Uniform global asymptotic stability of time-varying Lotka-Volterra predator-prey systems

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Abstract

The model to be dealt in this paper is

$$\begin{aligned} N' &= (a + ch(t) - dh(t)N - bP)N, \\ P' &= (-c + dN)P. \end{aligned}$$

Here, h is a nonnegative and locally integrable function. This model is a predator-prey system of Lotka-Volterra type with variable coefficients and it has a single interior equilibrium $(c/d, a/b)$. Sufficient conditions are given for the interior equilibrium to be uniformly globally asymptotically stable. One of them is described by using a certain uniform divergence condition on h. Our result is proved by examining in details the behaviour of all solutions of a planar system equivalent to this model.

Key words: Uniform global asymptotic stability; Lotka-Volterra predator-prey model; Uniform divergence; Growth condition; Time-varying system
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1. Introduction

The population of species in ecological models is one of the most important issues in biomathematics. To understand the dynamics of species populations, mathematical models are often used. For example, the classical Lotka-Volterra predator-prey system was first raised up to describe the population of sharks and fish in the Adriatic Sea. Afterwards, this model was progressively improved so that it can be used not only for ecology but also for analysis of physical phenomena and economical theory. In the process, it was pointed out that this model is structurally unstable and there is a gap with natural phenomena. For this reason, many researchers have paid various efforts to find more reasonable models that describe nature. (for example, see [1, 2, 3, 4, 5, 8, 9, 10, 11] and the references cited therein). Among them, the idea that the seasonal change should be emphasised has arisen. Because the environment, the habitat state and other related factors can change over season, this idea that alternation of season is one of the basic factors that can affect population ecology will be reasonable.

In this paper, taking into account that the prey is more susceptible to the seasonal change than the predator, we deal with the time-varying system

$$\begin{aligned} N' &= \alpha(t)N - \beta(t)N^2 - bNP, \\ P' &= -cP + dNP, \end{aligned} \tag{E}$$

where $' = d/dt$, the letters N and P represent the density of prey and predator population, respectively. Functions α and β are the intrinsic growth rate of prey and the density limiting rate due to the intraspecific competition, respectively. Parameters b, c, and d are the predation rate of predator on prey, the death rate of predator, and the rate at which predator increases by consuming prey, respectively. Since population can not be negative, it is natural to consider model (E) in the first quadrant $Q \overset{\text{def}}{=} \{(N, P) \colon N \geq 0 \text{ and } P \geq 0\}$.

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Through this paper. we assume that

$$\alpha(t) = a + ch(t) \quad \text{and} \quad \beta(t) = dh(t), \tag{1}$$

where a is a positive constant, and h is a nonnegative and locally integrable function on $[0, \infty)$. Then, model (E) has a unique interior equilibrium $(c/d, a/b)$. The purpose of this paper is to present sufficient conditions on the coefficient h for the interior equilibrium to be uniformly globally asymptotically stable. The definition of uniform global asymptotic stability is divided into three parts. The interior equilibrium is *uniformly globally asymptotically stable* if it is uniformly stable and uniformly globally attractive, and if the solutions are uniformly bounded. About these definitions, refer to the books [6, 7, 12].

We can find many research reports about the global asymptotic stability of the interior equilibrium of ecological models, but the concept of uniform global asymptotic stability is largely different from that of global asymptotic stability; that is, the interior equilibrium is stable and every solution tends to the interior equilibrium as time increases. It is natural that the arrival time from the current point to a neighbourhood of the interior equilibrium depends on the location of the current point. In time-varying models such as (E), the arrival time depends on also the initial time. Hence, it is impossible to estimate the arrival time from the current point to a neighbourhood of the interior equilibrium even if the location of the current point becomes clear. Since each solution represents a change in the density of two species, the uncertainty of the arrival time of the solution implies that the future density is unpredictable. In such a sense, even if it is found that the interior equilibrium is globally asymptotically stable, it can be said that this fact is not so useful for practical use.

In order for the interior equilibrium to be uniform global asymptotic stability, we have to confirm that all solutions starting from the same position approach the interior equilibrium within the same time regardless of their initial time. In other words, the initial time does not affect the asymptotic speed of solutions to the interior equilibrium. Hence, when the interior equilibrium is uniformly globally asymptotically stable, if we measure the arrival time at which one solution approaches near the interior equilibrium once, it is possible to judge the arrival time to the interior equilibrium of every solution departing from the same location.

