Global asymptotic stability and equiasymptotic stability for a time-varying phytoplankton-zooplankton-fish system

Wei Zheng[a], Jitsuro Sugie*[,a]

[a]*Department of Mathematics, Shimane University, Matsue 690-8504, Japan*

Abstract

In this paper, we consider a three-species system which consist of phytoplankton, zooplankton and fish. It is well known that zooplankton are harvested by other creatures besides fish, and it has been reported that zooplankton can absorb energy from other microorganisms besides phytoplankton in recent decades. For this reason, we add the parts of extra energy source and harvesting effect in the model. Taking account of seasonal variation which can bring affects to many aspects of plankton population, we assume that some factors about the phytoplankton and the zooplankton are time-dependent. In particular, since the environment does not change regularly, we does not require these time-varied factors are periodic, which is quite different with most ecological models. The purpose of this paper is to present sufficient conditions which guarantee that the equilibrium of this three-species system is globally asymptotically stable. Moreover, we will show that the equilibrium is equiasymptotically stable under relatively weaker conditions.

Key words: Phytoplankton-zooplankton-fish system; Global asymptotic stability; Equiasymptotically stable; Weakly integrally positive
2010 MSC: 34D05; 34D23; 92D25

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1. Introduction

The dynamic analysis of population growth through mathematical modeling is one of the main topics of mathematical biology, and it contributes greatly to the understanding of the world of biology. Since the 1990s, research on phytoplankton and zooplankton has attracted the interest of many researchers (e.g., see [2, 9, 10, 14, 16, 20]).

Phytoplankton, usually called primary producers, are autotrophic prokaryotic or eukaryotic algae that are primarily found in surface waters where there is sufficient light to support photosynthesis. The decrease of phytoplankton is mainly caused by zooplankton predation. In other words, phytoplankton and zooplankton form a grazing food chain.

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*Corresponding author

Email addresses: `vividloky@yahoo.co.jp` (Wei Zheng),
`jsugie@riko.shimane-u.ac.jp` (Jitsuro Sugie)

Preprint submitted to Nonlinear Analysis: Real World Applications *September 7, 2018*

Zooplankton include small protozoans and metazoans that mainly food on phytoplankton. Some of the eggs and freshly hatched larvae of larger nektonic organisms such as fish, crustaceans, and annelids, are also thought to be in the same position as zooplankton for a few days before they can swim against the water current. Zooplankton ingest nutritions also from bacterial plankton and other microorganisms in addition to predation of phytoplankton. Zooplankton and those microorganisms form a microbial food chain (refer to [1, 12]). Zooplankton store a large amount of organic material as the primary consumers, and by being eaten by higher nutritional values such as fish, bivalves, and jellyfish, zooplankton carry energy to them. Some kind of zooplankton are harvested by humans, such as neomysis awatschensis, krill and whitebait (e.g., fry of sardines and herring).

As the bottom of the marine food chain, the amount of phytoplankton and zooplankton may have significant impact on other aquatic living and water quality. For this reason, the theoretical and experimental researches about phytoplankton-zooplankton systems have become important themes for the marine science and environmental protection. It is pointed out in [5, 7, 13] that the interaction between phytoplankton and zooplankton is closely related to the supply of nutrients, such as nitrate, phosphate and dissolved silicic acid. It has also been reported that phytoplankton and zooplankton population are extensively exploited by humans in the real world. In [6, 8], paying attention to this fact, the harvesting effect was added to a phytoplankton-zooplankton system which is the classical Rosenzweig-MacArthur model, and an appropriate harvesting strategy was proposed. Moreover, since the presence of fish that ingest plankton may have a significant influence on the plankton community, many studies have focused on the predation effect of fish in the food web and considered some tritrophic food chain models of plankton-fish interaction (for example, see [3, 15]).

Focusing on the above-mentioned interactions among phytoplankton, zooplankton, fish, we propose a multiple species system consisting of lowest level preys, middle predators, and high level predators. In this paper, we also consider the fact that zooplankton ingest nutrients from the microbial food chain. Since the seasonal factors such as temperature and sunshine duration have influence on the activity of phytoplankton and zooplankton, we assume that the coefficients for phytoplankton and zooplankton are time-dependent. However, due to the indeterminacy of the environment, these factors do not always change periodically. We do not use any periodic property of these factors in this paper.

We consider the model

$$\begin{cases} P' = \gamma(t)P - \dfrac{h(t)}{K}P^2 - g(t)PZ, \\ Z' = aZ + f(t)PZ - bZF - j(t)Z, \\ F' = -cF + dZF \end{cases}$$

where the prime denotes d/dt; the letters P, Z and F mean the population densities of phytoplankton, zooplankton and fish, respectively. To be exact, the densities at time t

are represented by $P(t)$, $Z(t)$ and $F(t)$, which are usually measured in milligrams of dry weight per millilitre. We assume that the unit of time t is day and the unit of densities P, Z and F is mg·ml^{-1}.

Here, we explain the parameters a, b, c, d and K, and the functions f, g, h, γ and j. All parameters are positive constants. Parameter a represents the growth rate of zooplankton which is supported by the nutrient from the microbial food chain. Parameter b is the rate of zooplankton eaten by fish. Parameter c is the natural mortality rate of fish. Parameter d is the rate at which fish increase by preying zooplankton. We may consider that $d < b$. Parameter K represents the carrying capacity of phytoplankton. The unit of a and c is day^{-1}, the unit of b and d is ml·mg^{-1}·day^{-1} and the unit of K is mg·ml^{-1}. All functions f, g, h, γ and j are defined on $[0, \infty)$ and nonnegative. The function f is the growth rate of zooplankton by preying phytoplankton, the function g is the decay rate of phytoplankton due to be ingested by zooplankton and the function h is the density limiting rate for the carrying capacity K due to the intraspecific competition of phytoplankton. The function γ is the intrinsic growth rate of phytoplankton population. The function j is the mortality rate of zooplankton due to the harvest effect by human beings and the ingesting by other marine organisms except fish. The unit of f and g is ml·mg^{-1}·day^{-1} and the unit of h, γ and j is day^{-1}. Hence, the unit of both sides of each equation of the system above is ml·mg^{-1}·day^{-1}.

We have only to consider the system above in the octant

$$O = \{(P, Z, F) \in \mathbb{R}^3 : P > 0,\ Z > 0,\ F > 0\},$$

because it is a biological model. The intrinsic growth rate γ is often assumed to be the same as the density limiting rate h, such as in the logistic equation which predicts the variation of the population of the organism (population size) when a certain single kind of organism grows in a constant environment. However, in this paper, in addition to the relationship between γ and h, we think that the intrinsic growth rate will increase when zooplankton become active and take much phytoplankton, conversely, the intrinsic growth rate will decrease when zooplankton become inactive and do not prey on phytoplankton. Hence, we suppose that $\gamma(t) = h(t) + \frac{c}{d} g(t)$. Also, we think that the mortality rate of zooplankton due to the harvest effect will increases when zooplankton become active and eat more phytoplankton, conversely, the mortality rate due to the harvest effect will decreases when zooplankton become inactive and do not take much phytoplankton. Hence, we suppose that $j(t) = K f(t)$. As a result, our model becomes

$$\begin{cases} P' = \left(h(t) + \dfrac{c}{d} g(t)\right) P - \dfrac{h(t)}{K} P^2 - g(t) P Z, \\ Z' = aZ + f(t) P Z - bZF - K f(t) Z, \\ F' = -cF + dZF. \end{cases} \tag{E}$$

It is easy to check that (E) has a unique interior equilibrium $(K, c/d, a/b)$.

To describe our results, we need some notations. We say that a continuous and nonnegative function p belongs to $\mathcal{F}_{[\mathrm{WIP}]}$ if

$$\sum_{n=1}^{\infty}\int_{\tau_n}^{\sigma_n} p(t)dt = \infty$$

for every pair of sequences $\{\tau_n\}$ and $\{\sigma_n\}$ satisfying $\tau_n < \sigma_n < \tau_{n+1}$,

$$\liminf_{n\to\infty}(\sigma_n - \tau_n) > 0 \quad \text{and} \quad \limsup_{n\to\infty}(\tau_{n+1} - \sigma_n) < \infty.$$

