Philos-type oscillation criteria for linear differential equations with impulsive effects

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Abstract

The present paper deals with the problem of oscillation for a second-order linear differential equation in which an impulsive effect is considered. This equation can be regarded as an equation of motion in which the moving speed of a mass point changes sharply by some influence. It is proved that there is a case that the mass point may oscillate due to the influence of the impulsive effect even if the mass point does not oscillate in the model removing the impulsive effect. It is also shown that the obtained results extend some previous ones through the use of an example.

Key words: Oscillation problem; Impulse; Riccati transformation; Integral averaging technique

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1. Introduction

We consider the second-order linear impulsive differential equation

$$\begin{aligned} &x'' + a(t)x = 0, \qquad t \neq \theta_k; \\ &\Delta x'(\theta_k) + b_k x(\theta_k) = 0, \end{aligned} \tag{1.1}$$

where $' = d/dt$; the sequence $\{\theta_k\}_{k\in\mathbb{N}}$ is strictly increasing and satisfies that $\theta_k \geq t_0$ for some $t_0 > 0$ and θ_k tends to ∞ as $k \to \infty$; $\Delta z(\theta_k) = z(\theta_k^+) - z(\theta_k^-)$; the coefficient $a\colon [t_0,\infty) \to \mathbb{R}$ is *left piecewise continuous*; namely, a is continuous on intervals $[t_0,\theta_1)$ and (θ_k,θ_{k+1}), the left-side limit $a(\theta_k^-)$ and the right-side limit $a(\theta_k^+)$ of a exist, and a satisfies $a(\theta_k) = a(\theta_k^-)$ for all $k \in \mathbb{N}$; the coefficient $\{b_k\}$ is a sequence of real numbers. A function x is said to be a *solution* of (1.1) if x is continuous on the interval $[t_0,\infty)$, x' is left piecewise continuous on the interval $[t_0,\infty)$ and equation (1.1) is satisfied for all $t \geq t_0$. A nontrivial solution of (1.1) is said to be *oscillatory* if it has arbitrarily large zeros, and

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otherwise it is said to be *nonoscillatory*. According to the custom, equation (1.1) is called *oscillatory* if all nontrivial solutions are oscillatory.

The second equation of (1.1) represents an impulsive effect. If $b_k = 0$ for all $k \in \mathbb{N}$, the impulsive effect disappears and equation (1.1) becomes a linear differential equation of the normal form

$$x'' + a(t)x = 0, \qquad t \geq t_0. \tag{1.2}$$

A large number of results have been reported on the oscillation of solutions of (1.2) since a long time ago. Among them, the following result given by Wintner [27] is well known (see also [1, pp. 40–41]).

Theorem A. *Equation* (1.2) *is oscillatory if* a *is continuous and*

$$\lim_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \int_{t_0}^{s} a(\tau)\,d\tau ds = \infty. \tag{1.3}$$

In Theorem A, the double integral of the coefficient a plays an important role in the oscillation of solutions of (1.2). Many efforts were paid to improve condition (1.3) (for example, see [7, 11, 26]). Kamenev [9] gave the following oscillation theorem.

Theorem B. *Equation* (1.2) *is oscillatory if* a *is continuous and*

$$\limsup_{t\to\infty} \frac{1}{t^{\alpha}} \int_{t_0}^{t} (t-s)^{\alpha} a(s)\,ds = \infty \tag{1.4}$$

for some $\alpha > 1$.

Note that condition (1.4) is expressed by using the upper limit instead of the limit. He also pointed out that condition (1.3) implies condition (1.4). Hence, Theorem B is a complete generalization of Theorem A. Condition (1.4) has been extended to allow more general equations to be applied, including equation (1.2) (for example, see [20, 21, 28, 29, 32, 33, 34, 35, 37]). Later, Philos [22] gave the following theorem to improve some results of previous research.

Theorem C. *Let* $D = \{(t,s) : t \geq s \geq t_0\}$ *and let* $H : D \to [0,\infty)$ *be a continuous function, which is such that*

$$H(t,t) = 0 \quad \textit{for } t \geq t_0, \qquad H(t,s) > 0 \quad \textit{for } t > s \geq t_0$$

and has a continuous and nonpositive partial derivative on D *with respect to the second variable* s. *Moreover, let* $h : D \to [0,\infty)$ *be a continuous function with*

$$-\frac{\partial}{\partial s} H(t,s) = h(t,s)\sqrt{H(t,s)} \quad \textit{for all } (t,s) \in D.$$

Then equation (1.2) *is oscillatory if*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left(H(t,s)a(s) - \frac{1}{4} h^2(t,s) \right) ds = \infty. \tag{1.5}$$

Consider the case that $H(t,s) = (t-s)^\alpha$ with $\alpha \geq 2$. Then we can check that all assumptions of H are satisfied with $h(t,s) = \alpha(t-s)^{(\alpha-2)/2}$ and condition (1.5) becomes condition (1.4). For the proof, see [22, pp. 489–490]. Hence, Theorem C is a partial generalization of Theorem B. Another interesting case is

$$H(t,s) = \left(\ln\frac{t}{s}\right)^n$$

with any integer $n \geq 2$. All assumptions of H are also satisfied in this cases. At the present time, oscillation criteria of Philos-type are derived for more general equations such as delay differential equations, quasilinear functional differential equations, matrix differential systems, nonlinear differential equations with a damping term, dynamic equations on time scales, nonlinear elliptic equations (for example, see [2, 8, 13, 23, 25, 30, 31]).

