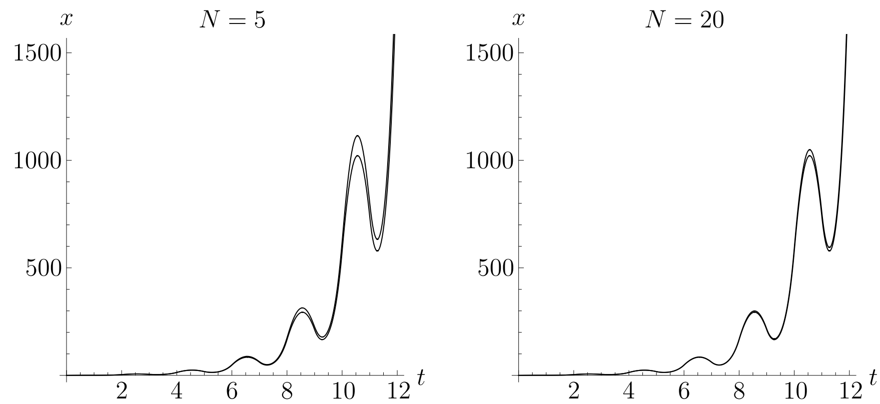

Fig. 6 Comparison between the solutions of (4.1) and (4.4) satisfying the initial condition $(x(0), x'(0)) = (0, 1)$ when $\alpha = 2$ and $\beta = 20/\pi$

Recently, Ishibashi and the author [15] have considered the second-order differential equations

$$x'' + (-\alpha + \beta\cos(\rho t))x = 0 \tag{4.5}$$

and

$$x'' + (-\alpha + \beta\sin(\rho t))x = 0, \tag{4.6}$$

where α, β and ρ are real parameters and $\rho > 0$, and reported the following results.

Theorem A *If*

$$|\beta| \geq \rho\sqrt{2\alpha} + \alpha \quad \textit{for} \;\; \alpha \geq 0, \tag{4.7}$$

then all nontrivial solutions of (4.5) (*or* (4.6)) *are oscillatory*.

Theorem B. *If*

$$|\beta| \leq \frac{\rho\sqrt{2\alpha}}{2} + \alpha \quad \textit{for} \;\; \alpha \geq 0, \tag{4.8}$$

then all nontrivial solutions of (4.5) (*or* (4.6)) *are nonoscillatory*.

They proved Theorems A and B by using an oscillation theorem given by Sugie and Matsumura [29, Theorem 3.1] and a nonoscillation theorem given by Kwong and Wong [18, Theorem 1], respectively. Theorem A (or Theorem B) gives a parametric oscillation (or nonoscillation) region for Mathieu's equations (4.5) and (4.6). Parametric oscillation and nonoscillation regions have already been studied by several researchers (for example, see [9, 16, 17, 20, 30, 32]).

Comparing equation (1.1) with equation (4.5) (or (4.6)), we see that the periodic function c in equation (1.1) corresponds to the monomial $\cos(\rho t)$ (or $\sin(\rho t)$) in equation (4.5) (or (4.6)). Theorem 1.1 can be applied even if the function c is represented by many number of terms as long as c is periodic of mean value zero. This is the feature of Theorem 1.1 that Theorems A and B do not have. Of course, we can apply Theorem 1.1 to equations (4.5) and (4.6). In the case when $c(t) = \cos(\rho t)$ (or $\sin(\rho t)$), we may regard 1 and $1/\rho$ as c^* and C^*, respectively. Hence, from Theorem 1.1, we see that if

$$|\beta| \leq \begin{cases} \rho\sqrt{\alpha} & \text{if } \; 0 \leq \alpha < \dfrac{1}{4}\rho^2, \\[2ex] \alpha + \dfrac{1}{4}\rho^2 & \text{if } \; \alpha \geq \dfrac{1}{4}\rho^2, \end{cases} \tag{4.9}$$

then all nontrivial solutions of (4.5) (or (4.6)) are nonoscillatory. Let us compare inequalities (4.8) and (4.9). If $\alpha = 1$ and $\beta = \rho = 4$, then the inequality (4.9) is satisfied because

$$0 \leq \alpha = 1 < 4 \quad \text{and} \quad \beta = 4 = 4 \times \sqrt{1} = \rho\sqrt{\alpha}.$$