Hatvani [4] had first named that the functions belonging to $\mathcal{F}_{[\mathrm{WIP}]}$ are weakly integrally positive. From this definition, any function with a positive lower bound belongs to the family of functions $\mathcal{F}_{[\mathrm{WIP}]}$. Moreover, it is possible that p belongs to $\mathcal{F}_{[\mathrm{WIP}]}$ even if p does not have a positive lower bound. For example, the functions $1/(1+t)$ and $\sin^2 t/(1+t)$ approach zero as t tends to ∞, but both functions belong to $\mathcal{F}_{[\mathrm{WIP}]}$ (for the proof, see [19, Proposition 2.1]). For any function $q\colon [0,\infty) \to \mathbb{R}$, let

$$q_-(t) = \max\{0, -q(t)\} \quad \text{and} \quad q_+(t) = \max\{0, q(t)\}.$$

In biology, the ratio f/g means the conversion rate from biomass of phytoplankton into biomass of zooplankton. Judging from the biological significance of the functions f, g and h, it is natural to assume that g is not less than f and these functions are bounded, namely, there exist positive constants $\underline{f}$, $\overline{g}$ and $\overline{h}$ such that

$$\underline{f} \le f(t) \le g(t) \le \overline{g} \quad \text{and} \quad 0 \le h(t) \le \overline{h} \tag{1.1}$$

for all $t \ge 0$. Throughout this paper, we assume that the ratio f/g is differentiable on $[0,\infty)$. Define the function $\ell\colon [0,\infty) \to \mathbb{R}$ by

$$\ell(t) = \left(\ln\frac{f(t)}{g(t)}\right)'.$$

From (1.1), we can choose positive constants k_1 and k_2 so that

$$k_1 \le \frac{f(t)}{g(t)} \le k_2 \quad \text{for } t \ge 0. \tag{1.2}$$

Note that $k_1 \ge \underline{f}/\overline{g}$ and $k_2 \le 1$. Define the function $\psi_\rho\colon [0,\infty) \to \mathbb{R}$ by

$$\psi_\rho(t) = \ell(t) - \frac{(1-e^{-\rho})^2}{\rho - 1 + e^{-\rho}}h(t)$$

for any $\rho > 0$. If

$$\int_0^\infty (\psi_\rho)_+(t)dt < \infty \tag{1.3}$$

holds for some $\rho > 0$, we can find an $L \geq 0$ so that

$$0 \leq \int_0^\infty (\psi_\rho)_+(t)dt \leq L.$$

Define the function $\Phi\colon \mathbb{R} \to [0,\infty)$ by

$$\Phi(w) = w - 1 + e^{-w}.$$

For any $\rho > 0$, let ρ_0 be a positive number satisfying

$$\Phi(-\rho_0) = \frac{m}{M} e^{-L}\Phi(\rho),$$

where

$$m = \min\left\{\frac{a}{dK}, \frac{c}{dK}, k_1\right\} \quad \text{and} \quad M = \max\left\{\frac{a}{dK}, \frac{c}{dK}, k_2\right\}.$$

Judging from the increase and decrease of Φ, we see that if ρ tends to ∞, then ρ_0 diverges to ∞.

We are now ready to state our main theorem. The following result gives conditions for the interior equilibrium of (E) to be uniformly stable and equiasymptotically stable.

Theorem 1.1. *Let condition* (1.1) *holds and suppose that there is a* $\rho > 0$ *with* (1.3) *and*

$$(\psi_\rho)_- \in \mathcal{F}_{[\mathrm{WIP}]}. \tag{1.4}$$

Then the interior equilibrium of (E) *is uniformly stable and equiasymptotically stable in the space*

$$S_{\rho_0} = \left\{(P,Z,F) \in \mathbb{R}^3 \colon \left(\ln\frac{P}{K}\right)^2 + \left(\ln\frac{dZ}{c}\right)^2 + \left(\ln\frac{bF}{a}\right)^2 < \rho_0^2\right\}.$$

About the definitions of uniform stability, equiasymptotic stability, and global asymptotic stability to be mentioned later, refer the books [11, 21] for example. Note that the interior equilibrium of (E) is equiasymptotically stable in the whole octant O if it is globally asymptotically stable.

In Theorem 1.1, the function ψ_ρ plays an important role. This function is composed of two functions ℓ and h. Note that even if the function ψ_ρ satisfies conditions (1.3) and (1.4), the function ℓ is not always absolutely integrable. In other words, Theorem 1.1 can be applied whether the function ℓ is absolutely integrable or not. If the function ℓ is absolutely integrable, we can obtain the following result which is more sharper.

Theorem 1.2. *Let condition* (1.1) *holds and suppose that*

$$\int_0^\infty \ell_+(t)dt < \infty \tag{1.5}$$

and

$$h \in \mathcal{F}_{[\mathrm{WIP}]}. \tag{1.6}$$

Then the interior equilibrium of (E) *is uniformly stable and globally asymptotically stable.*

We here show that Theorem 1.2 can be reduced to Theorem 1.1. Under the assumption (1.1), condition (1.5) implies that

$$\int_0^\infty \ell_-(t)dt < \infty.$$

In fact, if

$$\int_0^\infty \ell_+(t)dt < \infty \quad \text{and} \quad \int_0^\infty \ell_-(t)dt = \infty,$$

then

$$\int_0^\infty \ell(t)dt = \int_0^\infty (\ell_+(t) - \ell_-(t))dt = -\infty.$$

However, we have

$$\int_0^t \ell(s)ds = \ln\frac{f(t)}{g(t)} - \ln\frac{f(0)}{g(0)} = \ln\frac{g(0)f(t)}{f(0)g(t)}$$

for $t \geq 0$. From condition (1.1) it follows that $\ln\left((g(0)f(t))/(f(0)g(t))\right)$ does not diverge to $-\infty$ as $t \to \infty$. This is a contradiction. Hence, we have

$$\int_0^\infty |\ell(t)|dt = \int_0^\infty \ell_+(t)dt + \int_0^\infty \ell_-(t)dt < \infty.$$

This means that the function ℓ is absolutely integrable. Since $\psi_\rho(t) \leq \ell(t)$ for $t \geq 0$, we obtain

$$\int_0^\infty (\psi_\rho)_+(t)dt \leq \int_0^\infty |\ell(t)|dt < \infty,$$

namely, condition (1.3). Also, we have

$$\begin{aligned}\frac{\phi^2(\rho)}{\Phi(\rho)}h(t) - \ell_+(t) &\leq \frac{\phi^2(\rho)}{\Phi(\rho)}h(t) - \ell_+(t) + \ell_-(t) = -\psi_\rho(t)\\ &\leq (\psi_\rho)_-(t) \leq \frac{\phi^2(\rho)}{\Phi(\rho)}h(t) + \ell_-(t)\end{aligned}$$

for $t \geq 0$. We therefore conclude that condition (1.4) holds for any $\rho > 0$ if and only if condition (1.6) holds provided that conditions (1.1) and (1.5) are satisfied. It is clear that

$$\bigcup_{\rho_0>0} B_{\rho_0} = \mathbb{R}^3.$$

Thus, by the definitions of global asymptotic stability and equiasymptotic stability in the sphere B_{ρ_0}, we may consider that Theorem 1.2 is a special case of Theorem 1.1.

Hereafter, we will give only the proof of Theorem 1.1.

2. One-to-one transformation

Let

$$x = -\ln(bF/a), \quad y = -\ln(dZ/c) \quad \text{and} \quad z = -\ln(P/K).$$

Then model (E) becomes the system

$$\begin{cases} x' = c(1 - e^{-y}), \\ y' = -a(1 - e^{-x}) + Kf(t)(1 - e^{-z}), \\ z' = -\dfrac{c}{d}\,g(t)(1 - e^{-y}) - h(t)(1 - e^{-z}). \end{cases} \tag{2.1}$$

This transformation is a one-to-one correspondence from the octant O of (E) to the whole space $\mathbb{R}^3$ of (2.1). The interior equilibrium $(K, c/d, a/b)$ of (E) is transformed into the origin $(0, 0, 0)$ of (2.1) and the space S_{ρ_0} is transferred to the sphere

$$B_{\rho_0} = \left\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < \rho_0^2\right\}.$$

As can be seen from system (E), the terms which represent the transfer of energy due to predation are a bilinear form with or without time-varying. On the other hand, there is no product term of the variables x, y and z on the right side of (2.1). Although the above variable transformation may seem strange, it works to make the biological model structure easier. By using this variable transformation, it can be said that this research is clearly distinguished from others.

Thanks to this one-to-one transformation, in order to prove Theorem 1.1, it is sufficient to verify the following result.

Proposition 2.1. *Let conditions* (1.1), (1.3) *and* (1.4) *hold. Then the origin of* (2.1) *is uniformly stable and equiasymptotically stable in the sphere* B_{ρ_0}.

It would be meaningful to mention a little about the research related to Proposition 2.1 here. Linear approximation of (E) is the system

$$\begin{cases} x' = cy, \\ y' = -ax + Kf(t)z, \\ z' = -\dfrac{c}{d}\,g(t)y - h(t)z. \end{cases} \tag{2.2}$$

The second author and Ogami [17, 18] have already discussed global asymptotic stability of the origin for systems that are more general than system (2.2). We can derive the following result from the method that they have used.

Theorem A. *Let conditions* (1.1) *holds and suppose that*

$$\int_0^\infty (\psi)_+(t)\,dt < \infty \quad \text{and} \quad (\psi)_- \in \mathcal{F}_{[\mathrm{WIP}]},$$

where $\psi : [0, \infty) \to \mathbb{R}$ *is the function defined by* $\psi(t) = \ell(t) - 2h(t)$. *Then the origin of* (2.2) *is uniformly stable and globally asymptotically stable.*

Of course, we cannot obtain Propositon 2.1 from Theorem A directly. The big difference between system (2.2) and system (2.1) is that the former is linear but the latter is nonlinear. Although the mothod of linear approximation is useful for local theory, it is useless for nonlocal theory like Theorem 1.1 and Proposition 2.1. To prove Proposition 2.1, the idea in the proof of Theorem A helps, but a more detailed discussion is required than that of Theorem A.