Let us leave equation (1.2) without impulsive effect and return to equation (1.1) with impulsive effect. When considering the movement of a mass point, it is difficult to think that the position of the mass point instantaneously changes and it is discontinuous. However, it is sufficiently conceivable that the movement speed of the mass point changes discontinuously by some influences. For example, let us imagine that a small iron ball hung from a pivot in a space where magnetic force can be freely given or lost. We assume that there is no friction at the pivot of this pendulum. Equation (1.1) can be thought of as a model describing the motion of the mass point when shaking the pendulum slowly and giving a magnetic force instantaneously and occasionally to this space.

There are several styles to add impulsive effects to equations or systems. The interested reader can refer to the books [4, 5, 6, 12] and the references cited therein. It is well known that a new asymptotic behavior occurs by introducing an impulsive effect. Although it is not the theme of this paper, it has recently been reported that periodic solutions are generated by impulses. For example, see [10, 14, 15, 24]. The style of equation (1.1) seems to have been firstly given by Bainov and Simeonov [3]. Özbekler and Zafer are vigorously researching the oscillation theory of (1.1) lately (see [16, 17, 18, 19, 36]).

To conveniently describe one of the typical results of Özbekler and Zafer, we introduce some notation. Let us call θ_k a jumping time for each $k \in \mathbb{N}$. We pay attention to how many jumping times are in the interval $[t_0, t)$. Let $j(t)$ be the number of jumping times in the interval $[t_0, t)$; namely,

$$j(t) = \begin{cases} 0 & \text{if } t_0 \leq t \leq \theta_1, \\ k & \text{if } \theta_k < t \leq \theta_{k+1} \end{cases}$$

with $k \in \mathbb{N}$. By using the step function j, we can state an oscillation theorem given by Özbekler and Zafer [17, Corollary 2.6] as follows.

Theorem D. *Equation* (1.1) *is oscillatory if*

$$\lim_{t\to\infty} \frac{1}{t}\int_{t_0}^{t}\left\{\int_{t_0}^{s} a(\tau)d\tau + \sum_{i=1}^{j(s)} b_i\right\}ds = \infty. \tag{1.6}$$

Theorem D is a natural generalization of Wintner's oscillation criterion (Theorem A) in the sense that it can also be applied to the impulsive differential equation (1.1). From the form of condition (1.6), we can see that there are the factor related to the coefficient a that dominates the movement of the mass point and the factor related to jumping amount of the sequence $\{b_k\}$. It is reasonable to interpret that these factors affect whether all nontrivial solutions of (1.1) oscillate or not. Roughly speaking, if the average over the interval $[0,t]$ of the total of the integral of the left piecewise continuous function a and the sum of the impulsive coefficient $\{b_k\}$ diverges, all nontrivial solutions of (1.1) oscillate.

The purpose of this paper is to present Philos-type oscillation criteria concerning equation (1.1), such as the following.

Theorem 1. *Let H and h be the same functions as in Theorem C. Then equation* (1.1) *is oscillatory if*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \left\{ \int_{t_0}^{t} \left(H(t,s)a(s) - \frac{1}{4}h^2(t,s) \right) ds + \sum_{i=1}^{j(t)} b_i H(t,\theta_i) \right\} = \infty. \tag{1.7}$$

Remark 1. In Theorem 1, the mathematical symbol $\displaystyle\int_{t_0}^{t}$ means a proper integral form t_0 to t if a is a continuous function on $[t_0,\infty)$, and means

$$\int_{t_0}^{\theta_1} + \sum_{i=2}^{j(t)} \int_{\theta_{i-1}}^{\theta_i} + \int_{\theta_{j(t)}}^{t}$$

if a is a left piecewise continuous function.

The organization of this paper is as follows. We give the proof of Theorem 1 by using the so-called Riccati technique and integral averaging technique in the next section. In Section 3, we present two corollaries to make Theorem 1 easier to use. To illustrate our results, we also give an example in which equation (1.1) is oscillatory even though the factor related to the coefficient a works so as not to oscillate the mass point. In the final section, we clarify the relationship between conditions which guarantee that equation (1.1) is oscillatory and give a sharp example to compare the obtained result with previous works.