The study of uniform global asymptotic stability is considered to have a big advantage and merit as described above, but detailed analysis is necessary accordingly. For this reason, there are few researches on uniform global asymptotic stability of the interior equilibrium for ecological models.

Our main result is as follows.

Theorem 1. *Assume* (1). *Suppose that*

$$\liminf_{t\to\infty} \int_t^{t+d} h(s)ds > 0 \quad \textit{for every } d > 0. \tag{2}$$

and there exists a $\gamma > c$ *such that*

$$\lim_{t\to\infty} \int_\sigma^{t+\sigma} \frac{\int_\sigma^s e^{\gamma H(\tau)} d\tau}{e^{\gamma H(s)}} ds = \infty \quad \textit{uniformly with respect to } \sigma \geq 0. \tag{3}$$

Then the interior equilibrium of (E) *is uniformly globally asymptotically stable.*

Remark 1. As a related research, we can cite a result presented by Zheng and Sugie [13].. In their paper, a necessary and sufficient condition is given for global asymptotic stability of the interior equilibrium of (E) under the assumption (1) and a certain weaker condition than (2). This necessary and sufficient condition is represented by condition (3) with $\gamma = 1$ and $\sigma = 0$. However, unfortunately, this condition is not one for uniform global asymptotic stability. Hence, Theorem 1 is a whole new result.

2. Transformation

Let $x = -\ln(bP/a)$ and $y = -\ln(dN/c)$. Then, model (E) becomes the system

$$\begin{aligned} x' &= c(1 - e^{-y}), \\ y' &= -a(1 - e^{-x}) - ch(t)(1 - e^{-y}) \end{aligned} \tag{4}$$

under the assumption (1). System (4) has the zero solution $(x(t), y(t)) \equiv (0, 0)$. This transformation shifts the interior equilibrium $(c/d, a/b)$ of (E) to the origin $(0, 0)$ of (4) and is a one-to-one correspondence from the first quadrant Q to the whole real plane $\mathbb{R}^2$. Hence, the interior equilibrium of (E) is uniformly globally asymptotically stable if and only if the zero solution of (4) is uniformly globally asymptotically stable. This means that Theorem 1 can be represented by the following result.

Proposition 2. *Suppose that* (2) *holds and there exists a* $\gamma > c$ *such that condition* (3) *holds. Then the zero solution of* (4) *is uniformly globally asymptotically stable.*

Let $t_0 \geq 0$ and $\mathbf{x}_0 = (x(t_0), y(t_0)) \in \mathbb{R}^2$. We denote the solution of (4) passing through a point $\mathbf{x}_0$ at the initial time t_0 by $\mathbf{x}(t; t_0, \mathbf{x}_0)$. Let $\|\cdot\|$ be the Euclidean norm. We will prove Proposition 2 with the following procedure.

(a) We first show that the zero solution of (4) is uniformly stable; namely, for any $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that $t_0 \geq 0$ and $\|\mathbf{x}_0\| < \delta$ imply $\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| < \varepsilon$ for all $t \geq t_0$.

(b) We next show that the zero solution of (4) is uniformly globally attractive, namely, for any $r \geq 0$ and any $\eta > 0$, there is a $T(r, \eta) > 0$ such that $t_0 \geq 0$ and $\|\mathbf{x}_0\| < r$ imply $\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| < \eta$ for all $t \geq t_0 + T$. To this end, we determine $T(r, \eta)$ for every $r > 0$ and $\eta > 0$. Using this T, we verify that $\|\mathbf{x}(t^*; t_0, \mathbf{x}_0)\| < \delta(\eta)$ for some $t^* \in [t_0, t_0 + T]$. This part is the core of the proof of Proposition 2.

(c) We finally show that the solutions of (4) are uniformly bounded, namely, for any $r > 0$, there exists a $B(r) > 0$ such that $t_0 \geq 0$ and $\|\mathbf{x}_0\| < r$ imply $\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| < B$ for all $t \geq t_0$.

The important thing in the above procedure is to find δ, T and B which are independent of the initial time t_0.