To verify that Proposition 2.1 is correct, we will proceed with the discussion as follows. In Section 3, we examine some properties of the functions ψ_ρ and $\Phi(w)$ which are useful to prove Proposition 2.1. Section 4 is divided into two subsections. In the first subsection, we show that the origin of (2.1) is uniformly stable, namely, for any $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that $t_0 \geq 0$ and $\|\mathbf{x}_0\| < \delta$ imply $\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| < \varepsilon$ for all $t \geq t_0$, where $\mathbf{x}_0 = (x(t_0), y(t_0), z(t_0))$ and $\mathbf{x}(\cdot\,; t_0, \mathbf{x}_0)$ is the solution of (2.1) starting from the initial point $\mathbf{x}_0$ at the initial time t_0. In the second subsection, we prove that the origin of (2.1) is equiattractive in the sphere B_{ρ_0}, namely, for any $t_0 \geq 0$ and any $\eta > 0$, there is a $T(t_0, \eta) > 0$ such that $\|\mathbf{x}_0\| < \rho_0$ implies $\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| < \eta$ for all $t \geq t_0 + T$. To prove the equiattractivity, it is enough to show that $t_0 \geq 0$ and $\mathbf{x}_0 \in B_{\rho_0}$ imply $\|\mathbf{x}(t; t_0, \mathbf{x}_0)\|$ tends to 0 as $t \to \infty$. In Section 5, we make a conclusion and give two examples together with figures in which are drawn a solution curve.

3. Preliminaries

Recall that $\Phi(w) = w - 1 + e^{-w}$ for $w \in \mathbb{R}$. It is clear that $\Phi(0) = 0$. For convenience, we write

$$\phi(w) = \frac{d}{dw}\Phi(w).$$

Since $\phi(w) = 1 - e^{-w}$, we see that the derivative ϕ of Φ is strictly increasing and satisfies

$$\phi(-w) < \phi(0) = 0 < \phi(w)$$

for $w > 0$. Hence, Φ is strictly increasing on $[0, \infty)$ and strictly decreasing on $(-\infty, 0]$. We also see that

$$|\phi(w)| \leq -\phi(-\mu) \quad \text{for} \ \ w \in [-\mu, \mu], \tag{3.1}$$

where $\mu > 0$. For other properties of Φ and ϕ, refer to [22].

Consider the ratio $\phi^2(w)/\Phi(w)$ for $w \neq 0$. Then it is clear that

$$\lim_{w\to\infty} \frac{\phi^2(w)}{\Phi(w)} = 0 \quad \text{and} \quad \lim_{w\to-\infty} \frac{\phi^2(w)}{\Phi(w)} = \infty.$$

By L'Hôpital's rule, we have

$$\lim_{w\to 0} \frac{\phi^2(w)}{\Phi(w)} = \lim_{w\to 0} 2\phi'(w) = 2.$$

The ratio is nonnegative and strictly decreasing on $\mathbb{R}$. In fact,

$$\begin{aligned}\frac{d(\phi^2(w)/\Phi(w))}{dw} &= \frac{\phi(w)}{\Phi^2(w)}\left(-1+2we^{-w}+e^{-2w}\right)\\ &= \frac{\phi(w)\,e^{-w}}{\Phi^2(w)}\,(-e^{w}+2w+e^{-w})\\ &= -\frac{\phi(w)\,e^{-w}}{\Phi^2(w)}\left(\sum_{n=1}^{\infty}\frac{2w^{2n+1}}{(2n+1)!}\right)<0\end{aligned}$$

for $w \neq 0$. Hence, we see that

$$\frac{\phi^2(w)}{\Phi(w)} > \frac{\phi^2(\rho)}{\Phi(\rho)} \quad \text{for } \ |w| < \rho, \tag{3.2}$$

where ρ is an arbitrary positive number. The function ψ_ρ can be written as

$$\psi_\rho(t) = \ell(t) - \frac{\phi^2(\rho)}{\Phi(\rho)}h(t)$$

for $t \geq 0$. From the above property of ϕ^2/Φ and the nonnegativity of h, it turns out that

$$\psi_{\rho_1}(t) \leq \psi_{\rho_2}(t) \leq \ell(t)$$

for any numbers ρ_1 and ρ_2 with $0 < \rho_1 < \rho_2$, and therefore,

$$(\psi_{\rho_1})_+(t) \leq (\psi_{\rho_2})_+(t) \leq \ell_+(t).$$

Hence, if condition (1.5) is satisfied, then condition (1.3) also holds for any $\rho > 0$.

Define functions α and $\beta\colon \mathbb{R} \to [0,\infty)$ by

$$\alpha(w) = \begin{cases} \dfrac{\Phi(w)}{w^2} & \text{if } \ w \neq 0,\\ 1/2 & \text{if } \ w = 0 \end{cases} \quad \text{and} \quad \beta(w) = \begin{cases} \dfrac{\phi(w)}{w} & \text{if } \ w \neq 0,\\ 1 & \text{if } \ w = 0. \end{cases}$$

Then the functions α and β are continuously differentiable and strictly decreasing on $\mathbb{R}$. In fact, by L'Hôpital's rule, we have

$$\lim_{w\to 0}\alpha(w) = \frac{1}{2} \quad \text{and} \quad \lim_{w\to 0}\beta(w) = 1.$$

It is also clear that

$$\alpha'(w) = \frac{w\phi(w) - 2\Phi(w)}{w^3} \quad \text{and} \quad \beta'(w) = \frac{w\phi'(w) - \phi(w)}{w^2}$$

for $w \neq 0$. By a straightforward calculation, we see that

$$\alpha'(w) = \begin{cases} -\dfrac{e^{-w}}{w}\displaystyle\sum_{n=1}^{\infty}\frac{n\,w^n}{(n+2)!} < 0 & \text{if } \ w \geq 0,\\ \dfrac{1}{w}\displaystyle\sum_{n=1}^{\infty}\frac{n\,(-w)^n}{(n+2)!} < 0 & \text{if } \ w < 0 \end{cases}$$

and

$$\beta'(w) = \begin{cases} -\dfrac{e^{-w}}{w}\displaystyle\sum_{n=1}^{\infty}\dfrac{w^n}{(n+1)!} < 0 & \text{if } w \geq 0, \\ \dfrac{1}{w}\displaystyle\sum_{n=1}^{\infty}\dfrac{n\,(-w)^n}{(n+1)!} < 0 & \text{if } w < 0. \end{cases}$$

We therefore conclude that

$$\alpha(\varepsilon)w^2 \leq \Phi(w) \leq \alpha(-\varepsilon)w^2 \quad \text{for } w \in (-\varepsilon, \varepsilon) \tag{3.3}$$

and

$$\beta(\rho)w^2 \leq w\phi(w) \leq \beta(-\rho)w^2 \quad \text{for } w \in (-\rho, \rho). \tag{3.4}$$

The inequalities (3.3) and (3.4) will be used in Subsections 4.1 and 4.2, respectively.

4. Proof of Theorem 1.1

4.1. *Uniform stability*

To prove uniform stability of the origin of system (2.1), we will use only an idea of Lyapunov direct method, because using the standard Lyapunov theorem is inconvenient for proving the equiattractivity.

For any $0 < \varepsilon \leq \rho$, we take

$$\Phi(-\delta(\varepsilon)) = \frac{m}{M}e^{-L}\Phi(\varepsilon),$$

where m, M and L are constants given in Section 1. Note that $\delta(\rho) = \rho_0$. Since $m \leq M$, $L \geq 0$ and

$$\Phi(-\varepsilon) - \Phi(\varepsilon) = -2\varepsilon + e^{\varepsilon} - e^{-\varepsilon} = \sum_{n=1}^{\infty}\frac{2\varepsilon^{2n+1}}{(2n+1)!} > 0,$$

we see that

$$\Phi(-\delta(\varepsilon)) < \Phi(\varepsilon) < \Phi(-\varepsilon).$$

By taking into account that Φ is strictly decreasing on $(-\infty, 0]$, it turns out that

$$\delta(\varepsilon) < \varepsilon.$$

Let $t_0 \geq 0$ be the initial time and let $\mathbf{x}(\cdot\,; t_0, \mathbf{x}_0) = (x(t), y(t), z(t))$ be the solution of (2.1) starting from the initial point $\mathbf{x}_0 = (x(t_0), y(t_0), z(t_0)) \in \mathbb{R}^3$ at the initial time t_0. We will show that $\|\mathbf{x}_0\| < \delta$ implies $\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| < \varepsilon$ for all $t \geq t_0$. For simplicity, we write $\mathbf{x}(\cdot\,; t_0, \mathbf{x}_0)$ as $\mathbf{x}(\cdot)$.

