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Sufficient conditions for convergence of solutions of damped elliptic equations

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Abstract This paper gives several sets of sufficient conditions which guarantee that all radially symmetric solutions of

$$\operatorname{div}(D(u)\nabla u) + \frac{k(\|\mathbf{x}\|)}{\|\mathbf{x}\|}\,\mathbf{x}\cdot(D(u)\nabla u) + \omega^p |u|^{p-2}u = 0$$

converge to zero as $\|\mathbf{x}\| \to \infty$. Here, $\mathbf{x}$ is an N-dimensional vector in an exterior domain and $N \in \mathbb{N}\setminus\{1\}$; $D(u) = \|\nabla u\|^{p-2}$ with $p > 1$; k is a nonnegative and locally integrable function on $[a,\infty)$; ω is a positive constant. All of the obtained sufficient conditions have the advantage that it is possible to check relatively easily. In that sense, our results are practical enough. The relationships between those sufficient conditions are also clarified. To achieve our purpose, we discuss the asymptotic stability of the equilibrium of the equation

$$\left(|x'|^{p-2}x'\right)' + h(t)|x'|^{p-2}x' + \omega^p|x|^{p-2}x = 0,$$

where $h\colon [0,\infty) \to [0,\infty)$ is locally integrable.

Keywords Convergence of solutions · Damped elliptic equation · Radially symmetric solutions · Asymptotic stability · Growth condition · p-Laplacian

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1 Introduction

The nonlinear elliptic equation

$$\operatorname{div}(D(u)\nabla u) + f(\mathbf{x}, u, \nabla u) = 0$$

and various more general forms including this equation have been studied in a very broad field. Here, $\mathbf{x}$ is an N-dimensional vector with $N \ge 2$; $D(u)$ means $\|\nabla u\|^{p-2}$ with a real number $p > 1$; $\|\cdot\|$ is the usual Eclidean norm; ∇ is the usual nabla operator. For example, the subject of those researches are the behavior of weak solutions to the Dirichlet problem with boundary condition (Alkhutov and Borsuk [1]); the existence of a positive solution and a negative solution to the Dirichlet boundary value problem (Faraci et al. [7]); regularity and qualitative properties of weak solutions (Pucci and Servadei [20]); maximum principle and comparison theorems for weak solutions to the Dirichlet problem on complete Riemannian manifolds (Antonini et al. [2]). We can also find researches on the oscillation of (classical) solutions on an exterior domain in $\mathbb{R}^N$. Those results can be obtained by using the so-called generalized Riccati transformation, integral average techniques and Picone-type inequalities (for example, see [17, 18, 35–38, 40]).

One reason that the research field is wide seems to be that the steady solution of the reaction-diffusion equation

$$u_t = \operatorname{div}(D(u)\nabla u) + f(\mathbf{x}, u, \nabla u)$$

satisfies the above elliptic equations. The diffusion term $\operatorname{div}(D(u)\nabla u)$ is usually called the *p-Laplacian*. As is well known, the diffusion causes energy dissipation.

In this paper, we consider the equation

$$\operatorname{div}(D(u)\nabla u) + \frac{k(\|\mathbf{x}\|)}{\|\mathbf{x}\|}\,\mathbf{x}\cdot(D(u)\nabla u) + \omega^p \phi_p(u) = 0, \tag{1.1}$$

where $\mathbf{x}$ is in an exterior domain $G_a \overset{\text{def}}{=} \{\mathbf{x} \in \mathbb{R}^N : \|\mathbf{x}\| \ge a\}$ for some $a > 0$; k is a nonnegative and locally integrable function on $[a, \infty)$; ω is a positive constant; ϕ_p is a nonlinear function defined by

$$\phi_p(u) = \begin{cases} |u|^{p-2}u & \text{if } u \ne 0, \\ 0 & \text{if } u = 0. \end{cases}$$

In equation (1.1), the reaction term $f(\mathbf{x}, u, \nabla u)$ consists of two parts. The first part and the second part may be called a damping and a restoration, respectively.

If there is no damping, namely, the damping coefficient k is identically zero, then every radially symmetric solution u of (1.1) converges to zero as $\|\mathbf{x}\|$ tends to ∞ together with $\|\nabla u\|$, because the diffusion occurs energy loss. Then, what kind of influence does the damping have on convergence of solutions of (1.1)? As well as the diffusion, will the damping always promote convergence of solutions? The purpose of this paper is to answer these questions.

Let p^* be the conjugate number of p; namely,

$$\frac{1}{p} + \frac{1}{p^*} = 1.$$

Then p^* is also greater than 1. Note that ϕ_{p^*} is the inverse function of ϕ_p. Recently, Sugie and Minei [29, Theorem 1.1] have presented a necessary and sufficient condition for convergence of all radially symmetric solutions of quasilinear elliptic equations including equation (1.1). By applying their result to equation (1.1), we have the following result.

Theorem A *Suppose that*

$$\begin{aligned} &\text{there exists an } \varepsilon_0 > 0 \text{ and a } \delta_0 > 0 \text{ such that} \\ &|k(t) - k(s)| < \varepsilon_0 \quad \text{for all } t \ge a \text{ and } s \ge a \text{ with } |t-s| < \delta_0. \end{aligned} \tag{1.2}$$

Then, every radially symmetric solution u *of* (1.1) *satisfies the property that* $u(\mathbf{x})$ *and* $\|\nabla u(\mathbf{x})\|$ *tend to zero as* $\|\mathbf{x}\| \to \infty$ *if and only if*

$$\int_a^\infty \phi_{p^*}\left(\frac{\int_a^t e^{K(s)} ds}{e^{K(t)}}\right) dt = \infty, \tag{1.3}$$

where $K(t) = \int_a^t k(s)ds + (N-1)\log t$ *for* $t \ge a$.