2. Proof of the main result

To prove Theorem 1 by contradiction, we assume that equation (1.1) has a nonoscillatory solution x. Hence, there exists a $T \ge t_0$ such that $x(t) \ne 0$ for $t \ge T$, and therefore, we can define

$$u(t) = \frac{x'(t)}{x(t)} \quad \text{for } t \ge T$$

(This is generally called Riccati transformation). From the definition of solutions of (1.1), we see that the function u is left piecewise continuous on the interval $[T,\infty)$. The function u also satisfies

$$\begin{aligned} &u' + u^2 + a(t) = 0, \qquad t \neq \theta_k; \\ &\Delta u(\theta_k) + b_k = 0. \end{aligned} \tag{2.1}$$

In fact, it follows from (1.1) that

$$u'(t) = \frac{x''(t)x(t) - \big((x'(t))\big)^2}{x^2(t)} = \frac{x''(t)}{x(t)} - \left(\frac{x'(t)}{x(t)}\right)^2 = -a(t) - u^2(t)$$

for $t \neq \theta_k$. Since x is continuous on $[T,\infty)$, we see that

$$\begin{aligned} \Delta u(\theta_k) = u(\theta_k^+) - u(\theta_k^-) &= \frac{x'(\theta_k^+)}{x(\theta_k^+)} - \frac{x'(\theta_k^-)}{x(\theta_k^-)} \\ &= \frac{x'(\theta_k^+) - x'(\theta_k^-)}{x(\theta_k)} = \frac{\Delta x'(\theta_k)}{x(\theta_k)} = -b_k. \end{aligned}$$

We choose an $m \in \mathbb{N}$ so that $\theta_{m-1} \leq T < \theta_m$, where $\theta_0 = t_0$. For t sufficiently large, let $n = j(t)$. In other words, n is an integer which satisfies that $\theta_n \leq t < \theta_{n+1}$. Let I be a set excluded the points θ_m, θ_{m+1}, $\dots$, θ_n from the interval $[T,t]$. From the first equation of (2.1) it is clear that

$$\int_T^t H(t,s)a(s)ds = \int_I H(t,s)a(s)ds = -\int_I H(t,s)u'(s)ds - \int_I H(t,s)u^2(s)ds. \tag{2.2}$$

Since u is a left piecewise continuous function and H is a continuous function, we see that

$$\begin{aligned} H(t,t)u(t) - H(t,T)u(T) &= H(t,t)u(t) - H(t,\theta_n^+)u(\theta_n^+) \\ &\quad + H(t,\theta_n^+)u(\theta_n^+) - H(t,\theta_n^-)u(\theta_n^-) \\ &\quad + H(t,\theta_n^-)u(\theta_n^-) - H(t,\theta_{n-1}^+)u(\theta_{n-1}^+) \\ &\quad + H(t,\theta_{n-1}^+)u(\theta_{n-1}^+) - H(t,\theta_{n-1}^-)u(\theta_{n-1}^-) \\ &\quad + H(t,\theta_{n-1}^-)u(\theta_{n-1}^-) - H(t,\theta_{n-2}^+)u(\theta_{n-2}^+) \\ &\quad + \cdots\cdots + H(t,\theta_m^+)u(\theta_m^+) - H(t,\theta_m^-)u(\theta_m^-) \\ &\quad + H(t,\theta_m^-)u(\theta_m^-) - H(t,T)u(T) \\ &= \int_I \frac{\partial}{\partial s}\big(H(t,s)u(s)\big)ds \\ &\quad + \sum_{i=m}^{j(t)}\big(H(t,\theta_i^+)u(\theta_i^+) - H(t,\theta_i^-)u(\theta_i^-)\big) \\ &= \int_I \frac{\partial}{\partial s}\big(H(t,s)u(s)\big)ds \\ &\quad + \sum_{i=m}^{j(t)} H(t,\theta)\big(u(\theta_i^+) - u(\theta_i^-)\big) \end{aligned}$$

$$= \int_I \frac{\partial}{\partial s}\big(H(t,s)u(s)\big)ds + \sum_{i=m}^{j(t)} H(t,\theta_i)\Delta u(\theta_i).$$

From the assumption that $H(t,t)=0$ and the second equation of (2.1) it turns out that

$$\int_I \frac{\partial}{\partial s}\big(H(t,s)u(s)\big)ds = -H(t,T)u(T) + \sum_{i=m}^{j(t)} b_i H(t,\theta_i). \tag{2.3}$$

Using (2.2), (2.3) and the assumption of H and h, we obtain

$$\begin{aligned}
\int_T^t H(t,s)a(s)ds &= -\int_I \frac{\partial}{\partial s}\big(H(t,s)u(s)\big)ds + \int_I \left(\frac{\partial}{\partial s}H(t,s)\right)u(s)ds - \int_I H(t,s)u^2(s)ds \\
&= H(t,T)u(T) - \sum_{i=m}^{j(t)} b_i H(t,\theta_i) \\
&\quad - \int_I h(t,s)\sqrt{H(t,s)}u(s)ds - \int_I H(t,s)u^2(s)ds \\
&= H(t,T)u(T) - \sum_{i=m}^{j(t)} b_i H(t,\theta_i) + \frac{1}{4}\int_T^t h^2(t,s)ds \\
&\quad - \int_I \left(\frac{1}{2}h(t,s) + \sqrt{H(t,s)}u(s)\right)^2 ds \\
&\le H(t,T)u(T) - \sum_{i=m}^{j(t)} b_i H(t,\theta_i) + \frac{1}{4}\int_T^t h^2(t,s)ds.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\int_{t_0}^t \left(H(t,s)a(s) - \frac{1}{4}h^2(t,s)\right)ds + \sum_{i=1}^{j(t)} b_i H(t,\theta_i) \\
&= \int_{t_0}^T \left(H(t,s)a(s) - \frac{1}{4}h^2(t,s)\right)ds + \sum_{i=1}^{m-1} b_i H(t,\theta_i) \\
&\quad + \int_T^t \left(H(t,s)a(s) - \frac{1}{4}h^2(t,s)\right)ds + \sum_{i=m}^{j(t)} b_i H(t,\theta_i) \\
&\le \int_{t_0}^T \left(H(t,s)a(s) - \frac{1}{4}h^2(t,s)\right)ds + \sum_{i=1}^{m-1} b_i H(t,\theta_i) + H(t,T)u(T).
\end{aligned}$$