By way of contradiction, suppose that there exists a $t_1 > t_0$ with $\|\mathbf{x}(t_1)\| = \varepsilon$ and

$$\|\mathbf{x}(t)\| = \sqrt{x^2(t) + y^2(t) + z^2(t)} < \varepsilon \quad \text{for } t_0 \leq t < t_1. \tag{4.1}$$

Define the function $v\colon [0,\infty) \to [0,\infty)$ by

$$v(t) = \frac{a}{dK}\Phi(x(t)) + \frac{c}{dK}\Phi(y(t)) + \frac{f(t)}{g(t)}\Phi(z(t)).$$

From (1.2), (3.3) and (4.1) it follows that

$$\begin{aligned} m\alpha(\varepsilon)\|\mathbf{x}(t)\|^2 &\le m\Big(\Phi(x(t)) + \Phi(y(t)) + \Phi(z(t))\Big) \\ &\le v(t) \le M\Big(\Phi(x(t)) + \Phi(y(t)) + \Phi(z(t))\Big) \\ &\le M\alpha(-\varepsilon)\|\mathbf{x}(t)\|^2 \end{aligned} \tag{4.2}$$

for $t_0 \le t \le t_1$. We have

$$\begin{aligned} v'(t) &= \frac{a}{dK}\phi(x(t))x'(t) + \frac{c}{dK}\phi(y(t))y'(t) + \left(\frac{f(t)}{g(t)}\right)'\Phi(z(t)) + \frac{f(t)}{g(t)}\phi(z(t))z'(t) \\ &= \left(\frac{f(t)}{g(t)}\right)'\Phi(z(t)) - \frac{f(t)h(t)}{g(t)}\phi^2(z(t)). \end{aligned}$$

From (3.2) and (4.1) it follows that

$$\begin{aligned} v'(t) &< \left(\frac{f(t)}{g(t)}\right)'\Phi(z(t)) - \frac{\phi^2(\rho)}{\Phi(\rho)}\frac{f(t)h(t)}{g(t)}\Phi(z(t)) \\ &= \left\{\left(\ln\frac{f(t)}{g(t)}\right)' - \frac{\phi^2(\rho)}{\Phi(\rho)}h(t)\right\}\frac{f(t)}{g(t)}\Phi(z(t)) \\ &= \psi_\rho(t)\frac{f(t)}{g(t)}\Phi(z(t)) \le (\psi_\rho)_+(t)\frac{f(t)}{g(t)}\Phi(z(t)) \le (\psi_\rho)_+(t)v(t) \end{aligned}$$

for $t_0 \le t \le t_1$. This estimation and (1.3) lead to

$$v(t) \le v(t_0)\exp\left(\int_{t_0}^{t}(\psi_\rho)_+(s)\,ds\right) \le v(t_0)e^L \quad \text{for} \quad t_0 \le t \le t_1.$$

Using (4.2) and the assumption that $\|\mathbf{x}_0\| < \delta$, we obtain

$$v(t_0) \le M\Big(\Phi(x(t_0)) + \Phi(y(t_0)) + \Phi(z(t_0))\Big) \le M\alpha(-\delta)\|\mathbf{x}_0\|^2 e^L.$$

Hence, by (4.2) again, we can estimate that

$$\begin{aligned} m\alpha(\varepsilon)\|\mathbf{x}(t)\|^2 &\le v(t) \le v(t_0)e^L \le M\alpha(-\delta)\|\mathbf{x}_0\|^2 e^L \\ &< M\frac{\Phi(-\delta)}{\delta^2}\delta^2 e^L = M\Phi(-\delta)e^L \\ &= m\Phi(\varepsilon) = m\alpha(\varepsilon)\varepsilon^2 \end{aligned}$$

for $t_0 \le t < t_1$. Hence, we have $\|\mathbf{x}(t)\| < \varepsilon$ for $t_0 \le t \le t_1$. This contradicts the assumption that $\|\mathbf{x}(t_1)\| = \varepsilon$. We therefore conclude that

$$\|\mathbf{x}(t)\| < \varepsilon \quad \text{for} \quad t \ge t_0,$$

namely, the origin of (2.1) is uniformly stable.

4.2. *Equiattractivity in B_{ρ_0}*

Since this subsection is the core part of this paper, we here need to a very careful discussion. Before proving that the origin of (2.1) is equiattractive in the sphere B_{ρ_0}, it would be helpful to mention its broad outline.

The proof is divided into four parts. To begin with, we will show that

(i) the function v given in Subsection 4.1 approaches a nonnegative value v^*.

If v^* is zero, then every solution of (2.1) tends to zero as time passes. This conclusion is desirable. To complete the proof, we show that v^* is zero by way of contradiction. For this purpose, we define an auxiliary function u by

$$u(t) = \frac{f(t)}{g(t)}\Phi(z(t)).$$

Since u is positive and is not greater than the function v, the inferior and the superior limit of u exist. Then, we prove that

(ii) $\liminf_{t\to\infty} u(t) = 0$;

(iii) $\limsup_{t\to\infty} u(t) = 0$.

The proof of part (ii) is simple, but that of part (iii) needs a detailed calculation and considerable patience. In the proof of part (iii), we assume that the superior limit of u is positive and then examine the behavior of the auxiliary function u in detail. Since $\liminf_{t\to\infty} u(t) = 0 < \limsup_{t\to\infty} u(t)$, the value of the function u repeats increasing and decreasing. Hence, we can find three sequences $\{s_n\}$, $\{\tau_n\}$ and $\{\sigma_n\}$ with $T_2 < \tau_n < s_n < \sigma_n \le \tau_{n+1}$ and $\tau_n \to \infty$ as $n \to \infty$ such that $u(s_n) = 2\varepsilon_2$, $u(\tau_n) = u(\sigma_n) = \varepsilon_2$ and

$$\varepsilon_2 < u(t) < 2\varepsilon_2 \quad \text{for } \tau_n < t < s_n,$$

$$u(t) \ge \varepsilon_2 \quad \text{for } \tau_n < t < \sigma_n,$$

$$u(t) < 2\varepsilon_2 \quad \text{for } \sigma_n < t < \tau_{n+1},$$

where ε_2 is a sufficiently small number. We next transform system (2.1) into an equivalent system by using cylindrical coordinates $(x, y, z) \to (r, \theta, z)$ by $x = r\cos\theta$ and $y = r\sin\theta$ and examine any solution $(r(t), \theta(t), z(t))$ of the transformed system in detail. We pay particular attention to the movement of $(r(t), \theta(t))$ on the time interval $[\sigma_n, \tau_{n+1}]$ for each $n \in \mathbb{N}$ and show that there is an upper limit for the lengths of these intervals. From this conclusion and the assumptions (1.3) and (1.4), we can derive $\liminf_{t\to\infty}(s_n - \tau_n) = 0$. This leads us to the conclusion that $\limsup_{t\to\infty} u(t) = 0$. From parts (ii) and (iii), we see that $\lim_{t\to\infty} u(t) = 0$. Using this fact and repeating the same argument as in part (iii), we can show that

(iv) $z(t)$ does not converge to zero as $t \to \infty$.

However, from the definition of u, we see that if u converges to zero, then z also converges. This contradicts part (iv). Hence, the case that v_0 is positive does not occur.

Now, let us go through step-by-step to prove that the origin of (2.1) is equiattractive in the sphere B_{ρ_0}.

Part (i): We will show that $t_0 \geq 0$ and $\mathbf{x}_0 \in B_{\rho_0}$ imply $\|\mathbf{x}(t; t_0, \mathbf{x}_0)\|$ tends to 0 as $t \to \infty$. As have shown in Subsection 4.1, if $\mathbf{x}_0 \in B_{\rho_0}$, then $\mathbf{x}(t; t_0, \mathbf{x}_0) \in B_\rho$ for all $t \geq t_0$, where

$$B_\rho = \left\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < \rho^2\right\}.$$

Taking account of this and repeating the same way as Subsection 4.1, we get that

$$m\alpha(\rho)\|\mathbf{x}(t)\|^2 \leq v(t) \leq M\alpha(-\rho)\|\mathbf{x}(t)\|^2 \tag{4.3}$$

and

$$v'(t) < \psi_\rho(t)\frac{f(t)}{g(t)}\Phi(z(t)) \leq (\psi_\rho)_+(t)v(t) \tag{4.4}$$

for $t \geq t_0$. From (1.3) and (4.4) it follows that $v(t) < v(t_0)e^L$ for $t \geq t_0$. Hence, by (4.4) again, we have

$$v'(t) < v(t_0)e^L(\psi_\rho)_+(t) \quad \text{for } \ t \geq t_0.$$

Since the right-hand side of this inequality is nonnegative, we see that

$$(v')_+(t) \leq v(t_0)e^L(\psi_\rho)_+(t) \quad \text{for } \ t \geq t_0.$$

Integrating both sides from t_0 to ∞ and using (1.3) again, we obtain

$$\int_{t_0}^{\infty}(v')_+(t)dt \leq v(t_0)e^L L < \infty.$$

On the other hand, since $v(t) \geq 0$ for $t \geq t_0$, we have

$$\int_{t_0}^{\infty}(v')_-(t)dt = \int_{t_0}^{\infty}(v')_+(t)dt - \int_{t_0}^{\infty}v'(t)dt < \infty.$$

From the two evaluations above, we see that

$$\int_{t_0}^{\infty}|v'(t)|dt = \int_{t_0}^{\infty}\left\{(v')_+(t) + (v')_-(t)\right\}dt < \infty,$$

namely, the derivative v' is absolutely integrable on $[t_0, \infty)$. Hence, the function v has a nonnegative limiting value $v^* \geq 0$.

If $v^* = 0$, then it follows from (4.3) that $\|\mathbf{x}(t)\|$ tends to 0 as $t \to \infty$. This is our desired conclusion. Therefore, in order to complete the proof, we have only to show that the case of $v^* > 0$ does not occur.

If $v^* > 0$, then there exists a $T \geq t_0$ such that

$$0 < \frac{1}{2}v^* < v(t) < \frac{3}{2}v^* \quad \text{for } \ t \geq T. \tag{4.5}$$

Let

$$u(t) = \frac{f(t)}{g(t)} \Phi(z(t))$$

for $t \geq T$. Since the function u is not greater than the function v, there exist the inferior limit and the superior limit of u as t tends to ∞.