If k is either uniformly continuous or bounded on $[a,\infty)$, then condition (1.2) is satisfied. Of course, the converse is not true. Condition (1.3) is a criterion related to the degree of growth of the damping coefficient k. We see that the growth condition (1.3) is satisfied when k has an upper bound $\bar{k}$ or the polynomial degree of k is less than or equal to $p-1$ (for the proof, see [39, Theorem 3.1]). On the other hand, if the degree of k is too large in the sense that condition (1.3) does not hold, then it can happen that a radially symmetric solution of (1.1) does not converge to zero. For example, let us consider the case that

$$k(\|\mathbf{x}\|) = \omega^p \|\mathbf{x}\|^{p-1}(1 + \|\mathbf{x}\|)^{p-1} + \frac{2p - N - 1}{\|\mathbf{x}\|}.$$

Then, equation (1.1) has a radially symmetric solution u satisfying

$$\big(u(\mathbf{x}), \nabla u(\mathbf{x})\big) = \left(\frac{1 + \|\mathbf{x}\|}{\|\mathbf{x}\|}, -\frac{\mathbf{x}}{\|\mathbf{x}\|^3}\right).$$

This radially symmetric solution does not converge to zero as $\|\mathbf{x}\| \to \infty$. Such a situation is caused by “overdamping”.

Condition (1.3) is necessary and sufficient to ensure that all radially symmetric solutions of (1.1) converge to zero, but it is difficult to ascertain whether condition (1.3) holds or not. To get rid of this inconvenience, we give other growth conditions concerning the damping coefficient k. Our main results are as follows.

Theorem 1.1 *Suppose that condition* (1.2) *holds. If*

$$\limsup_{t\to\infty} \frac{K(t)}{t^p} < \infty, \tag{1.4}$$

then every radially symmetric solution u of (1.1) *satisfies the property that $u(\mathbf{x})$ and $||\nabla u(\mathbf{x})||$ tend to zero as $||\mathbf{x}|| \to \infty$.*

Remark 1.1 Condition (1.4) is equivalent to the condition

$$\limsup_{t\to\infty} \frac{\phi_{p^*}(K(t))}{t^{p^*}} < \infty.$$

Theorem 1.2 *Suppose that condition* (1.2) *holds. If*

$$\sum_{i=m}^{\infty} \phi_{p^*}\left(\frac{1}{K(i+1)-K(i)}\right) = \infty$$

for any fixed integer $m \ge a$, then every radially symmetric solution u of (1.1) *satisfies the property that $u(\mathbf{x})$ and $||\nabla u(\mathbf{x})||$ tend to zero as $||\mathbf{x}|| \to \infty$.*

Let K^{-1} be the inverse function of K. Since $\lim_{t\to\infty} K(t) = \infty$, the inverse function K^{-1} is defined on $[0,\infty)$.

Theorem 1.3 *Suppose that condition* (1.2) *holds. If*

$$\sum_{n=1}^{\infty} \left(K^{-1}(n) - K^{-1}((n-1))\right)^{p^*} = \infty,$$

then every radially symmetric solution u of (1.1) *satisfies the property that $u(\mathbf{x})$ and $||\nabla u(\mathbf{x})||$ tend to zero as $||\mathbf{x}|| \to \infty$.*

2 Damped half-linear oscillators

Let u be any radially symmetric solution of (1.1), and let ξ be the function defined by $\xi(t) = u(\mathbf{x})$ and $t = ||\mathbf{x}|| \ge a$. Then, we have $\nabla u(\mathbf{x}) = \dfrac{\xi'(t)}{t}\mathbf{x}$, and therefore,

$$\begin{aligned}
\operatorname{div}(D(u(\mathbf{x}))\nabla u(\mathbf{x})) &= \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(||\nabla u(\mathbf{x})||^{p-2}\frac{\partial u}{\partial x_i}\right) \\
&= \left(|\xi'(t)|^{p-2}\xi'(t)\right)' + \frac{N-1}{t}|\xi'(t)|^{p-2}\xi'(t)
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{x}\cdot ||\nabla u(\mathbf{x})||^{p-2}\nabla u(\mathbf{x}) &= \sum_{i=1}^{N} x_i\,||\nabla u(\mathbf{x})||^{p-2}\frac{\partial u}{\partial x_i} \\
&= t\,|\xi'(t)|^{p-2}\xi'(t).
\end{aligned}$$

Hence, the function ξ is a solution of the second-order nonlinear differential equation

$$\left(\phi_p(x')\right)' + \left(k(t) + \frac{N-1}{t}\right)\phi_p(x') + \omega^p \phi_p(x) = 0, \tag{2.1}$$

where ϕ_p is the function given in Section 1. The only equilibrium of (2.1) is the origin $(x, x') = (0,0)$. The equilibrium is said to be *asymptotically stable* [AS] if

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} x'(t) = 0$$

for every solution x. As the above-mentioned transformation shows, the convergence of radially symmetric solutions of (1.1) is reduced to the asymptotic stability of the equilibrium of (2.1).

Hereafter, we consider the following more general form than equation (2.1),

$$\left(\phi_p(x')\right)' + h(t)\phi_p(x') + \omega^p \phi_p(x) = 0, \tag{2.2}$$

where h is a nonnegative and locally integrable function on $[a, \infty)$. Equation (2.2) is often called the *damped half-linear oscillator* when $p \neq 2$. It is well-known that the solution space of the damped half-linear oscillator is homogeneous, but not additive.