Since H is decreasing with respect to the second variable s, we see that

$$\begin{aligned}
&\int_{t_0}^t \left(H(t,s)a(s) - \frac{1}{4}h^2(t,s)\right)ds + \sum_{i=1}^{j(t)} b_i H(t,\theta_i) \\
&\quad \le \int_{t_0}^T H(t,s)|a(s)|ds + \sum_{i=1}^{m-1} |b_i| H(t,\theta_i) + H(t,T)|u(T)|
\end{aligned}$$

$$\leq H(t,t_0)\left(\int_{t_0}^{T}|a(s)|ds+\sum_{i=1}^{m-1}|b_i|+|u(T)|\right).$$

We therefore conclude that

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\left\{\int_{t_0}^{t}\left(H(t,s)a(s)-\frac{1}{4}h^2(t,s)\right)ds+\sum_{i=1}^{j(t)}b_iH(t,\theta_i)\right\}<\infty.$$

This contradicts condition (1.7). Thus, the proof of Theorem 1 is complete. □

Remark 2. As can be seen from the proof of Theorem 1, h does not necessarily have to be continuous in D. In Theorem C and Theorem 1, the assumption of h can be weakened as follows. Let $h\colon E \stackrel{\text{def}}{=} \left\{(t,s)\colon t>s\geq t_0\right\} \to [0,\infty)$ be a continuous function with

$$-\frac{\partial}{\partial s}H(t,s)=h(t,s)\sqrt{H(t,s)}\quad \text{for all } (t,s)\in E$$

and

$$\int_{t_0}^{t}h^2(t,s)ds<\infty \quad \text{for each fixed } t\geq t_0.$$

3. Corollaries and an example

As mentioned already, we can choose $(t-s)^\alpha$ with $\alpha\geq 2$ as the function H of Theorem 1. In this case, the continuous function h defined by $h(t,s)=\alpha(t-s)^{(\alpha-2)/2}$ satisfies

$$\begin{aligned}\frac{1}{H(t,t_0)}\int_{t_0}^{t}h^2(t,s)ds&=\frac{\alpha^2}{(t-t_0)^\alpha}\int_{t_0}^{t}(t-s)^{\alpha-2}ds\\&=\frac{\alpha^2}{(\alpha-1)(t-t_0)}\to 0\quad \text{as } t\to\infty.\end{aligned}$$

Moreover, we obtain

$$\frac{t^\alpha}{H(t,t_0)}=\left(\frac{t}{t-t_0}\right)^\alpha=\left(\frac{1}{1-t_0/t}\right)^\alpha\to 1\quad \text{as } t\to\infty.$$

Hence, we have the following result. If

$$\limsup_{t\to\infty}\frac{1}{t^\alpha}\left\{\int_{t_0}^{t}(t-s)^\alpha a(s)ds+\sum_{i=1}^{j(t)}b_i(t-\theta_i)^\alpha\right\}=\infty \tag{3.1}$$

for some $\alpha\geq 2$, then equation (1.1) is oscillatory.

By using the method gave by Kamenev, the range of power α can be extended from $[2,\infty)$ to $(1,\infty)$.

Corollary 2. *Equation* (1.1) *is oscillatory if condition* (3.1) *holds for some* $\alpha>1$.

Proof. As in the proof of Theorem 1, we assume that equation (1.1) has a nonoscillatory solution x such that $x(t) \neq 0$ for $t \geq T \geq t_0$, and we define

$$u(t) = \frac{x'(t)}{x(t)} \quad \text{for } t \geq T.$$

Then the function u becomes a solution of Riccati type impulsive differential equation (2.1). Let m be an integer such that $\theta_m - 1 \leq T < \theta_m$ and let $n = j(t)$ for t sufficiently large. Then we have

$$\int_T^t (t-s)^\alpha a(s)ds = \int_I (t-s)^\alpha a(s)ds = -\int_I (t-s)^\alpha u'(s)ds - \int_I (t-s)^\alpha u^2(s)ds$$

and

$$\int_I \frac{\partial}{\partial s}\big((t-s)^\alpha u(s)\big)ds = -(t-T)^\alpha u(T) + \sum_{i=m}^{j(t)} b_i (t-\theta_i)^\alpha.$$