Part (ii): We first suppose that $\liminf_{t\to\infty} u(t) > 0$. Then there exist an $\varepsilon_1 > 0$ and a $T_1 \geq T$ such that $u(t) > \varepsilon_1$ for $t \geq T_1$. Using (4.4) and (4.5), we obtain

$$\begin{aligned} v'(t) &< \psi_\rho(t)u(t) = (\psi_\rho)_+(t)u(t) - (\psi_\rho)_-(t)u(t) \\ &\leq (\psi_\rho)_+(t)v(t) - (\psi_\rho)_-(t)u(t) < \frac{3}{2}v^*(\psi_\rho)_+(t) - \varepsilon_1(\psi_\rho)_-(t) \end{aligned}$$

for $t \geq T_1$. Hence, by (4.5) again, we have

$$\begin{aligned} -\infty < \frac{1}{2}v^* - v(T_1) < v(t) - v(T_1) &= \int_{T_1}^{t} v'(s)ds \\ &\leq \frac{3}{2}v^*\int_{T_1}^{t}(\psi_\rho)_+(s)ds - \varepsilon_1\int_{T_1}^{t}(\psi_\rho)_-(s)ds \end{aligned}$$

for $t \geq T_1$. This contradicts (1.4). We therefore conclude that $\liminf_{t\to\infty} u(t) = 0$.

Part (iii): Next, we suppose that $\lambda \overset{\text{def}}{=} \limsup_{t\to\infty} u(t) > 0$. Let ε_2 be sufficiently small so as to satisfy that $0 < \varepsilon_2 < \min\left\{\sqrt{v^*}/2,\ \lambda/2\right\}$ and

$$\begin{aligned} &-\sqrt{\frac{2M\alpha(-\rho)}{v^* - 4\varepsilon_2}}\,\phi\left(-\sqrt{\frac{2\varepsilon_2}{k_1\alpha(\rho)}}\right) \\ &\qquad \leq \beta(\rho)\min\left\{\frac{dm}{2\overline{g}},\ \frac{c\underline{f}}{\sqrt{2}d\overline{h}} - \frac{3d^2mMK^2\beta(-\rho)}{\pi\omega\overline{h}}\sqrt{\frac{4M\alpha(-\rho)\varepsilon_2}{k_1\alpha(\rho)(v^* - 4\varepsilon_2)}}\right\}. \end{aligned} \tag{4.6}$$

Note that we can choose an ε_2 satisfying (4.6). In fact, the right-hand side of (4.6) approaches a positive number as ε_2 tends to 0. On the other hand, the left-hand side of (4.6) approaches zero as ε_2 tends to 0.

Since $\liminf_{t\to\infty} u(t) = 0$, we can find a $T_2 \geq T_1$ so that $u(T_2) < \varepsilon_2$. By the assumption that $\limsup_{t\to\infty} u(t) = \lambda > 2\varepsilon_2$, the function u will sometime exceed the value $2\varepsilon_2$. Let

$$s_1 = \inf\{t > T_2 : u(t) \geq 2\varepsilon_2\}.$$

Then it is clear that $u(s_1) = 2\varepsilon_2$ and $u(t) < 2\varepsilon_2$ for $T_2 \leq t < s_1$. Since $T_2 < s_1$ and $u(T_2) < \varepsilon_2$, the set $\{t < s_1 : u(t) \leq \varepsilon_2\}$ is not empty. Let

$$\tau_1 = \sup\{t < s_1 : u(t) \leq \varepsilon_2\}.$$

Then we see that $u(\tau_1) = \varepsilon_2$ and $\varepsilon_2 < u(t) < 2\varepsilon_2$ for $\tau_1 < t < s_1$. Because of $\liminf_{t\to\infty} u(t) = 0$, the function u becomes smaller than the value ε_2 again. Let

$$\sigma_1 = \inf\{t > s_1 : u(t) < \varepsilon_2\}.$$

Then we see that $u(\sigma_1) = \varepsilon_2$ and $u(t) \geq \varepsilon_2$ for $\tau_1 < t < \sigma_1$. Similarly, let

$$s_2 = \inf\{t > \sigma_1 : u(t) \geq 2\varepsilon_2\},$$
$$\tau_2 = \sup\{t < s_2 : u(t) \leq \varepsilon_2\},$$
$$\sigma_2 = \inf\{t > s_2 : u(t) < \varepsilon_2\}.$$

Then, $u(s_2) = 2\varepsilon_2$, $u(\tau_2) = u(\sigma_2) = \varepsilon_2$, and $u(t) < 2\varepsilon_2$ for $\sigma_1 < t < \tau_2$ and $\varepsilon_2 < u(t) < 2\varepsilon_2$ for $\tau_2 < t < s_2$. By repeating the same process as above, we can choose three sequences $\{s_n\}$, $\{\tau_n\}$ and $\{\sigma_n\}$ with $T_2 < \tau_n < s_n < \sigma_n \leq \tau_{n+1}$ and $\tau_n \to \infty$ as $n \to \infty$ such that $u(s_n) = 2\varepsilon_2$, $u(\tau_n) = u(\sigma_n) = \varepsilon_2$ and

$$\varepsilon_2 < u(t) < 2\varepsilon_2 \quad \text{for} \ \ \tau_n < t < s_n, \tag{4.7}$$

$$u(t) \geq \varepsilon_2 \quad \text{for} \ \ \tau_n < t < \sigma_n, \tag{4.8}$$

$$u(t) < 2\varepsilon_2 \quad \text{for} \ \ \sigma_n < t < \tau_{n+1}. \tag{4.9}$$

Recall that the function α is strictly decreasing on $\mathbb{R}$. Since $|z(t)| \leq \|\mathbf{x}(t)\| < \rho$ for $t \geq t_0$, we see that $\Phi(z(t)) = \alpha(z(t))z^2(t) \geq \alpha(\rho)z^2(t)$ for $t \geq t_0$. Hence, by (1.2) we have

$$u(t) = \frac{f(t)}{g(t)}\Phi(z(t)) \geq k_1\Phi(z(t)) \geq k_1\alpha(\rho)z^2(t) \quad \text{for} \ \ t \geq t_0.$$

From this estimation and (4.9) it follows that

$$|z(t)| \leq \sqrt{\frac{u(t)}{k_1\alpha(\rho)}} \leq \sqrt{\frac{2\varepsilon_2}{k_1\alpha(\rho)}} \quad \text{for} \ \ \sigma_n \leq t \leq \tau_{n+1}. \tag{4.10}$$

We transform system (2.1) into polar coordinates by

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.$$

Then we have

$$\begin{cases} r' = c\,\phi(y)\cos\theta - a\,\phi(x)\sin\theta + K f(t)\phi(z)\sin\theta, \\ \theta' = \dfrac{K f(t)\phi(z)}{r}\cos\theta - \dfrac{1}{r^2}\Big(ax\phi(x) + cy\phi(y)\Big), \\ z' = -\dfrac{c}{d}\,g(t)\phi(y) - h(t)\phi(z). \end{cases} \tag{4.11}$$

Let $(r(\cdot), \theta(\cdot), z(\cdot))$ be the solution of (4.11) corresponding to $\mathbf{x}(\cdot)$. Since $|x(t)| \leq \|\mathbf{x}(t)\| < \rho$ and $|y(t)| \leq \|\mathbf{x}(t)\| < \rho$ for $t \geq t_0$, we see that $\Phi(x(t)) \leq \alpha(-\rho)x^2(t)$ and $\Phi(y(t)) \leq \alpha(-\rho)y^2(t)$ for $t \geq t_0$. Hence, by (4.5), (4.9) and the definition of M, we have

$$\begin{aligned} \frac{1}{2}v^* - 2\varepsilon_2 < v(t) - u(t) &= \frac{a}{dK}\Phi(x(t)) + \frac{c}{dK}\Phi(y(t)) \\ &\leq M\alpha(-\rho)\Big(x^2(t) + y^2(t)\Big) = M\alpha(-\rho)r(t) \end{aligned}$$

for $\sigma_n \le t \le \tau_{n+1}$. Arranging this, we obtain

$$\sqrt{\frac{v^* - 4\varepsilon_2}{2M\alpha(-\rho)}} < r(t) \le \|\mathbf{x}(t)\| < \rho \quad \text{for} \ \ \sigma_n \le t \le \tau_{n+1}. \tag{4.12}$$

From (4.10) and (4.12), we see that the point $(r(t), \theta(t), z(t))$ stays in the rectangular solid

$$R = \left\{(r,\theta,z)\colon \sqrt{\frac{v^* - 4\varepsilon_2}{2M\alpha(-\rho)}} < r < \rho,\ -\pi < \theta < \pi \ \text{ and } \ |z| \le \sqrt{\frac{2\varepsilon_2}{k_1\alpha(\rho)}}\right\}$$

for $\sigma_n \le t \le \tau_{n+1}$. It is clear that R corresponds to a thin disc in the (x, y, z)-space. From (3.4) and the definitions of m and M it follows that

$$\begin{aligned} dmK\beta(\rho)\left(x^2(t) + y^2(t)\right) &\le a\beta(\rho)x^2(t) + c\beta(\rho)y^2(t) \\ &\le ax(t)\phi(x(t)) + cy(t)\phi(y(t)) \\ &\le a\beta(-\rho)x^2(t) + c\beta(-\rho)y^2(t) \le dMK\beta(-\rho)\left(x^2(t) + y^2(t)\right) \end{aligned}$$

for $t \ge t_0$. Hence, by (1.1), (4.11) and (4.12), we have

$$\begin{aligned} -K\bar{g}\sqrt{\frac{2M\alpha(-\rho)}{v^* - 4\varepsilon_2}}\,|\phi(z(t))| - dMK\beta(-\rho) &< -\frac{Kf(t)|\phi(z(t))|}{r(t)} - dMK\beta(-\rho) \\ &\le \theta'(t) \le \frac{Kf(t)|\phi(z(t))|}{r(t)} - dmK\beta(\rho) \\ &< K\bar{g}\sqrt{\frac{2M\alpha(-\rho)}{v^* - 4\varepsilon_2}}\,|\phi(z(t))| - dmK\beta(\rho) \end{aligned}$$

for $\sigma_n \le t \le \tau_{n+1}$. Using (3.1) and (4.10), we obtain

$$-\omega - dMK\beta(-\rho) < \theta'(t) < \omega - dmK\beta(\rho) \tag{4.13}$$

for $\sigma_n \le t \le \tau_{n+1}$, where

$$\omega = -K\bar{g}\sqrt{\frac{2M\alpha(-\rho)}{v^* - 4\varepsilon_2}}\,\phi\left(-\sqrt{\frac{2\varepsilon_2}{k_1\alpha(\rho)}}\right) > 0.$$