3. Proof of Proposition 2

To prove Proposition 2. it is convenient to define the function $f(z) = z - 1 + e^{-z}$ and its derivative $g(z) = 1 - e^{-z}$ for $z \in \mathbb{R}$. It is easy to check that

$$g^2(z) \geq g^2(\alpha) \quad \text{for } |z| \geq \alpha, \tag{5}$$

$$|g(z)| < |g(-\alpha)| \quad \text{for } |z| < \alpha, \tag{6}$$

$$f(z_2) - f(z_1) < f(-z_2) - f(-z_1) \quad \text{for } z_1 < z_2, \tag{7}$$

$$0 \leq f(z) \leq f(-\alpha) \quad \text{for } |z| \leq \alpha, \tag{8}$$

where α is any positive number. Let

$$w = \hat{f}(z) \overset{\text{def}}{=} f(z)\text{sgnz},$$

and $\hat{f}^{-1}(w)$ be the inverse function of $\hat{f}(z)$. Then we see that $\hat{f}^{-1}(w)$ is strictly increasing for $w \in \mathbb{R}$ and $\hat{f}^{-1}(0) = 0$, and

$$0 < -\hat{f}^{-1}(-w) < \hat{f}^{-1}(w) \quad \text{for } w > 0 \tag{9}$$

holds. For the details, refer to [13]. Now, we are ready to prove Proposition 2.

Proof of Proposition 2. Part (a): Let $m = \min\{a, c\}$ and $M = \max\{a, c\}$. For any $\varepsilon > 0$ sufficiently small, we choose

$$\delta(\varepsilon) = \sqrt{2}\hat{f}^{-1}\left(\frac{m}{M} f\left(\frac{\varepsilon}{\sqrt{2}}\right)\right).$$

From the definitions of m and M, we see that

$$\delta \leq \sqrt{2}\hat{f}^{-1}\left(f\left(\frac{\varepsilon}{\sqrt{2}}\right)\right) = \varepsilon.$$

Let $t_0 \geq 0$ and $\mathbf{x}_0 \in \mathbb{R}^2$ be given. We will show that $\|\mathbf{x}_0\| < \delta$ implies $\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| < \varepsilon$ for $t \geq t_0$. For convenience, we write $\mathbf{x}(t; t_0, \mathbf{x}_0) = (x(t), y(t))$ and define

$$v(t) = a\,f(x(t)) + c\,f(y(t)) \quad \text{for } t \geq t_0.$$

Then it is clear that $v'(t) = -c^2 h(t) g^2(y(t)) \leq 0$ for $t \geq t_0$. Hence, $v(t)$ is decreasing for $t \geq t_0$.

Let $w(x,y) = f(x) + f(y)$ on $\mathbb{R}^2$. To use the method of Lagrange multiplier, we take $L(x,y,\lambda) = f(x) + f(y) + \lambda(x^2 + y^2 - r^2)$ as the Lagrange function, where r is any positive number. Then, we can estimate that

$$2f\left(\frac{r}{\sqrt{2}}\right) \le w(x,y) \le 2f\left(-\frac{r}{\sqrt{2}}\right)$$

on the circle $\{(x,y)\colon x^2 + y^2 = r^2\}$. Hence, we have

$$2mf\left(\frac{\|\mathbf{x}(t;t_0,\mathbf{x}_0)\|}{\sqrt{2}}\right) \le mw(x(t),y(t)) \le v(t) \le Mw(x(t),y(t)) \le 2Mf\left(-\frac{\|\mathbf{x}(t;t_0,\mathbf{x}_0)\|}{\sqrt{2}}\right). \tag{10}$$

It follows from (10) and the property of the inverse function $\hat{f}^{-1}$ that

$$\|\mathbf{x}(t;t_0,\mathbf{x}_0)\| \le \sqrt{2}\hat{f}^{-1}\!\left(\frac{v(t)}{2m}\right) \le \sqrt{2}\hat{f}^{-1}\!\left(\frac{v(t_0)}{2m}\right) \le \sqrt{2}\hat{f}^{-1}\!\left(\frac{M}{m}f\left(-\frac{\|\mathbf{x}_0\|}{\sqrt{2}}\right)\right) \le \|\mathbf{x}_0\| < \delta \le \varepsilon \tag{11}$$

for $t \ge t_0$, namely, the zero solution of (4) is uniformly stable.