Using these equalities, we obtain

$$\begin{aligned}
\int_T^t (t-s)^\alpha a(s)ds &= (t-T)^\alpha u(T) - \sum_{i=m}^{j(t)} b_i(t-\theta_i)^\alpha \\
&\quad - \alpha \int_I (t-s)^{\alpha-1} u(s)ds - \int_I (t-s)^\alpha u^2(s)ds \\
&= (t-T)^\alpha u(T) - \sum_{i=m}^{j(t)} b_i(t-\theta_i)^\alpha + \frac{\alpha^2}{4}\int_T^t (t-s)^{\alpha-2}ds \\
&\quad - \int_I \left(\frac{\alpha}{2}(t-s)^{(\alpha-2)/2} + (t-s)^{\alpha/2}u(s)\right)^2 ds \\
&\leq (t-T)^\alpha u(T) - \sum_{i=m}^{j(t)} b_i(t-\theta_i)^\alpha + \frac{\alpha^2}{4(\alpha-1)}(t-T)^{\alpha-1}.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\frac{1}{t^\alpha}\left\{\int_{t_0}^t (t-s)^\alpha a(s)ds + \sum_{i=1}^{j(t)} b_i(t-\theta_i)^\alpha\right\} \\
&\leq \left(1-\frac{T}{t}\right)^\alpha u(T) + \frac{\alpha^2}{4(\alpha-1)}\left(1-\frac{T}{t}\right)^{\alpha-1}\frac{1}{t} + \frac{1}{t^\alpha}\sum_{i=1}^{m-1} b_i(t-\theta_i)^\alpha < \infty.
\end{aligned}$$

This contradicts condition (3.1). Thus, the proof of Corollary 2 is complete. □

It is clear that Corollary 2 is a generalization of Theorem B. The following result can be obtained by changing how to select the function H.

Corollary 3. *Equation* (1.1) *is oscillatory if*

$$\limsup_{t\to\infty}\frac{1}{(\ln t)^n}\left\{\int_{t_0}^{t}\Big(\ln\frac{t}{s}\Big)^n a(s)ds+\sum_{i=1}^{j(t)}b_i\Big(\ln\frac{t}{\theta_i}\Big)^n\right\}=\infty, \tag{3.2}$$

where n is an integer with $n\geq 2$.

Proof. For any integer $n \geq 2$, let

$$H(t,s)=\Big(\ln\frac{t}{s}\Big)^n.$$

Then the function H is continuous on the domain D and satisfies conditions

$$H(t,t)=0 \quad \text{for } t\geq t_0 \quad \text{and} \quad H(t,s)>0 \quad \text{for } t>s\geq t_0.$$

Also, H has a continuous and nonpositive partial derivative on D with respect to the second variable s. Moreover, the function

$$h(t,s)=\frac{n}{s}\Big(\ln\frac{t}{s}\Big)^{(n-2)/2}$$

is continuous on D and

$$-\frac{\partial}{\partial s}H(t,s)=h(t,s)\sqrt{H(t,s)}$$

holds for all $(t,s)\in D$. It is clear that

$$\frac{(\ln t)^n}{H(t,t_0)}=\left(\frac{\ln t}{\ln t-\ln t_0}\right)^n=\left(\frac{1}{1-(\ln t_0)/(\ln t)}\right)^n\to 1 \quad \text{as } t\to\infty.$$

Let $\tau = t/s$. Then we have

$$\begin{aligned}
\int_{t_0}^{t}h^2(t,s)ds &= n^2\int_{t_0}^{t}\frac{(\ln(t/s))^{n-2}}{s^2}ds=\frac{n^2}{t}\int_{1}^{t/t_0}(\ln\tau)^{n-2}d\tau\\
&=\frac{n^2}{t}\left[\tau\sum_{r=0}^{n-2}(-1)^r\frac{(n-2)!\,(\ln(\tau))^{n-2-r}}{(n-2-r)!}\right]_1^{t/t_0}\\
&=\frac{n^2}{t}\left\{\frac{t}{t_0}\sum_{r=0}^{n-2}(-1)^r\frac{(n-2)!\,(\ln(t/t_0))^{n-2-r}}{(n-2-r)!}-(-1)^{n-2}(n-2)!\right\}\\
&=\frac{n^2}{t_0}\Big(\ln\frac{t}{t_0}\Big)^{n-2}-\frac{n^2(n-2)}{t_0}\Big(\ln\frac{t}{t_0}\Big)^{n-3}+\frac{n^2(n-2)(n-3)}{t_0}\Big(\ln\frac{t}{t_0}\Big)^{n-4}\\
&\quad-\frac{n^2(n-2)}{t_0}\Big(\ln\frac{t}{t_0}\Big)^{n-5}+\cdots+(-1)^n\frac{n^2(n-2)!}{t_0}-(-1)^n\frac{n^2(n-2)!}{t}
\end{aligned}$$

for $t \geq t_0$. Hence, we see that

$$\begin{aligned}\frac{1}{H(t,t_0)}\int_{t_0}^t h^2(t,s)ds = \frac{n^2}{t_0(\ln(t/t_0))^n}&\left\{\left(\ln\frac{t}{t_0}\right)^{n-2} - (n-2)\left(\ln\frac{t}{t_0}\right)^{n-3}\right.\\ &+(n-2)(n-3)\left(\ln\frac{t}{t_0}\right)^{n-4} - (n-2)(n-3)(n-4)\left(\ln\frac{t}{t_0}\right)^{n-5}\\ &\left.+\cdots+(-1)^n(n-2)! - (-1)^n(n-2)!\,\frac{t_0}{t}\right\} \to 0 \quad \text{as } t\to\infty.\end{aligned}$$