From (4.6), we see that

$$\omega \le \frac{1}{2}dmK\beta(\rho).$$

Hence, because of the strictly decreasing property of β, we can estimate that

$$\begin{aligned} \frac{1}{2}dmK\beta(\rho) &\le dmK\beta(\rho) - \omega < dmK\beta(\rho) \\ &< dMK\beta(-\rho) < \omega + dMK\beta(-\rho) \\ &\le \frac{1}{2}dmK\beta(\rho) + dMK\beta(-\rho) < \frac{3}{2}dMK\beta(-\rho). \end{aligned} \tag{4.14}$$

Define a rectangle Ω by

$$\Omega=\left\{(r,\theta)\colon\ \sqrt{\frac{v^*-4\varepsilon_2}{2M\alpha(-\rho)}}<r<\rho,\ \frac{\pi}{2}\left(1-\frac{\omega}{dmK\beta(\rho)}\right)\le\theta\le\frac{\pi}{2}\left(1+\frac{\omega}{dmK\beta(\rho)}\right)\right\},$$

which corresponds to a sector in the (x,y)-plane. Since

$$\frac{1}{4}\pi\le\frac{\pi}{2}\left(1-\frac{\omega}{dmK\beta(\rho)}\right)<\frac{\pi}{2}<\frac{\pi}{2}\left(1+\frac{\omega}{dmK\beta(\rho)}\right)\le\frac{3}{4}\pi, \tag{4.15}$$

the central angle of this sector is not greater than $\pi/2$.

Here, we show that

$$\tau_{n+1}-\sigma_n\le\frac{2\pi}{dmK\beta(\rho)-\omega}\quad\text{for }\ n\in\mathbb{N} \tag{4.16}$$

If this inequality does not always hold for any natural number n, then there exists an $n_0\in\mathbb{N}$ such that $\tau_{n_0+1}-\sigma_{n_0}>2\pi/(dmK\beta(\rho)-\omega)$. Let us focus on the movement of the point $(r(t),\theta(t))$ in detail. From (4.12) it follows that the point $(r(t),\theta(t))$ remains within the rectangle

$$\Lambda=\left\{(r,\theta)\colon\ \sqrt{\frac{v^*-4\varepsilon_2}{2M\alpha(-\rho)}}<r<\rho,\ -\pi<\theta<\pi\right\}\supset\Omega\pmod{2\pi}$$

for $\sigma_{n_0}\le t\le\tau_{n_0+1}$. The rectangle Λ corresponds to an annulus in the (x,y)-plane. Integrating (4.13) from σ_{n_0} to τ_{n_0+1}, we obtain

$$\theta(\sigma_{n_0})-\theta(\tau_{n_0+1})=-\int_{\sigma_{n_0}}^{\tau_{n_0+1}}\theta'(t)dt>(dmK\beta(\rho)-\omega)(\tau_{n_0+1}-\sigma_{n_0})>2\pi.$$

This means that the point $(r(t),\theta(t))$ penetrates the rectangle Ω. Let t_2 and t_3 be constants with $\sigma_{n_0}\le t_2<t_3\le\tau_{n_0+1}$ such that

$$\theta(t_2)=\frac{\pi}{2}\left(1+\frac{\omega}{dmK\beta(\rho)}\right)\quad\text{and}\quad\theta(t_3)=\frac{\pi}{2}\left(1-\frac{\omega}{dmK\beta(\rho)}\right),$$

and

$$(r(t),\theta(t))\in\Omega\quad\text{for }\ t_2<t<t_3. \tag{4.17}$$

By (4.13) and (4.14), we get

$$\theta(t_2)-\theta(t_3)<(\omega+dMK\beta(-\rho))(t_3-t_2)<\frac{3}{2}dMK\beta(-\rho)(t_3-t_2).$$

Hence, we have

$$t_3-t_2>\frac{2(\theta(t_2)-\theta(t_3))}{3dMK\beta(-\rho)}=\frac{2\pi\omega}{3d^2mMK^2\beta(\rho)\beta(-\rho)}. \tag{4.18}$$

From (4.15) and (4.17), we see that

$$\begin{aligned}|y(t)| = r(t)|\sin\theta(t)| &> \sqrt{\frac{v^* - 4\varepsilon_2}{2M\alpha(-\rho)}}\sin\left(\frac{\pi}{2}\left(1 - \frac{\omega}{dmK\beta(\rho)}\right)\right)\\ &\geq \sqrt{\frac{v^* - 4\varepsilon_2}{2M\alpha(-\rho)}}\sin\frac{\pi}{4} = \sqrt{\frac{v^* - 4\varepsilon_2}{4M\alpha(-\rho)}} \end{aligned}\tag{4.19}$$

for $t_2 \leq t \leq t_3$. Using the third equation of (2.1), we obtain

$$|z'(t)| \geq \frac{c}{d}g(t)|\phi(y(t))| - h(t)|\phi(z(t))| = \frac{c}{d}g(t)\left|\frac{\phi(y(t))}{y(t)}\right||y(t)| - h(t)|\phi(z(t))|$$

for $t \geq t_0$. From (1.1) and (3.4) it follows that

$$|z'(t)| \geq \frac{c}{d}\underline{f}\,\beta(\rho)|y(t)| - \overline{h}\,|\phi(z(t))| \quad \text{for} \ \ t_2 \leq t \leq t_3.$$

Combining (3.1), (4.6), (4.10) and (4.19), we get

$$\begin{aligned}|z'(t)| &> \frac{c}{d}\underline{f}\,\beta(\rho)\sqrt{\frac{v^* - 4\varepsilon_2}{4M\alpha(-\rho)}} + \overline{h}\,\phi\left(-\sqrt{\frac{2\varepsilon_2}{k_1\alpha(\rho)}}\right)\\ &\geq \frac{3d^2mMK^2\beta(\rho)\beta(-\rho)}{\pi\omega}\sqrt{\frac{2\varepsilon_2}{k_1\alpha(\rho)}} > 0\end{aligned}\tag{4.20}$$

for $t_2 \leq t \leq t_3$. Since $z'(\cdot)$ is continuous on $[t_0, \infty)$, it does not change sign on $[t_2, t_3]$. Hence, the equality

$$\left|\int_{t_2}^{t_3} z'(t)dt\right| = \int_{t_2}^{t_3}|z'(s)|ds$$

holds. By (4.10), (4.18) and (4.20), we have

$$\begin{aligned}2\sqrt{\frac{2\varepsilon_2}{k_1\alpha(\rho)}} &\geq |z(t_2)| + |z(t_3)| \geq \int_{t_2}^{t_3}|z'(t)|dt\\ &> \left(\frac{3d^2mMK^2\beta(\rho)\beta(-\rho)}{\pi\omega}\sqrt{\frac{2\varepsilon_2}{k_1\alpha(\rho)}}\right)(t_3 - t_2) > 2\sqrt{\frac{2\varepsilon_2}{k_1\alpha(\rho)}}.\end{aligned}$$

This is a contradiction. Thus, it was verified that (4.16) holds.

Using (4.4), (4.5) and the same way as above, we obtain

$$v'(t) < \psi_\rho(t)u(t) \leq (\psi_\rho)_+(t)v(t) - (\psi_\rho)_-(t)u(t) < \frac{3}{2}v^*(\psi_\rho)_+(t) - (\psi_\rho)_-(t)u(t)$$

for $t \geq T_2$. From (4.5) and (4.8), we see that

$$\begin{aligned}
-\infty < \frac{1}{2}v^* - v(T_2) &< \lim_{t\to\infty} v(t) - v(T_2) = \int_{T_2}^{\infty} v'(t)dt \\
&< \frac{3}{2}v^* \int_{T_2}^{\infty} (\psi_\rho)_+(t)dt - \int_{T_2}^{\infty} (\psi_\rho)_-(t)u(t)dt \\
&\leq \frac{3}{2}v^* \int_{T_2}^{\infty} (\psi_\rho)_+(t)dt - \varepsilon_2 \sum_{n=1}^{\infty} \int_{\tau_n}^{\sigma_n} (\psi_\rho)_-(t)dt.
\end{aligned}$$