To describe some results on the asymptotic stability of the equilibrium of (2.2), we need to define the following family of functions. A function $h\colon [a,\infty) \to [0,\infty)$ is said to belong to $\mathcal{F}_{[\mathrm{WIP}]}$ if

$$\sum_{n=1}^{\infty} \int_{\tau_n}^{\sigma_n} h(t)dt = \infty$$

for every pair of sequences $\{\tau_n\}$ and $\{\sigma_n\}$ satisfying $\tau_n < \sigma_n < \tau_{n+1}$,

$$\liminf_{n\to\infty} (\sigma_n - \tau_n) > 0 \quad \text{and} \quad \limsup_{n\to\infty} (\tau_{n+1} - \sigma_n) < \infty.$$

The concept of the weak integral positivity was first published in Hatvani [8]. Even if $\liminf_{t\to\infty} h(t) = 0$, the damping coefficient h is allowed to belong to $\mathcal{F}_{[\mathrm{WIP}]}$. For example, the functions $1/t$ and $\sin^2 t/t$ belong to $\mathcal{F}_{[\mathrm{WIP}]}$ (for the proof, see [28, Proposition 2.1]). However, the function $1/t^2$ no longer belongs to $\mathcal{F}_{[\mathrm{WIP}]}$. From these fact, we see that the weak integral positivity plays a role in prohibiting too fast decline of the damping coefficient h.

Let

$$H(t) = \int_a^t h(s)ds$$

for $t \geq a$. Using a growth condition of Smith-type, Sugie and Minei [29, Theorem 2.5] gave a necessary and sufficient condition for the equilibrium of (2.2) to be asymptotically stable (refer to [21] for Smith's criterion).

Theorem B *Suppose that*

$$\begin{aligned}&\textit{there exists an } \varepsilon_0 > 0 \textit{ and a } \delta_0 > 0 \textit{ such that}\\ &|h(t) - h(s)| < \varepsilon_0 \ \textit{ for all } t \geq a \textit{ and } s \geq a \textit{ with } |t-s| < \delta_0\end{aligned} \tag{2.3}$$

and h belongs to $\mathcal{F}_{[\mathrm{WIP}]}$. Then, the equilibrium of (2.2) *is asymptotically stable if and only if*

$$\int_a^\infty \phi_{p^*}\left(\frac{\int_a^t e^{H(s)}ds}{e^{H(t)}}\right)dt = \infty. \tag{S$_p$}$$

As mentioned in Section 1, the Smith-type condition (S_p) is asuumed to prohibit too growth of the damping coefficient h. In general, however, it is hard to check whether the growth condition (S_p) holds or not. For this reason, we present different growth conditions which are easy to check.

Theorem 2.1 *Suppose that condition* (2.3) *holds and h belongs to $\mathcal{F}_{[\mathrm{WIP}]}$. If*

$$\limsup_{t\to\infty}\frac{H(t)}{t^p} < \infty, \tag{A$_p$}$$

then the equilibrium of (2.2) *is asymptotically stable.*

Theorem 2.2 *Suppose that condition* (2.3) *holds and h belongs to $\mathcal{F}_{[\mathrm{WIP}]}$. If*

$$\begin{aligned}&H(i+1) > H(i) \ \text{ for any integer } i \ge a \text{ and}\\ &\text{there exists an integer } m \ge a \text{ such that } \sum_{i=m}^{\infty}\phi_{p^*}\left(\frac{1}{H(i+1)-H(i)}\right) = \infty,\end{aligned} \tag{H$_p$}$$

then the equilibrium of (2.2) *is asymptotically stable.*

If h belongs to $\mathcal{F}_{[\mathrm{WIP}]}$, then H diverges to ∞. Hence, we can define

$$H^{-1}(s) = \min\{t \in [a,\infty) : H(t) \ge s\}$$

for all $s \in [0,\infty)$. The function H^{-1} is a generalization of the usual inverse function.

Theorem 2.3 *Suppose that condition* (2.3) *holds and h belongs to $\mathcal{F}_{[\mathrm{WIP}]}$. If*

$$\sum_{n=1}^{\infty}\Big(H^{-1}(n) - H^{-1}((n-1))\Big)^{p^*} = \infty, \tag{D$_p$}$$

then the equilibrium of (2.2) *is asymptotically stable.*

Recall that the nonlinear elliptic equation (1.1) is reduced to equation (2.1) which is a special case of (2.2). Since the damping coefficient h of (2.2) corresponds to $k(t) + (N-1)/t$ in equation (2.1), condition (1.2) implies that

$$\begin{aligned}|h(t) - h(s)| &\le |k(t) - k(s)| + \left|\frac{N-1}{t} - \frac{N-1}{s}\right|\\ &< \varepsilon_0 + \frac{N-1}{a^2}\delta_0 \stackrel{\text{def}}{=} \varepsilon_1.\end{aligned}$$

for $t \ge a$ and $s \ge a$ with $|t-s| < \delta_0$. Hence, condition (2.3) is satisfied with ε_1 and δ_0. Since $h(t) > (N-1)/t$ for $t \ge a$, the damping coefficient h naturally belongs to $\mathcal{F}_{[\mathrm{WIP}]}$. In this special case, we have

$$K(t) = \int_a^t k(s)ds + (N-1)\log t = \int_a^t h(s)ds = H(t)$$

for $t \ge a$ and

$$K(i+1) > K(i) \quad \text{for any integer } \; i \ge a.$$

Hence, Theorems 1.1, 1.2 and 1.3 are derived from Theorems 2.1, 2.2 and 2.3, respectively.