Part (b): By using condition (2) with $d = 1$, we can find an a $\ell > 0$ and a $\hat{t} > 0$ such that

$$\int_t^{t+1} h(s)ds \ge \ell \quad \text{for} \;\; t \ge \hat{t}. \tag{12}$$

For any $\eta > 0$, we choose

$$\mu(\eta) = \min\left\{-\hat{f}^{-1}\!\left(-\frac{m^2}{cM}f\left(\frac{\eta}{\sqrt{2}}\right)\right),\; ag\left(\hat{f}^{-1}\!\left(\frac{m^2}{aM}f\left(\frac{\eta}{\sqrt{2}}\right)\right)\right),\; -ag\left(\hat{f}^{-1}\!\left(-\frac{m^2}{aM}f\left(\frac{\eta}{\sqrt{2}}\right)\right)\right)\right\}.$$

Note that $\mu > 0$. Using this number μ, we define

$$\tau_1(r,\eta) = \hat{t} + \left[\frac{2Mf\left(-r/\sqrt{2}\right)}{\ell c^2 g^2(\mu/2)}\right] + 1$$

for any $r > 0$, where $[\cdot]$ means the greatest integer which is not great than a real number $(\cdot)$. From condition (3), we can find a $\tau_2(r,\eta) > 0$ such that

$$\int_\sigma^{t+\sigma}\frac{\int_\sigma^s e^{\gamma H(\tau)}d\tau}{e^{\gamma H(s)}}ds \ge \max\left\{\frac{\hat{f}^{-1}\!\left(2Mf\left(-r/\sqrt{2}\right)\big/a\right)}{acg\left(\hat{f}^{-1}\!\left(m^2f\left(\eta/\sqrt{2}\right)\big/(aM)\right)\right)},\; \frac{\mu\hat{f}^{-1}\!\left(-2Mf\left(-r/\sqrt{2}\right)\big/a\right)}{acg(\mu)g\left(\hat{f}^{-1}\!\left(-m^2f\left(\eta/\sqrt{2}\right)\big/(aM)\right)\right)}\right\} \tag{13}$$

for $t \ge \tau_2 - 1$. Without loss of generality, we assume that $\tau_2 > 1$. Let

$$\omega(r,\eta) = \frac{f(\mu) - f(\mu/2)}{ag\left(-\hat{f}^{-1}\!\left(2Mf\left(-r/\sqrt{2}\right)\big/a\right)\right)g(-\mu)}.$$

Taking into account of the properties of the functions f, g and $\hat{f}^{-1}$ and the fact that $\mu > 0$, we see that the number ω is also positive. Let

$$\rho = \liminf_{t\to\infty}\frac{c^2g^2(\mu/2)}{4Mf\left(-r/\sqrt{2}\right)}\int_t^{t+\omega}h(s)ds.$$

The number ρ depends only on r and η. From condition (2) it turns out that ρ is positive. Hence, we can find a $\tau_3(r,\eta) > 0$ such that

$$\int_t^{t+\omega}h(s)ds \ge \frac{2\rho Mf\left(-r/\sqrt{2}\right)}{c^2g^2\left(\mu/2\right)} \quad \text{for} \;\; t \ge \tau_3. \tag{14}$$

Using numbers τ_1, τ_2 and τ_3 depending only on r and η, we define

$$T(r,\eta) = \tau_3 + \left(\left[\frac{1}{\rho}\right] + 1\right)(\tau_1 + \tau_2).$$

To prove that the zero solution of (4) is uniformly globally attractive, we have only to show that if $\|\mathbf{x}_0\| < r$, then there exists a $t^* \in [t_0, t_0 + T(r, \eta)]$ such that

$$\|\mathbf{x}(t^*; t_0, \mathbf{x}_0)\| < \delta(\eta). \tag{15}$$

In fact, by letting $\mathbf{x}(t^*; t_0, \mathbf{x}_0)$ to $\mathbf{x}^*$, the conclusion of part (a) and the inequality (15) lead that $\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| = \|\mathbf{x}(t; t^*, \mathbf{x}^*)\| < \eta \quad \text{for} \quad t \geq t^*$.