By (1.3), we have

$$\int_{T_2}^{\infty} (\psi_\rho)_+(t)dt < \infty,$$

and therefore,

$$\sum_{n=1}^{\infty} \int_{\tau_n}^{\sigma_n} (\psi_\rho)_-(t)dt < \infty.$$

If $\liminf_{t\to\infty}(\sigma_n - \tau_n) > 0$, then from (1.4) and (4.16) it turns out that

$$\sum_{n=1}^{\infty} \int_{\tau_n}^{\sigma_n} (\psi_\rho)_-(s)ds = \infty.$$

This is a contradiction. Hence, we see that $\liminf_{t\to\infty}(\sigma_n - \tau_n) = 0$. Since $[\tau_n, s_n] \subset [\tau_n, \sigma_n]$ for any $n \in \mathbb{N}$, we get

$$\liminf_{t\to\infty}(s_n - \tau_n) = 0. \tag{4.21}$$

Since $|y(t)| < \rho$ and $|z(t)| < \rho$ for $t \geq t_0$, we can estimate that

$$\begin{aligned}
u'(t) &= \left(\frac{f(t)}{g(t)}\right)' \Phi(z(t)) + \frac{f(t)}{g(t)}\phi(z(t))z'(t) \\
&= \left(\frac{f(t)}{g(t)}\right)' \Phi(z(t)) - \frac{f(t)h(t)}{g(t)}\phi^2(z(t)) - \frac{c}{d} f(t)\phi(y(t))\phi(z(t)) \\
&= v'(t) - \frac{c}{d} f(t)\phi(y(t))\phi(z(t)) \leq |v'(t)| + \frac{c\bar{g}}{d}\phi^2(-\rho)
\end{aligned}$$

for $t \geq t_0$. Integrating this inequality from τ_n to s_n, we obtain

$$\varepsilon_2 = u(s_n) - u(\tau_n) \leq \int_{\tau_n}^{s_n} |v'(t)|dt + \frac{c\bar{g}}{d}\phi^2(-\rho)(s_n - \tau_n)$$

for each $n \in \mathbb{N}$. However, by (4.21), the right-hand side of the above inequality approaches zero as $n \to \infty$. This is a contradiction. We therefore conclude that

$$\limsup_{t\to\infty} u(t) = 0.$$

Part (iv): Since $u(t)$ tends to zero as $t \to \infty$, there exists a $T_3 \geq t_0$ such that

$$u(t) < 2\varepsilon_2 \quad \text{for } \ t \geq T_3.$$

Using this inequality instead of (4.9) and repeating the same process as above, we can choose two numbers t_4 and t_5 with $T_3 < t_4 < t_5$ such that

$$t_5 - t_4 > \frac{2\pi\omega}{3d^2mMK^2\beta(\rho)\beta(-\rho)}$$

and

$$|z'(t)| > \frac{3d^2mMK^2\beta(\rho)\beta(-\rho)}{\pi\omega}\sqrt{\frac{2\varepsilon_2}{k_1\alpha(\rho)}} > 0 \quad \text{for } \ t_4 \leq t \leq t_5.$$

Hence, we have

$$\begin{aligned} |z(t_4)| + |z(t_5)| &\geq \int_{t_4}^{t_5} |z'(t)| dt \\ &> \left\{\frac{3d^2mMK^2\beta(\rho)\beta(-\rho)}{\pi\omega}\sqrt{\frac{2\varepsilon_2}{k_1\alpha(\rho)}}\right\}(t_5 - t_4) > 2\sqrt{\frac{2\varepsilon_2}{k_1\alpha(\rho)}}. \end{aligned}$$

On the other hand, by (1.1) we have

$$u(t) \geq \frac{\underline{f}}{\overline{g}}\,\Phi(z(t))$$

for $t \geq T$. Hence, $z(t)$ also tends to zero as $t \to \infty$. This is a contradiction. Thus, it turns out that the case of $v_0 > 0$ does not happen. We therefore conclude that the origin of (2.1) is equiattractive in the sphere B_{ρ_0}.

The proof of Proposition 2.1 is now complete. □

5. Conclusion and examples

In this paper, we proposed a time-varying phytoplankton-zooplankton-fish system (E), taking into account the energy transfer by the food chain of microorganisms and the harvesting effect of zooplankton. We presented two theorems. One guarantees that the interior equilibrium of (E) is equiasymptotically stable in a bounded domain. Another relates to global asymptotic stability of the interior equilibrium.

As a common condition of two theorems, we assumed that the functions f, g and h are bounded. The intrinsic growth rate of phytoplankton and the harvesting effect are also assumed to be bounded, namely,

$$\frac{c}{d}\underline{f} < \gamma(t) < \overline{h} + \frac{c}{d}\overline{g} \quad \text{and} \quad K\underline{f} < j(t) < K\overline{g}.$$

According to the former assumption, phytoplankton is supplied with the nutrients necessary for proliferation, but explosive growth due to excessive nutrients does not occur.

Also, the latter assumption means that there is also an intake of zooplankton by other marine organisms and human beings, except fish, but it is limited. It would be reasonable to think that suitable nutritional supply and well-balanced food distribution will helps to keep the ecological system sustainable.

To get our results, we focused on the function ℓ defined by

$$\ell(t) = \left(\ln \frac{f(t)}{g(t)}\right)'.$$

Here, the ratio f/g is the conversion rate from individuals of phytoplankton into individuals of zooplankton. There are two cases to be considered: (i) the function ℓ is absolutely integrable; (ii) the function ℓ is not absolutely integrable. As explained in Section 3, conditions (1.1) and (1.5) yield that ℓ is absolutely integrable. Hence, the integral

$$\int_0^t |\ell(s)| ds$$

converges to some value as t tends to ∞, and therefore, there exists a constant C such that $\lim_{t\to\infty} f(t)/g(t) = C$. This means that the conversion rate f/g approaches a steady state in the future. In addition to conditions (1.1) and (1.5), if condition (1.6) is satisfied, then the interior equilibrium of (E) (or the origin of (2.1)) becomes globally asymptotically stable (see Theorem 1.2). On the other hand, when the function ℓ is not absolutely integrable, Theorem 1.2 is not available. There is a possibility to be able to apply Theorem 1.1 in the case that the function ℓ is not absolutely integrable. To be exact, under the assumption (1.1), if there exists a positive ρ such that

$$\psi_\rho(t) = \ell(t) - \frac{(1 - e^{-\rho})^2}{\rho - 1 + e^{-\rho}} h(t)$$

satisfies conditions (1.3) and (1.4), then the interior equilibrium of (E) (or the origin of (2.1)) is equiasymptotically stable in a bounded domain.

Needless to say, whether such a ρ exists or not depend on h even if ℓ is the same. For example, consider the case that

$$f(t) = \exp\left(\sqrt{2} \sin\left(t + \frac{\pi}{4}\right) - 1\right) \quad \text{and} \quad g(t) = \exp(\sin t + 1).$$

Then we have

$$\frac{f(t)}{g(t)} = \exp(\cos t - 2) < 1 \quad \text{and} \quad \ell(t) = \left(\ln \frac{f(t)}{g(t)}\right)' = -\sin t$$

for $t \geq 0$. Hence, it is clear that the function ℓ is not absolutely integrable and

$$\underline{f} \overset{\text{def}}{=} \frac{1}{e^{1+\sqrt{2}}} \leq f(t) < g(t) \leq e^2 \overset{\text{def}}{=} \overline{g}$$

for $t \geq 0$. If $h(t) = (2+t)/(1+t)$ for $t \geq 0$, then the upper bound $\overline{h}$ is 2. Since $2.7 < e < 2.8$, we see that

$$\frac{\phi^2(1)}{\Phi(1)} = \frac{(e-1)^2}{e} > \frac{(2.7-1)^2}{2.8} > 1.$$

Let $\rho = 1$. Then we have

$$\begin{aligned}\psi_1(t) = \ell(t) - \frac{(1-e^{-1})^2}{1-1+e^{-1}}h(t) &= -\sin t - \frac{\phi^2(1)}{\Phi(1)}\left(1 + \frac{1}{1+t}\right)\\ &< -\sin t - 1 - \frac{1}{1+t} \leq -\frac{1}{1+t}\end{aligned}$$

for $t \geq 0$, and therefore, $(\psi_1)_+(t) \equiv 0$ and $(\psi_1)_-(t) > 1/(1+t)$ for $t \geq 0$. Hence, conditions (1.3) and (1.4) hold for $\rho = 1$. If $h(t) = 1/(1+t)$ for $t \geq 0$, then the upper bound $\overline{h}$ is 1. As have shown in Section 3, since $\phi^2(\rho)/\Phi(\rho) < 2$ for any $\rho > 0$, we see that

$$\begin{aligned}\psi_\rho(t) = \ell(t) - \frac{(1-e^{-\rho})^2}{\rho-1+e^{-\rho}}h(t) &= -\sin t - \frac{\phi^2(\rho)}{\Phi(\rho)}\frac{1}{1+t}\\ &> -\sin t - \frac{2}{1+t}\end{aligned}$$

for $t \geq 0$. Hence, it is not difficult to show that

$$\int_0^\infty (\psi_\rho)_+(t)dt = \infty \quad \text{and} \quad (\psi_\rho)_- \notin \mathcal{F}_{[\mathrm{WIP}]}$$

for any $\rho > 0$, namely, we cannot find a $\rho > 0$ such that ψ_ρ satisfies conditions (1.3) and (1.4). As can be seen from the above example, in the case that the function ℓ is not absolutely integrable, $(\psi_\rho)_-$ may not belong to $\mathcal{F}_{[\mathrm{WIP}]}$ even if h belongs to $\mathcal{F}_{[\mathrm{WIP}]}$. For this reason, conditions (1.3) and (1.4) are required in Theorem 1.1 instead of conditions (1.5) and (1.6).