It would be meaningful to touch a little bit on the background of this research. Equation (2.2) contains naturally the damped harmonic oscillator

$$x'' + h(t)x' + \omega^2 x = 0. \tag{2.4}$$

Many attempts have been made to provide sufficient conditions and necessary conditions for the asymptotic stability of (2.4) (or more general forms). For example, refer to [3–6, 8, 10–16, 19, 21–27, 30–34]. Among them, it would be allowed to say that Levin and Nohel [16, Theorem 1] was a pioneering work. They dealt with a little more general equations than equation (2.4). To apply their result, we have to assume the existence of an upper bound and a positive lower bound of h. At almost the same period of time, Smith [21, Theorems 1 and 2] proved that condition

$$\int_a^\infty \frac{\int_a^t e^{H(s)}ds}{e^{H(t)}}dt = \infty. \tag{S$_2$}$$

is a necessary and sufficient condition for the equilibrium of (2.4) to be asymptotically stable, under the assumption that the damping coefficient h has a positive lower bound. In other words, he removed the upper limit of h and gave a criteria for the degree of divergence of h that ensures the asymptotic stability of the equilibrium of (2.4). Condition (S$_2$) has a form of double integral. We call this double integral a growth condition in this paper. Incidentally, we can rewrite this growth condition to

$$\int_a^\infty u(t)dt = -\infty,$$

where u is a solution of the first-order linear differential equation

$$u' + h(t)u + 1 = 0$$

satisfying the initial condition $u(a) = 0$. After that, Artstein and Infante [3] gave the different growth condition

$$\limsup_{t\to\infty} \frac{H(t)}{t^2} < \infty \tag{A$_2$}$$

for the asymptotic stability of (2.4). They also showed that $H(t)/t^2$ cannot be replaced by $H(t)/t^{2+\varepsilon}$ for any $\varepsilon > 0$ in their condition. Condition (A$_2$) is easier to handle than condition (S_2), but it is not a necessary and sufficient condition. For example, consider the case that $h(t) = t\log(1+t)$ for $t \ge a = 1$. Then, we have

$$H(t) = \int_1^t s\log(1+s)ds = \frac{1}{2}(1+t)^2\log(1+t) - \frac{1}{4}(1+t)^2 - (1+t)\log(1+t) + t.$$

Hence, condition (A_2) is not satisfied. On the other hand, from Theorem 3.3 in [39] it turns out that condition (S_2) holds in this case (see also [4, Corollary 7]). We have to mention results given by Hatvani, Krisztin and Totik [13, Theorem 1.1] as well. They clarified that the growth condition (S_2) is equivalent to the discrete condition

$$\sum_{n=1}^{\infty}\left(H^{-1}(n) - H^{-1}(n-1)\right)^2 = \infty \qquad (\mathrm{D}_2)$$

provided that $H(t)$ diverges to ∞ as $t \to \infty$. However, please note that these are not the original form. Using this result, they also gave several sufficient conditions for the asymptotic stability of (2.4). As one of them, we can cite the condition that

$$\begin{aligned}&H(i+1) > H(i) \text{ for any integer } i \ge a \text{ and}\\ &\text{there exists an integer } m \ge a \text{ such that } \sum_{i=m}^{\infty}\frac{1}{H(i+1)-H(i)} = \infty\end{aligned} \qquad (\mathrm{H}_2)$$

(see Corollary 3.6 in [13]).

As can be seen from the above, the growth conditions (S_p), (A_p), (H_p), and (D_p) are natural extensions of (S_2), (A_2), (H_2), and (D_2), respectively.

3 Proofs of Theorems 2.1–2.3

To clarify the relationships between the growth conditions defined in Section 2, we prepare the following propositions.

Proposition 3.1 *Suppose that $H(i+1) > H(i)$ for any integer $i \ge a$. Then condition (A_p) implies condition (H_p).*

Proposition 3.2 *Suppose that $\lim_{t\to\infty} H(t) = \infty$. Then condition (H_p) implies condition (D_p).*

Proposition 3.3 *Suppose that $\lim_{t\to\infty} H(t) = \infty$. Then condition (A_p) implies condition (D_p).*

Proposition 3.4 *Suppose that $\lim_{t\to\infty} H(t) = \infty$. Then condition (D_p) implies condition (S_p).*

Remark 3.1 In order for condition (D_p) to hold, $H^{-1}(n)$ has to exist for each $n \in \mathbb{N}$. Hence, in Proposition 3.4, it is guaranteed that H is a monotone divergent function.

Proposition 3.1 can be easily proved by using the following lemma that is a generalization of an idea in Artstein and Infante [3].

Lemma 3.5 *Let* $\{a_n\}$ *be a sequence. If*

$$\begin{aligned}&\textit{there exist a } K>0 \textit{ and an } m\in\mathbb{N} \textit{ such that}\\ &a_n>0 \ \textit{ for } n\ge m \ \textit{ and } \sum_{i=m}^{\ell} a_i \le K(\ell+1)^p \ \textit{ for } \ell\ge m,\end{aligned}$$

then

$$\sum_{i=m}^{\infty}\phi_{p^*}\left(\frac{1}{a_i}\right)=\infty.$$

Proof of Lemma 3.5 For any fixed integer $n\ge m$, let $b_j=a_{2^n+j}>0$ with $j=m,\dots,2^n$. Then, by assumption, we have

$$\sum_{j=m}^{2^n} b_j=\sum_{j=m}^{2^n} a_{2^n+j}=\sum_{i=2^n+m}^{2^{n+1}} a_i<\sum_{i=m}^{2^n+m-1} a_i+\sum_{i=2^n+m}^{2^{n+1}} a_i=\sum_{i=m}^{2^{n+1}} a_i\le K\left(2^{n+1}+1\right)^p.$$