We therefore conclude that

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\left\{\int_{t_0}^t\left(H(t,s)a(s)-\frac{1}{4}h^2(t,s)\right)ds+\sum_{i=1}^{j(t)}b_iH(t,\theta_i)\right\}\\ &=\limsup_{t\to\infty}\frac{1}{(\ln t)^n}\left\{\int_{t_0}^t\left(\ln\frac{t}{s}\right)^n a(s)ds+\sum_{i=1}^{j(t)}b_i\left(\ln\frac{t}{\theta_i}\right)^n\right\}=\infty,\end{aligned}$$

Thus, by Theorem 1, equation (1.1) is oscillatory. □

It is clear that equation (1.1) is nonoscillatory when the coefficient a is always negative and there is no impulse effect. The following is an example where equation (1.1) is oscillatory due to only the impulsive effect nevertheless the coefficient a is always negative.

Example 1. Let $t_0 = 1/2$. Suppose that $\theta_k = k$ and $b_k = 1/k$ for $k \in \mathbb{N}$, and $a(t) = -1/t^2$ for $t \geq t_0 = 1/2$. Then equation (1.1) is oscillatory.

By a straightforward calculation, we have

$$\begin{aligned}\frac{1}{(\ln t)^2}\int_{t_0}^t\left(\ln\frac{t}{s}\right)^2 a(s)ds &= -\frac{1}{(\ln t)^2}\int_{1/2}^t\frac{(\ln t-\ln s)^2}{s^2}ds\\ &= -\int_{1/2}^t\frac{1}{s^2}+\frac{2}{\ln t}\int_{1/2}^t\frac{\ln s}{s^2}ds-\frac{1}{(\ln t)^2}\int_{1/2}^t\frac{(\ln s)^2}{s^2}ds\\ &= \frac{1}{t}-2+\frac{4(1+\ln(1/2))}{\ln t}-\frac{2(\ln t+1)}{t\ln t}\\ &\quad+\frac{1}{t}+\frac{2}{t\ln t}+\frac{2}{t(\ln t)^2}-\frac{4(1+\ln(1/2))+2(\ln(1/2))^2}{(\ln t)^2}\\ &= -2+\frac{4(1+\ln(1/2))}{\ln t}+\frac{2}{t(\ln t)^2}\\ &\quad-\frac{4(1+\ln(1/2))+2(\ln(1/2))^2}{(\ln t)^2}.\end{aligned}$$

Hence, we see that

$$\limsup_{t\to\infty} \frac{1}{(\ln t)^2}\int_{t_0}^{t}\Big(\ln\frac{t}{s}\Big)^2 a(s)ds = -2.$$

We next estimate the impulsive effect. Recall that

$$j(t) = \begin{cases} 0 & \text{if } t_0 \le t \le \theta_1, \\ k & \text{if } \theta_k < t \le \theta_{k+1} \end{cases}$$

with $k\in\mathbb{N}$. Hence, for $k<t\le k+1$, we have

$$\begin{aligned}
\frac{1}{(\ln t)^2}\sum_{i=1}^{j(t)} b_i\Big(\ln\frac{t}{\theta_i}\Big)^2 &= \frac{1}{(\ln t)^2}\sum_{i=1}^{k} b_i\big(\ln t - \ln\theta_i\big)^2 \\
&= \frac{1}{(\ln t)^2}\sum_{i=1}^{k}\frac{1}{i}\Big((\ln t)^2 - 2(\ln i)(\ln t) + (\ln i)^2\Big) \\
&= \sum_{i=1}^{k}\frac{1}{i}\Big(1-\frac{2\ln i}{\ln t}+\frac{(\ln i)^2}{(\ln t)^2}\Big) \\
&> \sum_{i=1}^{k}\frac{1}{i} - \frac{2}{\ln k}\sum_{i=1}^{k}\frac{\ln i}{i} + \frac{1}{(\ln(k+1))^2}\sum_{i=1}^{k}\frac{(\ln i)^2}{i}.
\end{aligned}$$