However, condition (4.8) does not hold. In fact,

$$\beta = 4 > 2\sqrt{2} + 1 = \frac{4\sqrt{2 \times 1}}{2} + 1 = \frac{\rho\sqrt{2\alpha}}{2} + \alpha.$$

Finally, let us apply Theorem 1.1 to the differential equation

$$x'' + (-\alpha + \beta\cos^3(\rho t))x = 0. \tag{4.10}$$

It is clear that $c^* = 1$. Since

$$\int_0^t \cos^3(\rho s)ds = \frac{1}{12\rho}\sin(3\rho t) + \frac{3}{4\rho}\sin(\rho t),$$

we see that $\underline{C} = -2/(3\rho)$ and $\overline{C} = 2/(3\rho)$. It follows that $C^* = (\overline{C} - \underline{C})/2 = 2/(3\rho)$

$$|C(t)| = \left| \int_0^t c(s)ds \right| \le C^* \quad \text{for } t \ge 0.$$

Hence, condition (1.5) becomes

$$|\beta| \le \begin{cases} \dfrac{3}{2}\rho\sqrt{\alpha} & \text{if } 0 \le \alpha < \dfrac{9}{16}\rho^2, \\ \alpha + \dfrac{9}{16}\rho^2 & \text{if } \alpha \ge \dfrac{9}{16}\rho^2. \end{cases}$$

For example, if $\alpha = 1$, $\beta = 18$ and $\rho = 12$, then the inequality (4.11) holds because

$$0 \le \alpha = 1 < 81 = \frac{9\rho^2}{16} \quad \text{and} \quad \frac{3\rho}{2}\sqrt{\alpha} = \frac{3 \times 12}{2}\sqrt{1} = 18 = \beta.$$

Hence, from Theorem 1.1 it turns out that all nontrivial solutions of (4.10) are nonoscillatory (see Figure 7).

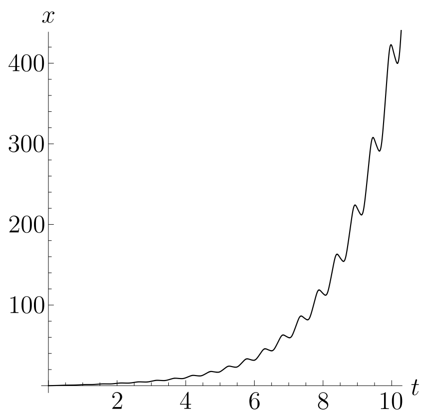

Fig. 7 The solution of (4.10) satisfying the initial condition $(x(0), x'(0)) = (0, 1)$ when $\alpha = 1$, $\beta = 18$ and $\rho = 12$

In contrast to the above situation, all nontrivial solutions of (4.5) are oscillatory when $\alpha = 1$, $\beta = 18$ and $\rho = 12$ (see Figure 8). In fact,

$$\beta = 18 > 12\sqrt{2 \times 1} + 1 = \rho\sqrt{2\alpha} + \alpha,$$

namely, condition (4.7) is satisfied. Hence, Theorem A is available.

As can be seen from the above comparison of equations (4.5) and (4.10), parametric nonoscillation region about (α, β) spreads as the value of C^* decreases provided that c^* is a constant value. For example, if $c(t) = \cos^5(\rho t)$, then $c^* = 1$ and

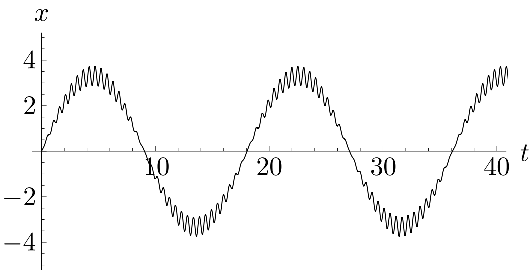

Fig. 8 The solution of (4.5) satisfying the initial condition $(x(0), x'(0)) = (0, 1)$ when $\alpha = 1$, $\beta = 18$ and $\rho = 12$

$C^* = 8/(15\rho)$; if $c(t) = \cos^7(\rho t)$, then $c^* = 1$ and $C^* = 16/(35\rho)$. Hence, condition (1.5) becomes

$$|\beta| \le \begin{cases} \dfrac{15}{8}\rho\sqrt{\alpha} & \text{if } 0 \le \alpha < \dfrac{225}{256}\rho^2, \\ \alpha + \dfrac{225}{256}\rho^2 & \text{if } \alpha \ge \dfrac{225}{256}\rho^2. \end{cases}$$

and

$$|\beta| \le \begin{cases} \dfrac{35}{16}\rho\sqrt{\alpha} & \text{if } 0 \le \alpha < \dfrac{1225}{1024}\rho^2, \\ \alpha + \dfrac{1225}{1024}\rho^2 & \text{if } \alpha \ge \dfrac{1225}{1024}\rho^2. \end{cases}$$

respectively.