Hence, it follows from the Hölder inequality that

$$\begin{aligned}2^n-m+1&=\sum_{j=m}^{2^n}\left(b_j\right)^{1/p}\left(\frac{1}{b_j}\right)^{1/p}\le\left(\sum_{j=m}^{2^n}\left(\left(b_j\right)^{1/p}\right)^p\right)^{1/p}\left(\sum_{j=m}^{2^n}\left(\left(\frac{1}{b_j}\right)^{1/p}\right)^{p^*}\right)^{1/p^*}\\ &=\left(\sum_{j=m}^{2^n} b_j\right)^{1/p}\left(\sum_{j=m}^{2^n}\phi_{p^*}\left(\frac{1}{b_j}\right)\right)^{1/p^*}<K^{1/p}\left(2^{n+1}+1\right)\left(\sum_{j=m}^{2^n}\phi_{p^*}\left(\frac{1}{b_j}\right)\right)^{1/p^*}.\end{aligned}$$

Thus, we obtain

$$\sum_{j=m}^{2^n}\phi_{p^*}\left(\frac{1}{b_j}\right)>\left(\frac{2^n-m+1}{K^{1/p}\left(2^{n+1}+1\right)}\right)^{p^*}.$$

We therefore conclude that

$$\begin{aligned}\sum_{i=m}^{\infty}\phi_{p^*}\left(\frac{1}{a_i}\right)&>\sum_{k=0}^{2^m-1}\phi_{p^*}\left(\frac{1}{a_{m+k}}\right)+\sum_{n=m}^{\infty}\left(\sum_{j=m}^{2^n}\phi_{p^*}\left(\frac{1}{a_{2^n+j}}\right)\right)>\sum_{n=m}^{\infty}\left(\sum_{j=m}^{2^n}\phi_{p^*}\left(\frac{1}{b_j}\right)\right)\\ &>\sum_{n=m}^{\infty}\left(\frac{2^n-m+1}{K^{1/p}\left(2^{n+1}+1\right)}\right)^{p^*}=\frac{1}{K^{p^*/p}}\sum_{n=m}^{\infty}\left(\frac{1-(m-1)/2^n}{2+1/2^n}\right)^{p^*}.\end{aligned}$$

Since $\dfrac{1-(m-1)/2^n}{2+1/2^n}\nearrow\dfrac{1}{2}$ as $n\to\infty$, we see that

$$\sum_{i=m}^{\infty}\phi_{p^*}\left(\frac{1}{a_i}\right)=\infty.$$

This completes the proof. □

Using Lemma 3.5, we give the proof of Proposition 3.1.

Proof of Proposition 3.1 From condition (A_p), we can find a $K > 0$ and a $T \geq a$ such that

$$H(t) < Kt^p \quad \text{for } t \geq T. \tag{3.1}$$

Let m be an integer satisfying $m \geq T$. Define

$$a_i = H(i+1) - H(i)$$

for any integer $i \geq m$. Then, from the assumption of H we see that $a_n > 0$ for $n \geq m$. By (3.1), we have

$$\sum_{i=m}^{\ell} a_i = H(\ell+1) - H(m) \leq H(\ell+1) < K(\ell+1)^p$$

for $\ell \geq m$. Hence, from Lemma 11 we obtain

$$\sum_{i=m}^{\infty} \phi_{p^*}\left(\frac{1}{H(i+1)-H(i)}\right) = \sum_{i=m}^{\infty} \phi_{p^*}\left(\frac{1}{a_i}\right) = \infty;$$

namely, condition (H_p). The proof is complete. □

Proof of Proposition 3.2 By the assumption of H, the generalized inverse function $H^{-1}(n)$ exists for each $n \in \mathbb{N}$. Let $t_0 = a$ and $t_n = H^{-1}(n)$. Then, the sequence $\{t_n\}$ is strictly increasing with respect to $n \in \mathbb{N}$ and diverges to ∞ as $n \to \infty$. Define $\Delta t_n = t_n - t_{n-1} \geq 0$ for each $n \in \mathbb{N}$.

If Δt_n does not converge to 0 as $n \to \infty$, then condition (D_p) obviously holds. Consider the case that Δt_n converges to 0 as $n \to \infty$. For any integer $i \geq a$, let $n_i = \min\{n \in \mathbb{N} : t_n \geq i\}$ and $k_i = \max\{k \in \mathbb{N} : t_{n_i+k} \leq i+1\}$; that is, n_i is the smallest positive integer satisfying $t_{n_i} \geq i \geq a$ and k_i is the largest positive integer satisfying $t_{n_i+k_i} \leq i+1$. Since $\Delta t_n \to 0$ as $n \to \infty$, there exists an integer $N \geq a$ such that $i \geq N$ implies $\Delta t_{n_i} < 1/3$. Hence, we can estimate that

$$\begin{aligned} 1 = i - (i-1) &= i - t_{n_i+k_i} + t_{n_i+k_i} - t_{n_i} + t_{n_i} - (i-1) \\ &< \Delta t_{n_i+k_i+1} + t_{n_i+k_i} - t_{n_i} + \Delta t_{n_i} \\ &< \frac{1}{3} + t_{n_i+k_i} - t_{n_i} + \frac{1}{3} \end{aligned}$$

for $i \geq N$, namely,

$$t_{n_i+k_i} - t_{n_i} > \frac{1}{3} \quad \text{for } i \geq N. \tag{3.2}$$

Taking into account that H is an increasing function on $[a, \infty)$ and $a \leq i \leq t_{n_i} < t_{n_i+k_i} \leq i+1$, we have

$$H(i+1) - H(i) \geq H(t_{n_i+k_i}) - H(t_{n_i}) = n_i + k_i - n_i = k_i.$$

Hence, we obtain

$$\phi_{p^*}\left(\frac{1}{k_i}\right) \ge \phi_{p^*}\left(\frac{1}{H(i+1)-H(i)}\right) \quad \text{for each integer } i \ge a. \tag{3.3}$$