Since

$$\sum_{i=1}^{k}\frac{1}{i} > \int_1^{k+1}\frac{1}{x}dx = \ln(k+1),$$

$$\begin{aligned}
\sum_{i=1}^{k}\frac{\ln i}{i} &= \frac{\ln 1}{1}+\frac{\ln 2}{2}+\frac{\ln 3}{3}+\cdots+\frac{\ln k}{k} < \frac{\ln 2}{2}+\frac{\ln 3}{3}+\int_3^k\frac{\ln x}{x}dx \\
&= \frac{\ln 2}{2}+\frac{\ln 3}{3}+\frac{(\ln k)^2}{2}-\frac{(\ln 3)^2}{2}
\end{aligned}$$

and

$$\begin{aligned}
\sum_{i=1}^{k}\frac{(\ln i)^2}{i} &= \frac{(\ln 1)^2}{1}+\frac{(\ln 2)^2}{2}+\frac{(\ln 3)^2}{3}+\cdots+\frac{(\ln k)^2}{k} \\
&> \frac{(\ln 2)^2}{2}+\frac{(\ln 3)^2}{3}+\cdots+\frac{(\ln 7)^2}{7}+\int_8^{k+1}\frac{(\ln x)^2}{x}dx \\
&= \frac{(\ln 2)^2}{2}+\frac{(\ln 3)^2}{3}+\cdots+\frac{(\ln 7)^2}{7}+\frac{(\ln(k+1))^3}{3}-\frac{(\ln 8)^3}{3},
\end{aligned}$$

we see that

$$\begin{aligned}
\frac{1}{(\ln t)^2}\sum_{i=1}^{j(t)} b_i\big(\ln t-\ln\theta_i\big)^2 > \ln(k+1) &- \frac{2}{\ln k}\left(\frac{\ln 2}{2}+\frac{\ln 3}{3}+\frac{(\ln k)^2}{2}-\frac{(\ln 3)^2}{2}\right) \\
&+ \frac{1}{(\ln(k+1))^2}\left(\frac{(\ln 2)^2}{2}+\frac{(\ln 3)^2}{3}+\cdots+\frac{(\ln 7)^2}{7}\right)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{(\ln(k+1))^2}\left(\frac{(\ln(k+1))^3}{3}-\frac{(\ln 8)^3}{3}\right)\\
&=\frac{4}{3}\ln(k+1)-\ln k-\frac{2}{\ln k}\left(\frac{\ln 2}{2}+\frac{\ln 3}{3}-\frac{(\ln 3)^2}{2}\right)\\
&+\frac{1}{(\ln(k+1))^2}\left(\frac{(\ln 2)^2}{2}+\cdots+\frac{(\ln 7)^2}{7}-\frac{(\ln 8)^3}{3}\right)\\
&=\ln\left\{\left(1+\frac{1}{k}\right)^{4/3}k^{1/3}\right\}-\frac{2}{\ln k}\left(\frac{\ln 2}{2}+\frac{\ln 3}{3}-\frac{(\ln 3)^2}{2}\right)\\
&+\frac{1}{(\ln(k+1))^2}\left(\frac{(\ln 2)^2}{2}+\cdots+\frac{(\ln 7)^2}{7}-\frac{(\ln 8)^3}{3}\right)
\end{aligned}
$$

which diverges to infinity as k tends to ∞. When t diverges to infinity, the number k of jumping times also diverges. Hence, we see that

$$
\limsup_{t\to\infty}\frac{1}{(\ln t)^2}\sum_{i=1}^{j(t)} b_i\left(\ln\frac{t}{\theta_i}\right)^2=\infty.
$$

We therefore conclude that condition (3.2) holds with $n=2$. Thus, by Corollary 3, equation (1.1) is oscillatory.

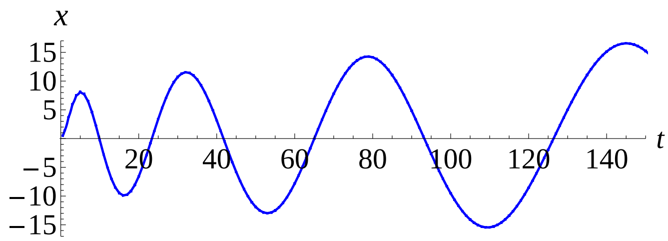

Figure 1: This graph is the solution curve of (1.1) with the initial time $t_0 = 1/2$ and the initial value $(x(1/2), x'(1/2)) = (1/2, 1)$. The drawing range is $[1/2, 150]$.

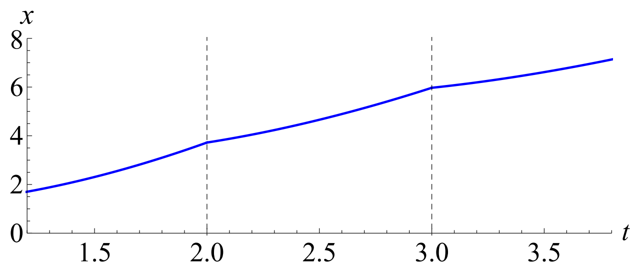

Figure 2: This graph is the same solution curve of (1.1) as in Figure1. However, the drawing range is $[1.2, 3.8]$. The solution curve is not smooth at $t = 2.0$ and $t = 3.0$.

As can be seen from Figure 1, the solution curve is waving and its amplitude is gradually increasing. Since the solution curve intersects the t-axis many times, we see that the nontrivial solution of (1.1) is oscillatory. Figure 2 is a lateral enlargement of the solution curve drawn in Figure 1. Although the solution curve looks like smooth overall in Figure 1, it can be seen from Figure 2 that the solution curve is a piecewise continuously differentiable function.