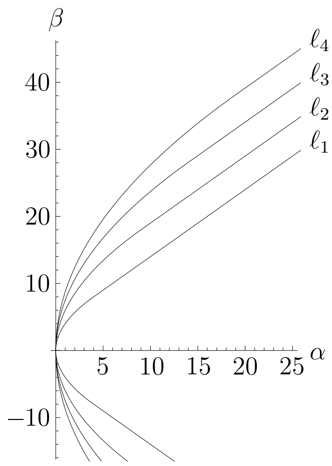

Fig. 9 The curves defined by (3.2) when $c^* = 0.7854$, $C^* = 0.3927$ and $k = 2.5, 3, 3.5, 4, 5$

Figure 9 shows parametric nonoscillation regions about (α,β) for the differential equation

$$x'' + (-\alpha + \beta\cos^{2m-1}(\rho t))x = 0, \tag{4.11}$$

where $m = 1,2,3,4$. The equations of the curved lines ℓ_1, ℓ_2, ℓ_3 and ℓ_4 are

$$\beta = \begin{cases} 4\sqrt{\alpha} & \text{if } 0 \le \alpha < 4, \\ \alpha + 4 & \text{if } \alpha \ge 4, \end{cases}$$

$$\beta = \begin{cases} 6\sqrt{\alpha} & \text{if } 0 \le \alpha < 9, \\ \alpha + 9 & \text{if } \alpha \ge 9, \end{cases}$$

$$\beta = \begin{cases} \dfrac{15}{2}\sqrt{\alpha} & \text{if } 0 \le \alpha < \dfrac{225}{16}, \\ \alpha + \dfrac{225}{16} & \text{if } \alpha \ge \dfrac{225}{16}, \end{cases}$$

and

$$\beta = \begin{cases} \dfrac{35}{4}\sqrt{\alpha} & \text{if } 0 \le \alpha < \dfrac{1225}{64}, \\ \alpha + \dfrac{1225}{64} & \text{if } \alpha \ge \dfrac{1225}{64}, \end{cases}$$

respectively. Parametric nonoscillation region for equation (4.11) spreads out like a fan with increase of m and covers the half-plane $\{(\alpha,\beta) : \alpha > 0 \text{ and } \beta \in \mathbb{R}\}$ as m tends to infinity.

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References

1. Agarwal, R.P., Grace, S.R., O'Regan, D.: Oscillation Theory for Second Order Linear, Half-linear, Superlinear and Sublinear Dynamic Equations. Kluwer Academic Publishers, Dordrecht (2002)
2. Benjamin, T.B., Ursell, F.: The stability of the plane free surface of a liquid in vertical periodic motion. Proc. Roy. Soc. London Ser. A **225**, 505–515 (1954)
3. Binney, J.: Twisted and warped disks as consequences of heavy halos. Mon. Not. R. astr. Soc. **183**, 779–797 (1978)
4. Binney, J.: Resonant excitation of motion perpendicular to galactic planes. Mon. Not. R. astr. Soc. **196**, 445–467 (1981)
5. Çakmak, D.: A note on M. K. Kwong and J. S. W. Wong's paper. Dynam. Systems Appl. **15**, 409-414 (2006)
6. Cerda, E.A., Tirapegui, E.L.: Faraday's instability in viscous fluid. J. Fluid Mech. **368**, 195–228 (1998)
7. Coppel, W.A.: Disconjugacy, Lecture Notes in Mathematics. vol. **220**, Springer-Verlag, Berlin–New York (1971)