From the Hölder inequality, we see that

$$\begin{aligned}(t_{n_i+k_i}-t_{n_i})^{p^*} &= \left(\sum_{j=1}^{k_i} \Delta t_{n_i+j}\right)^{p^*} \le \left(\sum_{j=1}^{k_i} 1^p\right)^{p^*/p} \sum_{j=1}^{k_i} (\Delta t_{n_i+j})^{p^*} \\ &= \phi_{p^*}(k_i) \sum_{j=1}^{k_i} (\Delta t_{n_i+j})^{p^*} \end{aligned} \tag{3.4}$$

for each integer $i \ge a$. Using (3.2)–(3.4), we conclude that

$$\begin{aligned}\sum_{n=1}^{\infty}\left(H^{-1}(n)-H^{-1}(n-1)\right)^{p^*} &= \sum_{n=1}^{\infty}(\Delta t_n)^{p^*} \ge \sum_{i=N}^{\infty}\sum_{j=1}^{k_i}(\Delta t_{n_i+j})^{p^*} \\ &\ge \sum_{i=N}^{\infty}\phi_{p^*}\left(\frac{1}{k_i}\right)(t_{n_i+k_i}-t_{n_i})^{p^*} \\ &\ge \left(\frac{1}{3}\right)^{p^*}\sum_{i=N}^{\infty}\phi_{p^*}\left(\frac{1}{H(i+1)-H(i)}\right).\end{aligned}$$

Hence, condition (H_p) implies condition (D_p). □

Proof of Proposition 3.3 Recall that from condition (A_p) it follows that inequality (3.1) holds for a $K > 0$ and a $T > a$. Let $\{\tau_i\}$ be an strictly increasing sequence satisfying $\tau_1 \ge T+1$ and $\tau_{i+1} = 2\tau_i$ for each $i \in \mathbb{N}$. Then, from (3.1) we obtain

$$\tau_{i+1} - \tau_i = \tau_i \ge \tau_1 \ge T+1 > 1 \tag{3.5}$$

and

$$\begin{aligned}(H(\tau_{i+1})-H(\tau_i))^{p^*-1} &= \left(\int_{\tau_i}^{\tau_{i+1}} h(s)ds\right)^{p^*-1} \le \left(\int_0^{\tau_{i+1}} h(s)ds\right)^{p^*-1} \\ &< \left(K\tau_{i+1}^p\right)^{p^*-1} = \phi_{p^*}(K)\,\tau_{i+1}^{p^*} = \phi_{p^*}(K)\,(2(\tau_{i+1}-\tau_i))^{p^*} \\ &= 2^{p^*}\phi_{p^*}(K)\,(\tau_{i+1}-\tau_i)^{p^*}\end{aligned} \tag{3.6}$$

for each $i \in \mathbb{N}$.

Since H is a monotone divergent function, $H^{-1}(n)$ exists for each $n \in \mathbb{N}$. As in the proof of Proposition 3.2, we define $t_0 = a$, $t_n = H^{-1}(n)$ and $\Delta t_n = t_n - t_{n-1}$. Then, we see that the sequence $\{t_n\}$ is strictly increasing and diverges to ∞ as $n \to \infty$, and $\Delta t_n \ge 0$ for each $n \in \mathbb{N}$. Since condition (D_p) inevitably holds if Δt_n does not converge to 0 as $n \to \infty$, we consider the opposite case. Then, there exists an $N \in \mathbb{N}$ such that

$$\Delta t_n < 1/3 \quad \text{for } n \ge N. \tag{3.7}$$

Let $n_i = \min\{n \geq N : t_n \geq \tau_i\}$ and $k_i = \max\{k \in \mathbb{N} : t_{n_i+k} \leq \tau_{i+1}\}$. Then, we have

$$t_{n_i-1} < \tau_i \leq t_{n_i} < t_{n_i+k_i} \leq \tau_{i+1} < t_{n_i+k_i+1}.$$

From (3.7) it follows that

$$t_{n_i} - \tau_i < \Delta t_{n_i} < \frac{1}{3} \quad \text{and} \quad \tau_{i+1} - t_{n_i+k_i} < \Delta t_{n_i+k_i+1} < \frac{1}{3}$$

for each $i \in \mathbb{N}$. Hence, we obtain

$$t_{n_i+k_i} - t_{n_i} = -(\tau_{i+1} - t_{n_i+k_i}) + \tau_{i+1} - \tau_i - (t_{n_i} - \tau_i) > \tau_{i+1} - \tau_i - \frac{2}{3}$$

for each $i \in \mathbb{N}$. From (3.5), we see that

$$t_{n_i+k_i} - t_{n_i} > \tau_{i+1} - \tau_i - \frac{2}{3}(\tau_{i+1} - \tau_i) = \frac{1}{3}(\tau_{i+1} - \tau_i) \tag{3.8}$$

for each $i \in \mathbb{N}$. Since H is an increasing function on $[a, \infty)$ and $a < \tau_i \leq t_{n_i} < t_{n_i+k_i} \leq \tau_{i+1}$, we have

$$H(\tau_{i+1}) - H(\tau_i) \geq H(t_{n_i+k_i}) - H(t_{n_i}) = n_i + k_i - n_i = k_i.$$

Hence, we obtain

$$\phi_{p^*}\left(\frac{1}{k_i}\right) \geq \frac{1}{(H(\tau_{i+1}) - H(\tau_i))^{p^*-1}} \quad \text{for each } i \in \mathbb{N}. \tag{3.9}$$