Using the method of proof by contradiction, we will show that (15) holds. Suppose that $\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| \geq \delta(\eta)$ for $t \in [t_0, t_0 + T]$. Then it follows from (10) that

$$\frac{2m^2}{M} f\left(\frac{\eta}{\sqrt{2}}\right) = 2mf\left(\frac{\delta(\eta)}{\sqrt{2}}\right) \leq 2mf\left(\frac{\|\mathbf{x}(t; t_0, \mathbf{x}_0)\|}{\sqrt{2}}\right) \leq v(t) \leq v(t_0) \leq 2Mf\left(-\frac{\|\mathbf{x}_0\|}{\sqrt{2}}\right) < 2Mf\left(-\frac{r}{\sqrt{2}}\right) \tag{16}$$

for $t \in [t_0, t_0 + T]$. After this, we will divide our argument into three steps and examine the behaviour of $y(t)$ which is the second component of the solution $\mathbf{x}(t; t_0, \mathbf{x}_0)$.

Step 1. If $|y(t)| \geq \mu/2$ for $t \in [\alpha, \beta] \subset [t_0, t_0 + T]$, then $\beta - \alpha < \tau_1$. In fact, suppose that there exists an interval $[\alpha_1, \beta_1] \subset [t_0, t_0 + T]$ with $\beta_1 - \alpha_1 \geq \tau_1$ such that $|y(t)| \geq \mu/2$ for $t \in [\alpha_1, \beta_1]$. Since $v'(t) = -c^2 h(t) g^2(y(t)) \leq 0$ for $t \geq t_0$, by (5) and (16) we have

$$c^2 g^2\left(\frac{\mu}{2}\right)\int_{\alpha_1}^{\beta_1} h(t)dt \leq c^2 \int_{\alpha_1}^{\beta_1} h(t) g^2(y(t))dt = -\int_{\alpha_1}^{\beta_1} v'(t)dt = v(\alpha_1) - v(\beta_1) < 2Mf\left(-\frac{r}{\sqrt{2}}\right). \tag{17}$$

On the other hand, from (12) and the definition of τ_1, we see that

$$\int_{\alpha_1}^{\beta_1} h(t)dt \geq \int_{\alpha_1+\hat{t}}^{\alpha_1+\tau_1} h(t)dt = \sum_{i=0}^{\left[\frac{2Mf(-r/\sqrt{2})}{\ell c^2 g^2(\mu/2)}\right]} \int_{\alpha_1+\hat{t}+i}^{\alpha_1+\hat{t}+i+1} h(t)dt \geq \left(\left[\frac{2Mf\left(-r/\sqrt{2}\right)}{\ell c^2 g^2(\mu/2)}\right] + 1\right)\ell \geq \frac{2M}{c^2} f\left(-\frac{r}{\sqrt{2}}\right) g^{-2}\left(\frac{\mu}{2}\right).$$

This contradicts (17). Thus, the assertion at the beginning of this step is correct.

Step 2. If $|y(t)| \leq \mu$ for $t \in [\alpha, \beta] \subset [t_0, t_0 + T]$, then $\beta - \alpha < \tau_2$. We will show this assertion. Since $\mu \leq -\hat{f}^{-1}\left(-m^2 f\left(\eta/\sqrt{2}\right)/(cM)\right)$, by (8) and (16) we have

$$af(x(t)) = v(t) - cf(y(t)) \geq \frac{2m^2}{M} f\left(\frac{\eta}{\sqrt{2}}\right) - cf(-\mu) \geq \frac{2m^2}{M} f\left(\frac{\eta}{\sqrt{2}}\right) - \frac{m^2}{M} f\left(\frac{\eta}{\sqrt{2}}\right) = \frac{m^2}{M} f\left(\frac{\eta}{\sqrt{2}}\right)$$

for $t \in [\alpha, \beta]$. Hence, there are two possibilities: $x(t) \geq \hat{f}^{-1}\left(m^2 f\left(\eta/\sqrt{2}\right)/(aM)\right)$ for $t \in [\alpha, \beta]$; $x(t) \leq \hat{f}^{-1}\left(-m^2 f\left(\eta/\sqrt{2}\right)/(aM)\right)$ for $t \in [\alpha, \beta]$. We consider only the former, because the proof of the latter is carried out in the same way as that of the former. Let

$$k_1 = \frac{1 - e^{-\mu}}{\mu} \quad \text{and} \quad k_2 = \min\left\{\frac{\gamma}{c}, \ \frac{e^{\mu} - 1}{\mu}\right\}.$$