8. Csörgő, S., Hatvani, L.: Stability properties of solutions of linear second order differential equations with random coefficients. J. Differential Equations **248**, 21–49 (2010)
9. El-Sayed, M.A.: An oscillation criterion for a forced second order linear differential equation. Proc. Amer. Math. Soci. **118**, 813–817 (1993)
10. Faraday, M.: On a peculiar class of acoustical figures; and on certain forms assumed by groups of particles upon vibrating elastic surfaces. Phil. Trans. R. Soc. Lond. **121**, 299–340 (1831)
11. Fite, W.B.: Concerning the zeros of the solutions of certain differential equations. Trans. Amer. Math. Soc. **19**, 341–352 (1918)
12. Hatvani, L.: On the critical values of parametric resonance in Meissner's equation by the method of difference equations, Electron. J. Qual. Theory Differ. Equ. **2009**, Special Edition I, No. 13, 10 pp.
13. Hill, G.W.: On the part of the motion of the lunar perigee which is a function of the mean motions of the sun and the moon. Acta. Math. **8**, 1–36 (1886)
14. Hochstadt, H.: A special Hill's equation with discontinuous coefficients. Amer. Math. Monthly **70**, 18–26 (1963)
15. Ishibashi, K., Sugie, J.: Simple conditions for parametrically excited oscillations of generalized Mathieu equations. J. Math. Anal. Appl. **446**, 233–247 (2017)
16. Komkov, V.: A technique for the detection of oscillation of second order ordinary differential equation. Pacific J. Math. **42**, 105-115 (1972)
17. Kong, Q., Pašić, M.: Second-order differential equations some significant results due to James S. W. Wong. Diff. Eqn. Appl. **6**, 99–163 (2014)
18. Kwong, M.K., Wong, J.S.W.: Oscillation and nonoscillation of Hill's equation with periodic damping. J. Math. Anal. Appl. **288**, 15–19 (2003)
19. Kwong, M.K., Wong, J.S.W.: On the oscillation of Hill's equations under periodic forcing. J. Math. Anal. Appl. **320**, 37–55 (2006)
20. Leighton, W.: Hill's equaiton revisited. J. Math. Anal. Appl. **114**, 497–502 (1986)
21. Magnus, W., Winkler, S.: Hill's Equation. Dover, New York (1979)
22. Mathieu, É.: Mémoire sur le mouvement vibratoire d'une membrane de forme elliptique. J. Math. Pure. Appl. **13**, 137–203 (1868)
23. McLachan, N.W.: Theory and Application of Mathieu functions. Dover, New York (1964)
24. Meissner, E.: Ueber Schüttelerscheinungen in Systemen periodisch veränderlicher Elastizität, Schweiz. Bauzeit. **72**, 95–98 (1918)
25. Nayfeh, A.H., Mook, D.T.: Nonlinear Oscillations. John Wiley & Sons, New York (1995)
26. Rajchenbach, J., Clamond, D.: Faraday waves: their dispersion relation, nature of bifurcation and wavenumber selection revisited. J. Fluid Mech. **777**, (2015), R2, 12 pages.
27. Sato, C.: Correction of stability curves in Hill-Meissner's equation. Math. Comp. **20**, 98–106 (1966)
28. Sugie, J.: Geometrical conditions for oscillation of second-order half-linear differential equations. Acta. Math. Hungar. **118**, 369–394 (2008)
29. Sugie, J., Matsumura, K.: A nonoscillation theorem for half-linear differential equations with periodic coefficients. Appl. Math. Comput. **199**, 447–455 (2008)
30. Sun, Y.G., Ou, C.H., Wong, J.S.W.: Interval oscillation theorems for a second-order linear differential equation. Comput. Math. Appl. **48**, 1693–1699 (2004)
31. Swanson, C.A.: Comparison and Oscillation Theory of Linear Differential Equations. Mathematics in Science and Engineering, vol. **48**, New York and London, Academic Press (1968)
32. Wong, J.S.W.: Second order nonlinear forced oscillations. SIAM J. Math. Anal. **19**, 667–675 (1988)
33. Wong, J.S.W.: On a theorem of Sobol. Bull. Inst. Math. Acad. Sinica **27**, 253–264 (1999)
34. Wong, J.S.W.: On Kamenev-type oscillation theorems for second-order differential equations with damping. J. Math. Anal. Appl. **258**, 244–257 (2001)
35. Yan, J.: A note on an oscillation criterion for an equation with damped term. Proc. Amer. Math. Soci. **90**, 277–280 (1984)
36. Yan, J.: Oscillation theorems for second order linear differential equations with damping. Proc. Amer. Math. Soci. **98**, 276–282 (1986)
37. Zheng, Z.: Note on Wong's paper. J. Math. Anal. Appl. **274**, 466–473 (2002)