Using the Hölder inequality with (3.6), (3.8) and (3.9), we conclude that

$$\begin{aligned}
\sum_{n=1}^{\infty}\left(H^{-1}(n) - H^{-1}(n-1)\right)^{p^*} &= \sum_{n=1}^{\infty}(\Delta t_n)^{p^*} \geq \sum_{i=1}^{\infty}\sum_{j=1}^{k_i}(\Delta t_{n_i+j})^{p^*} \\
&\geq \sum_{i=1}^{\infty}\phi_{p^*}\left(\frac{1}{k_i}\right)(t_{n_i+k_i} - t_{n_i})^{p^*} \\
&\geq \left(\frac{1}{3}\right)^{p^*}\sum_{i=1}^{\infty}\frac{(\tau_{i+1} - \tau_i)^{p^*}}{(H(\tau_{i+1}) - H(\tau_i))^{p^*-1}} \\
&> \left(\frac{1}{6}\right)^{p^*}\sum_{i=1}^{\infty}\frac{1}{\phi_{p^*}(K)} = \infty.
\end{aligned}$$

Hence, condition (D_p) holds. □

Proof of Proposition 3.4 Let $t_0 = a$, $t_n = H^{-1}(n)$ and $\Delta t_n = t_n - t_{n-1}$ for each $n \in \mathbb{N}$. From the monotone divergence of H, we see that the sequence $\{t_n\}$ is strictly increasing and diverges to ∞ as $n \to \infty$. Hence, $\Delta t_n \ge 0$ and $t_{n-1} \ge a$ for each $n \in \mathbb{N}$. Since H is an increasing function on $[a, \infty)$, we can estimate that

$$\begin{aligned}
\int_a^\infty \phi_{p^*}\left(\frac{\int_a^t e^{H(s)}ds}{e^{H(t)}}\right)dt &= \int_a^\infty e^{-(p^*-1)H(t)}\left(\int_a^t e^{H(s)}ds\right)^{p^*-1}dt \\
&\ge \sum_{n=1}^\infty \int_{t_{n-1}}^{t_n} e^{-(p^*-1)H(t)}\left(\int_{t_{n-1}}^t e^{H(s)}ds\right)^{p^*-1}dt \\
&\ge \sum_{n=1}^\infty \int_{t_{n-1}}^{t_n} e^{-(p^*-1)H(t_n)}\left(e^{H(t_{n-1})}(t-t_{n-1})\right)^{p^*-1}dt \\
&= \sum_{n=1}^\infty e^{-(p^*-1)(H(t_n)-H(t_{n-1}))}\int_{t_{n-1}}^{t_n}(t-t_{n-1})^{p^*-1}dt \\
&= \frac{e^{-(p^*-1)}}{p^*}\sum_{n=1}^\infty\left(H^{-1}(n)-H^{-1}((n-1))\right)^{p^*}.
\end{aligned}$$

Hence, condition (D_p) implies condition (S_p). □

We can sum up Theorem B and Propositions 3.1–3.4 as follows.

$$\begin{array}{c}
H(i+1) > H(i) \text{ for any integer } i \ge a \\
\downarrow \\
(\mathrm{A_p}) \Longrightarrow (\mathrm{H_p}) \\
\lim_{t\to\infty} H(t) = \infty \longrightarrow \Downarrow \quad \swarrow\!\!\!\swarrow \longleftarrow \lim_{t\to\infty} H(t) = \infty \\
(\mathrm{D_p}) \Longrightarrow (\mathrm{S_p}) \Longleftrightarrow [\mathrm{AS}] \\
\uparrow \qquad\qquad \uparrow \\
\lim_{t\to\infty} H(t) = \infty \quad (2.3)\ \&\ h \in \mathcal{F}_{[\mathrm{WIP}]}
\end{array}$$

Fig. 1 The marks "$\longrightarrow$", "$\Longrightarrow$", "$\Longleftrightarrow$" and [AS] mean "addition to", "implies", "if and only if" and the asymptotic stability of (2.2), respectively.

Taking into account that $h \in \mathcal{F}_{[\mathrm{WIP}]}$ implies $\lim_{t\to\infty} H(t) = \infty$, we can give the proofs of Theorems 2.1–2.3 by combining Propositions 3.1–3.4 with Theorem B (we omit the details).

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Appendix – About condition (1.2)

Here, we examine condition (1.2) from several angles. If k is uniformly continuous on $[a,\infty)$; namely, for each $\varepsilon > 0$, there is a $\delta(\varepsilon) > 0$ such that $|k(t) - k(s)| < \varepsilon$ for all $t \ge a$ and $s \ge a$ with $|t - s| < \delta$, then condition (1.2) is satisfied. In fact, let ε_0 be any positive fixed number. Then we can find $\delta_0 = \delta(\varepsilon_0)$ so that $|k(t) - k(s)| < \varepsilon_0$ for all $t \ge 0$ and $s \ge 0$ with $|t - s| < \delta_0$. Hence, the uniform continuity of k implies condition (1.2). Of course, the converse is not always true. If k is a bounded function on $[a,\infty)$, then there exists a $\bar{k} > 0$ such that $0 \le k(t) \le \bar{k}$ for $t \ge a$. Hence, $|k(t) - k(s)| \le |k(t)| + |k(s)| \le 2\bar{k}$ for all $t \ge a$ and $s \ge a$, and therefore, the bounded function k satisfies condition (1.2) with respect to $\varepsilon_0 = 2\bar{k}$ and any $\delta_0 > 0$. Condition (1.2) may be satisfied even if k is unbounded and not uniformly continuous. For example, consider the case that