Note that $0 < k_1 < 1 < k_2$. From the form of model (4), we see that if a solution curve intersects the positive x-axis, then it passes through the x-axis only once vertically from top to bottom. Hence, only the following three cases could happen: (i) there exists a $\tilde{t} \in (\alpha, \beta)$ such that $y(\tilde{t}) = 0$, $0 < y(t) \leq \mu$ for $t \in [\alpha, \tilde{t})$ and $-\mu \leq y(t) < 0$ for $t \in (\tilde{t}, \beta]$; (ii) $0 < y(t) \leq \mu$ for $t \in [\alpha, \beta]$; (iii) $-\mu \leq y(t) < 0$ for $t \in [\alpha, \beta]$.

Case (i). Since $\left(y(t)e^{ck_1 H(t)}\right)' = (y'(t) + ck_1 h(t) y(t))e^{ck_1 H(t)} \leq (y'(t) + ch(t)g(y(t)))e^{ck_1 H(t)} = -ag(x(t))e^{ck_1 H(t)} \leq -ag\left(\hat{f}^{-1}\left(m^2 f\left(\eta/\sqrt{2}\right)/(aM)\right)\right)e^{ck_1 H(t)}$ for $t \in [\alpha, \tilde{t}]$, we obtain

$$0 = y(\tilde{t}) \leq -ag\left(\hat{f}^{-1}\left(\frac{m^2}{aM} f\left(\frac{\eta}{\sqrt{2}}\right)\right)\right)\frac{\int_{\alpha}^{\tilde{t}} e^{ck_1 H(s)}ds}{e^{ck_1 H(\tilde{t})}} + y(\alpha)\frac{e^{ck_1 H(\alpha)}}{e^{ck_1 H(\tilde{t})}}.$$

Since H is monotonic increasing and $y(\alpha) \leq \mu \leq ag\left(\hat{f}^{-1}\left(m^2 f\left(\eta/\sqrt{2}\right)/(aM)\right)\right)$, we see that

$$0 \leq -ag\left(\hat{f}^{-1}\left(\frac{m^2}{aM} f\left(\frac{\eta}{\sqrt{2}}\right)\right)\right)(\tilde{t} - \alpha)\frac{e^{ck_1 H(\alpha)}}{e^{ck_1 H(\tilde{t})}} + \mu\frac{e^{ck_1 H(\alpha)}}{e^{ck_1 H(\tilde{t})}} \leq -a(\tilde{t} - \alpha - 1)g\left(\hat{f}^{-1}\left(\frac{m^2}{aM} f\left(\frac{\eta}{\sqrt{2}}\right)\right)\right)\frac{e^{ck_1 H(\alpha)}}{e^{ck_1 H(\tilde{t})}}.$$

Hence, it turns out that $\tilde{t} - \alpha < 1$. Similarly, we obtain

$$y(t) \leq -ag\left(\hat{f}^{-1}\left(\frac{m^2}{aM} f\left(\frac{\eta}{\sqrt{2}}\right)\right)\right) \frac{\int_{\tilde{t}}^{t} e^{ck_2 H(s)} ds}{e^{ck_2 H(t)}} \quad \text{for} \quad t \in [\tilde{t}, \beta].$$

From (16) it follows that $x(\tilde{t}) < \hat{f}^{-1}\left(2Mf(-r/\sqrt{2})/a\right)$. Suppose that $\beta - \tilde{t} \geq \tau_2 - 1$. Then we see that $x(\tilde{t} + \tau_2 - 1) \geq \hat{f}^{-1}\left(m^2 f(\eta/\sqrt{2})/(aM)\right) > 0$. Hence, we obtain

$$\begin{aligned} -\hat{f}^{-1}\left(\frac{2M}{a} f\left(-\frac{r}{\sqrt{2}}\right)\right) < -x(\tilde{t}) < x(\tilde{t} + \tau_2 - 1) - x(\tilde{t}) = \int_{\tilde{t}}^{\tilde{t}+\tau_2-1} x'(t)dt = \int_{\tilde{t}}^{\tilde{t}+\tau_2-1} c(1 - e^{-y(t)})dt \\ \leq \int_{\tilde{t}}^{\tilde{t}+\tau_2-1} cy(t)dt \leq -acg\left(\hat{f}^{-1}\left(\frac{m^2}{aM} f\left(\frac{\eta}{\sqrt{2}}\right)\right)\right) \int_{\tilde{t}}^{\tilde{t}+\tau_2-1} \frac{\int_{\tilde{t}}^{t} e^{ck_2 H(s)} ds}{e^{ck_2 H(t)}} dt. \end{aligned}$$