To illustrate our results, we give an example for each of the above two cases. In these two examples, the constants in the model (E) is fixed as follows: $a = 0.4$, $b = 0.1$, $c = 0.3$, $d = 0.2$ and $K = 3$. Hence, the interior equilibrium $(K, c/d, a/b)$ is $(3, 3/2, 4)$.

Example 5.1. Consider system (E) with

$$f(t) = \frac{1}{4}\exp\left(\sin\left(\frac{\pi t}{6} + 1\right)\right), \quad g(t) = \frac{1}{2}\exp\left(\sin\left(\frac{\pi t}{6} + \frac{3}{2}\right)\right) \quad \text{and} \quad h(t) = \frac{3}{5}. \tag{5.1}$$

Then the interior equilibrium is uniformly stable and equiasymptotically stable in the space S_1 (see Figure 1).

It is clear that condition (1.1) holds for $\underline{f} = 1/(4e)$, $\overline{g} = e/2$ and $\overline{h} = 3/5$. Since

$$\frac{f(t)}{g(t)} = \frac{1}{2}\exp\left(\sin\left(\frac{\pi t}{6} + 1\right) - \sin\left(\frac{\pi t}{6} + \frac{3}{2}\right)\right) = \frac{1}{2}\exp\left(-2\sin\frac{1}{4}\cos\left(\frac{\pi t}{6} + \frac{5}{4}\right)\right),$$

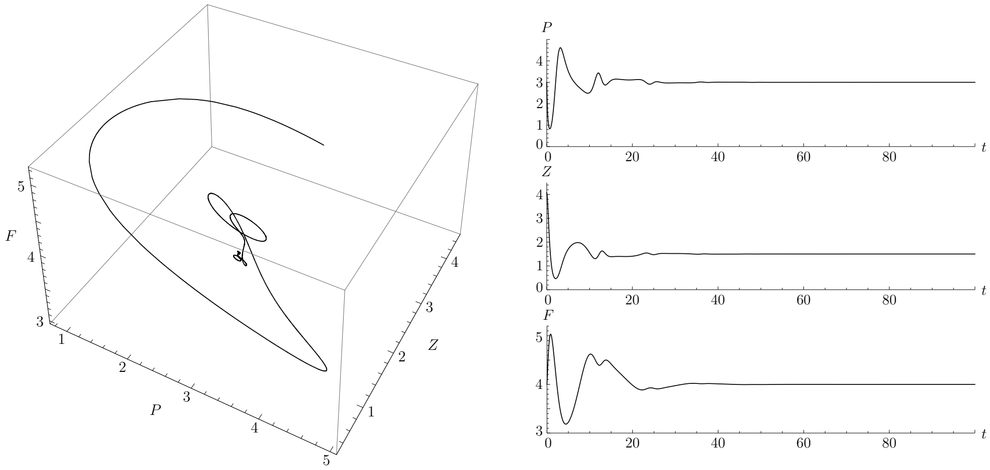

Figure 1: A solution curve of (E) with (5.1) is drawn in the left figure. The initial time t_0 is 0 and the initial point (P_0, Z_0, F_0) is $(3, 3e/2, 4) \in S_1$. The point $(P(t), Z(t), F(t))$ moves on the solution curve from the initial point $(1, 1, 1)$ toward the interior equilibrium $(3, 3/2, 4)$ according to increase of t. The right three figures show the movements of the components P, Z and F. The components P, Z and F finally converge to the values 3, 3/2 and 4, respectively.

we have

$$0.304 < \frac{1}{2}\exp\left(-2\sin\frac{1}{4}\right) \le \frac{f(t)}{g(t)} \le \frac{1}{2}\exp\left(2\sin\frac{1}{4}\right) < 0.821$$

for $t \ge 0$. Hence, we can choose 0.304 and 0.821 as k_1 and k_2, respectively. Hence, we see that

$$m = \min\left\{\frac{2}{3}, \frac{1}{2}, 0.304\right\} = 0.304 \quad\text{and}\quad M = \max\left\{\frac{2}{3}, \frac{1}{2}, 0.821\right\} = 0.821.$$

Since

$$\ell(t) = \left(\ln\frac{f(t)}{g(t)}\right)' = \frac{\pi}{3}\sin\frac{1}{4}\sin\left(\frac{\pi t}{6} + \frac{5}{4}\right)$$

for $t \ge 0$, it is a nontrivial periodic function whose period is 12 (of course, there is no special meaning in the fact that the period of ℓ is 12, but the value can be understood 12 months). Hence, the function ℓ is not absolutely integrable, and therefore, Theorem 1.2 is not available. However, we can apply Theorem 1.1 to this example. Let $\rho = 3$. Then we have

$$\begin{aligned}\psi_3(t) = \ell(t) - \frac{\phi^2(3)}{\Phi(3)}h(t) &\le \frac{\pi}{3}\sin\frac{1}{4} - \frac{3}{5}\frac{\phi^2(3)}{\Phi(3)}\\ &< 0.260 - 0.6 \times 0.44 = -0.004 < 0\end{aligned}$$

for $t \ge 0$. From this estimation it turns out that $(\psi_3)_+(t) = 0$ and $(\psi_3)_-(t) > 0.004$ for $t \ge 0$. Hence, conditions (1.3) and (1.4) hold for $\rho = 3$. Finally, let us evaluate the value of ρ_0. We may choose L to zero. Since

$$0.755 < \frac{0.304}{0.821} \times 2.04 < \frac{m}{M}e^{-L}\Phi(3) < \frac{0.304}{0.821} \times 2.05 < 0.760,$$

$$\Phi(-1.02) < 0.754 \quad \text{and} \quad \Phi(-1.03) > 0.771,$$

we can estimate that $1.02 < \rho_0 < 1.03$. We therefore conclude that all solution curves starting from space $S_1 \subset S_{\rho_0}$ asymptotically approach the interior equilibrium $\mathbf{x}^*$ of (E).

Example 5.2. Consider system (E) with

$$f(t) = \frac{1}{5} + \frac{1}{10}\sin\left(\frac{\pi t}{6}\right), \quad g(t) = \frac{3+2t}{1+t}\left(\frac{1}{5} + \frac{1}{10}\sin\left(\frac{\pi t}{6}\right)\right) \quad \text{and} \quad h(t) = \frac{5}{1+t} \tag{5.2}$$

for $t \geq 0$. Then the interior equilibrium is uniformly stable and globally asymptotically stable (see Figure 2).

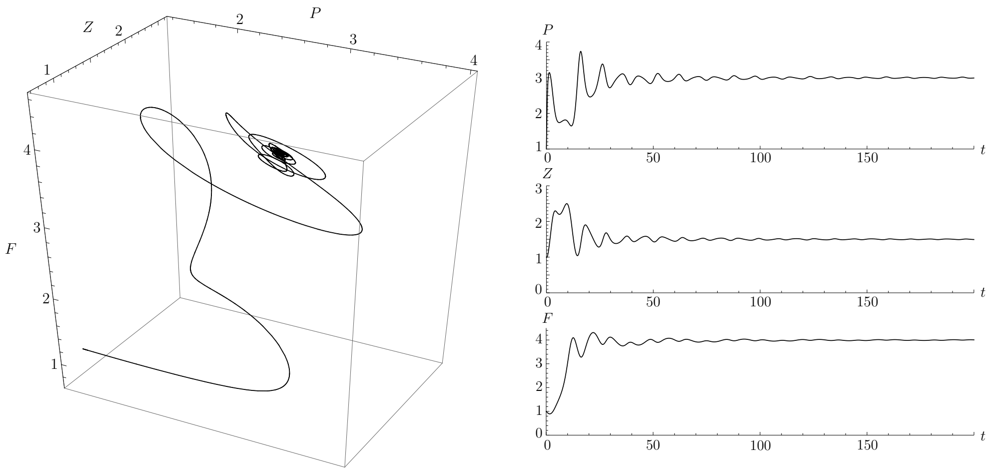

Figure 2: A solution curve of (E) with (5.2) is drawn in the left figure. The initial time t_0 is 0 and the initial point (P_0, Z_0, F_0) is $(1, 1, 1)$. The point $(P(t), Z(t), F(t))$ moves on the solution curve from the initial point $(1, 1, 1)$ toward the interior equilibrium $(3, 3/2, 4)$ according to increase of t. The right three figures show the movements of the components P, Z and F. The components P, Z and F oscillate up and down around 3, 3/2 and 4, respectively, and they finally approaches 3, 3/2 and 4, respectively.

It is clear that the functions f, g and h are positive and bounded, and

$$\frac{f(t)}{g(t)} = \frac{1+t}{3+2t} < \frac{1}{2} \quad \text{and} \quad t \geq 0.$$

Since

$$\ell(t) = \left(\ln \frac{f(t)}{g(t)}\right)' = \frac{1}{(1+t)(3+2t)} > 0$$

for $t \geq 0$, we see that $\ell_+(t) \equiv \ell(t)$ and $\ell_-(t) \equiv 0$. Hence, we have

$$\int_0^t \ell_+(s)ds = \ln \frac{1+t}{3+2t} < \ln \frac{1}{2} \quad \text{for} \quad t \geq 0.$$

It is also clear that $h \in \mathcal{F}_{[\text{WIP}]}$. Since we have confirmed that conditions (1.1), (1.5) and (1.6) hold, we can use Theorem 1.2.

Acknowledgements

The second author's work was supported in part by Grant-in-Aid for Scientific Research No. 17K05327 from the Japan Society for the Promotion of Science. The authors thank two anonymous reviewers for their valuable comments.

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