$$k(t) = \left(\sqrt{t} + \sin(t^2) + 1\right)\left(\sin\sqrt{t}\right)^2.$$

Then it is clear that k is a nonnegative and continuous function on $[1,\infty)$. However, the function k is not bounded. Since k contains the term $\sin(t^2)$, it is not uniformly continuous. On the other hand, condition (1.2) is satisfied. In fact, let $\varepsilon_0 = 5$ and $\delta_0 = 1$. Then, whenever $t \ge 1$, $s \ge 1$ and $|t - s| < \delta_0 = 1$, we have

$$\begin{aligned}
|k(t) - k(s)| &\le \left|\sqrt{t}\left(\sin\sqrt{t}\right)^2 - \sqrt{s}\left(\sin\sqrt{s}\right)^2\right| \\
&\quad + \left|\sin\left(t^2\right)\left(\sin\sqrt{t}\right)^2 - \sin\left(s^2\right)\left(\sin\sqrt{s}\right)^2\right| \\
&\quad + \left|\left(\sin\sqrt{t}\right)^2 - \left(\sin\sqrt{s}\right)^2\right| \\
&\le \left|\int_s^t \frac{d}{d\tau}\sqrt{\tau}\left(\sin\sqrt{\tau}\right)^2 d\tau\right| + 4 \\
&= \left|\int_s^t \left(\frac{\left(\sin\sqrt{\tau}\right)^2}{2\sqrt{\tau}} + \frac{\sin\left(2\sqrt{\tau}\right)}{2}\right) d\tau\right| + 4 \\
&\le |t - s| + 4 < 5 = \varepsilon_0.
\end{aligned}$$

Even if condition (1.2) is satisfied, the damping coefficient k does not always belong to $\mathcal{F}_{[\mathrm{WIP}]}$. For example, consider the case that

$$k(t) = \begin{cases} \alpha_{n+1}(t - n) + n^{p-1} & \text{if } n - 1/2^{n+1} \le t \le n, \\ -\beta_{n+1}(t - n) + n^{p-1} & \text{if } n < t \le n + 1/2^{n+1}, \\ 1/(1+t) & \text{otherwise}, \end{cases} \tag{A.1}$$

where

$$\alpha_n = 2^n\left((n-1)^{p-1} - \frac{2^n}{2^n n - 1}\right) \text{ and } \beta_n = 2^n\left((n-1)^{p-1} - \frac{2^n}{2^n n + 1}\right).$$

Since

$$\begin{aligned}\alpha_{n+1}(t-n)+n^{p-1}-\frac{1}{1+t} &\geq -\frac{\alpha_{n+1}}{2^{n+1}}+n^{p-1}-\frac{1}{1+n-1/2^{n+1}}\\ &= -n^{p-1}+\frac{2^{n+1}}{2^{n+1}(n+1)-1}+n^{p-1}-\frac{1}{1+n-1/2^{n+1}}=0\end{aligned}$$

for $n-1/2^{n+1}\leq t\leq n$, and

$$\begin{aligned}-\beta_{n+1}(t-n)+n^{p-1}-\frac{1}{1+t} &\geq -\frac{\beta_{n+1}}{2^{n+1}}+n^{p-1}-\frac{1}{1+n+1/2^{n+1}}\\ &= -n^{p-1}+\frac{2^{n+1}}{2^{n+1}(n+1)+1}+n^{p-1}-\frac{1}{1+n+1/2^{n+1}}=0\end{aligned}$$

for $n<t\leq n+1/2^{n+1}$, we see that $k(t)\geq 1/(1+t)$ for $t\geq 0$. From the fact that $1/(1+t)\in \mathcal{F}_{[\mathrm{WIP}]}$, we see that k also belongs to $\mathcal{F}_{[\mathrm{WIP}]}$. However, in this case, condition (1.2) does not hold. In fact, let $a=3/4$, $t_n=n$ and $s_n=n-1/2^{n+1}$. Then, it is clear that $t_n>s_n\geq a$, and

$$t_n-s_n\to 0 \quad \text{and} \quad k(t_n)-k(s_n)=n^{p-1}-\frac{1}{1+n-1/2^{n+1}}\to\infty \quad \text{as } n\to\infty.$$

Hence, there are no constants ε_0 and δ_0 satisfying condition (1.2). Note that $k(t)\leq t^{p-1}$ for $t\geq a=3/4$. Hence, the function k given by (A.1) satisfies the growth condition (S_p).

Conversely, even if k belongs to $\mathcal{F}_{[\mathrm{WIP}]}$, condition (1.2) is not always satisfied. For example, consider the case that

$$k(t)=\begin{cases}1 & \text{if } 2(n-1)\leq t<2n-1,\\ 0 & \text{if } 2n-1\leq t<2n\end{cases} \tag{A.2}$$

for $n\in\mathbb{N}$. In this case, the damping coefficient k is piecewise continuous on $[0,\infty)$. It is clear that condition (1.2) is satisfied with any $\varepsilon_0>1$ and any $\delta_0>0$. However, k does not belong to $\mathcal{F}_{[\mathrm{WIP}]}$. In fact, let $\tau_n=2n-1$ and $\sigma_n=2n$. Then we see that $\tau_n<\sigma_n<\tau_{n+1}$, $\sigma_n-\tau_n=\tau_{n+1}-\sigma_n=1$ and

$$\sum_{n=1}^{\infty}\int_{\tau_n}^{\sigma_n}k(t)dt=0.$$

Since the 'on-off' switching function k given by (A.2) is bounded on $[0,\infty)$, it satisfies the growth condition (S_p).

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