Since $ck_2 \leq \gamma$, it follows that

$$\int_{\tilde{t}}^{\tilde{t}+\tau_2-1} \frac{\int_{\tilde{t}}^{t} e^{ck_2 H(s)} ds}{e^{ck_2 H(t)}} dt \geq \int_{\tilde{t}}^{\tilde{t}+\tau_2-1} \frac{\int_{\tilde{t}}^{t} e^{\gamma H(s)} ds}{e^{\gamma H(t)}} dt.$$

Hence, using (13) at $\sigma = \tilde{t}$ and $t = \tau_2 - 1$, we can lead that

$$-\hat{f}^{-1}\left(\frac{2M}{a} f\left(-\frac{r}{\sqrt{2}}\right)\right) < -acg\left(\hat{f}^{-1}\left(\frac{m^2}{aM} f\left(\frac{\eta}{\sqrt{2}}\right)\right)\right) \frac{\hat{f}^{-1}\left(2Mf\left(-r/\sqrt{2}\right)/a\right)}{acg\left(\hat{f}^{-1}\left(m^2 f\left(\eta/\sqrt{2}\right)/(aM)\right)\right)} = -\hat{f}^{-1}\left(\frac{2M}{a} f\left(-\frac{r}{\sqrt{2}}\right)\right),$$

which is a contradiction. Hence, it turns out that $\beta - \tilde{t} < \tau_2 - 1$. We therefore conclude that $\beta - \alpha = \tilde{t} - \alpha + \beta - \tilde{t} < 1 + \tau_2 - 1 = \tau_2$ in this case.

Case (ii). Repeating the same argument as Case (i), we see that $\beta - \alpha < 1 < \tau_2$.

Case (iii). Repeating the same argument as Case (i), we see that $\beta - \alpha < \tau_2 - 1 < \tau_2$.

Step 3. We divide the interval $[t_0 + \tau_3, t_0 + T]$ as follows: $[t_0 + \tau_3, t_0 + T] = J_1 \cup J_2 \cup \cdots \cup J_{[1/\rho]+1}$, where $J_i = [t_0 + \tau_3 + (i-1)(\tau_1 + \tau_2), t_0 + \tau_3 + i(\tau_1 + \tau_2)]$ for $1 \leq i \leq [1/\rho] + 1$. To examine the behaviour of $|y(t)|$ in detail, we first subdivide J_1 into two intervals $[t_0 + \tau_3, t_0 + \tau_1 + \tau_3]$ and $[t_0 + \tau_1 + \tau_3, t_0 + \tau_1 + \tau_2 + \tau_3]$. Since the widths of the two intervals are τ_1 and τ_2, respectively, it turns out that from the assertions of Step 1 and Step 2 that there exist a $\underline{t} \in [t_0 + \tau_3, t_0 + \tau_1 + \tau_3]$ and a $\bar{t} \in [t_0 + \tau_1 + \tau_3, t_0 + \tau_1 + \tau_2 + \tau_3]$ such that $|y(\underline{t})| < \mu/2$ and $|y(\bar{t})| > \mu$. From the continuity of $|y(t)|$, we can find numbers t_1 and t_2 with $\underline{t} \leq t_1 < t_2 \leq \bar{t}$ such that $|y(t_1)| = \mu/2$, $|y(t_2)| = \mu$ and

$$\frac{\mu}{2} < |y(t)| < \mu \quad \text{for} \quad t \in (t_1, t_2). \tag{18}$$

Note that $y(t_1)$ and $y(t_2)$ have the same sign. From (16), we see that $0 < x(t) < \hat{f}^{-1}\left(2Mf(-r/\sqrt{2})/a\right)$ for $t \in [t_0, t_0 + T]$. Hence, it follows from (6) that $|g(y(t))| < |g(-\mu)|$ for $t \in (t_1, t_2)$ and $|g(x(t))| < \left|g\left(-\hat{f}^{-1}\left(2Mf(-r/\sqrt{2})/a\right)\right)\right|$ for $t \in [t_0, t_0 + T]$. Using these inequalities and (7), we obtain

$$\begin{aligned} f(\mu) - f(\mu/2) = f(|y(t_2)|) - f(|y(t_1)|) \leq f(y(t_2)) - f(y(t_1)) = \int_{t_1}^{t_2} f'(y(t))dt \\ = \int_{t_1}^{t_2} \left(-ag(x(t))g(y(t)) - ch(t)g^2(y(t))\right) dt \leq a \int_{t_1}^{t_2} |g(x(t))||g(y(t))|dt \\ < ag\left(-\hat{f}^{-1}\left(\frac{2M}{a} f\left(-\frac{r}{\sqrt{2}}\right)\right)\right) g(-\mu)(t_2 - t_1), \end{aligned}$$

namely, $\omega < t_2 - t_1$. Hence, together with (5), (14) and (18), we get

$$v(t_2) - v(t_1) = \int_{t_1}^{t_2} v'(t)dt = -\int_{t_1}^{t_2} c^2 h(t) g^2(y(t))dt \leq -c^2 g^2\left(\frac{\mu}{2}\right) \int_{t_1}^{t_1+\omega} h(t)dt \leq -2\rho M f\left(-\frac{r}{\sqrt{2}}\right).$$

Recall that $v'(t) = -c^2 h(t) g^2(y(t)) \leq 0$ for $t \geq t_0$. Then it is clear that

$$v(t_1) - v(t_0 + \tau_3) \leq 0 \quad \text{and} \quad v(t_0 + \tau_1 + \tau_2 + \tau_3) - v(t_2) \leq 0.$$

Combining these estimation of v, we obtain

$$\int_{J_1} v'(t)dt = v(t_0 + \tau_1 + \tau_2 + \tau_3) - v(t_2) + v(t_2) - v(t_1) + v(t_1) - v(t_0 + \tau_3) \leq -2\rho M f\left(-\frac{r}{\sqrt{2}}\right).$$

Repeating the same process, we can conclude that $\int_{J_i} v'(t)dt \leq -2\rho M f\left(-r/\sqrt{2}\right)$ for $i = 2, 3, \cdots, [1/\nu] + 1$. Hence, the loss of the total energy v in each interval J_i is at least $2\rho M f\left(-r/\sqrt{2}\right)$, and therefore,

$$v(t_0 + T) = v(t_0 + \tau_3) + \sum_{i=1}^{[1/\rho]+1} \int_{J_i} v'(t)dt < v(t_0 + \tau_3) - 2\rho\left(\left[\frac{1}{\rho}\right] + 1\right) M f\left(-r/\sqrt{2}\right) \leq v(t_0 + \tau_3) - 2M f\left(-r/\sqrt{2}\right).$$

However, it follows from (16) that $0 < v(t_0 + T) < v(t_0 + \tau_3) - 2M f\left(-r/\sqrt{2}\right) < 0$. This is a contradiction. Thus, the inequality (15) holds.

Part (c): For any $r > 0$, let

$$B(r) = \sqrt{2}\hat{f}^{-1}\left(\frac{M}{m} f\left(-\frac{r}{\sqrt{2}}\right)\right).$$

Note that f is strictly decreasing on $(-\infty, 0)$ and $\hat{f}^{-1}$ in strictly increasing on $\mathbb{R}$. From (11) it follows that if $\|\mathbf{x}_0\| < r$, then

$$\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| \leq \sqrt{2}\hat{f}^{-1}\left(\frac{M}{m} f\left(-\frac{\|\mathbf{x}_0\|}{\sqrt{2}}\right)\right) < \sqrt{2}\hat{f}^{-1}\left(\frac{M}{m} f\left(-\frac{r}{\sqrt{2}}\right)\right) = B$$

for $t \geq t_0$. Hence, the zero solution of (4) is uniformly bounded.

In conclusion, the solutions of (4) are uniformly globally asymptotically stable. □

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