Let $y = x'$ as a new variable. Then, equation (1.1) becomes the planar system

$$\begin{aligned} &x' = y \\ &y' = -a(t)x, \qquad t \neq \theta_k; \\ &\Delta y(\theta_k) + b_k x(\theta_k) = 0. \end{aligned} \tag{3.3}$$

In Figure 3, we show the orbit of (3.3) corresponding to the solution curve drawn in Figures 1 and 2. The orbit is composed of infinitely many curve segments $Q_{k-1}P_k$ with $k \in \mathbb{N}$ (we write only k from 1 to 12 in Figure 3). The point $(x(t), y(t))$ moves on the curve segments. It starts from $Q_0 = (x(1/2), y(1/2)) = (1/2, 1)$ and runs to P_1. After, that, the point $(x(t), y(t))$ jumps at P_1 and arrives at Q_1. Similarly, the point $(x(t), y(t))$ proceeds from P_k to Q_k on the curve segment $Q_{k-1}P_k$ and jumps at P_k. Note that the points P_1, P_2, $\ldots$, P_9 and Q_0, Q_1, $\ldots$, Q_9 are is in the right half plane, but the points P_{10}, P_{11}, $\ldots$, P_{12} and Q_{10}, Q_{11} are in the left half plane. The point $(x(t), y(t))$ moves from the right-hand side half plane (x, y) to the left-hand side half plane and then moves from the left-hand side half plane to the right-hand side half plane. Since this movement is repeated, we see that the nontrivial solution of (1.1) is oscillatory.

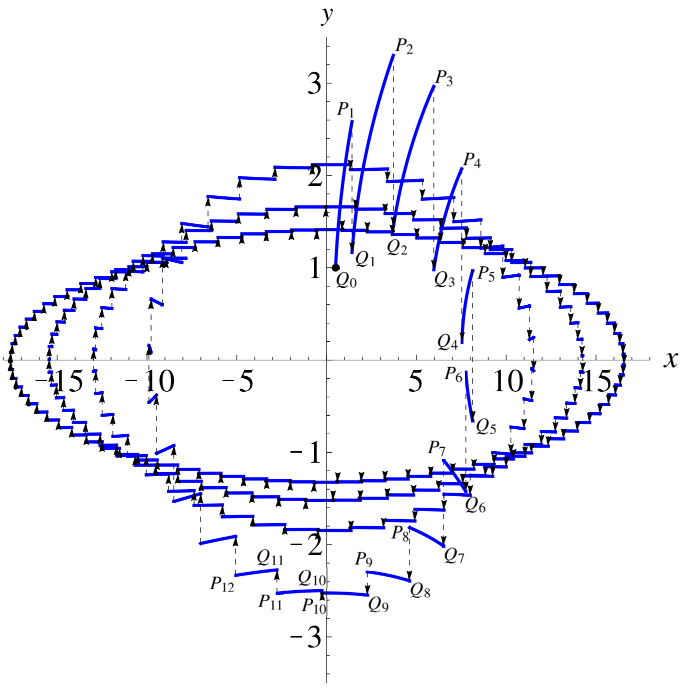

Figure 3: The orbit of (3.3) with the initial time $t_0 = 1/2$ and the initial value $Q_0 = (x(1/2), y(1/2)) = (1/2, 1)$

Consider the Euler differential equation

$$x'' + \frac{\lambda}{t^2} x = 0 \tag{3.4}$$

with $\lambda \in \mathbb{R}$. It is well known that

(i) if $\lambda \leq 1/4$, then equation (3.4) is nonoscillatory;

(ii) if $\lambda > 1/4$, then equation (3.4) is oscillatory.

Let us the impulsive effect of Example 1 to equation (3.4). Then, as shown in Example 1, all nontrivial solutions become to oscillate even if $\lambda \leq 1/4$.

4. Relationship between conditions

From the definition of $j(t)$ it follows that

$$\sum_{i=1}^{j(t)} b_i (t-\theta_i)^{\alpha-1} = \begin{cases} 0 & \text{if } t_0 \leq t \leq \theta_1, \\ \displaystyle\sum_{i=1}^{k} b_i (t-\theta_i)^{\alpha-1} & \text{if } \theta_k < t \leq \theta_{k+1} \end{cases}$$

with $\alpha \geq 1$ and $k \in \mathbb{N}$. Hence, we can obtain the following equality:

$$\alpha \int_{t_0}^{t} \sum_{i=1}^{j(s)} b_i (s-\theta_i)^{\alpha-1} ds = \sum_{i=1}^{j(t)} b_i (t-\theta_i)^{\alpha} \tag{4.1}$$

for $t \geq t_0$. In fact, we have

$$\begin{aligned}
\alpha \int_{t_0}^{t} \sum_{i=1}^{j(s)} b_i (s-\theta_i)^{\alpha-1} ds &= \alpha \int_{t_0}^{\theta_1} \sum_{i=1}^{j(s)} b_i (s-\theta_i)^{\alpha-1} ds + \alpha \int_{\theta_1}^{\theta_2} \sum_{i=1}^{j(s)} b_i (s-\theta_i)^{\alpha-1} ds \\
&\quad + \alpha \int_{\theta_2}^{\theta_3} \sum_{i=1}^{j(s)} b_i (s-\theta_i)^{\alpha-1} ds + \cdots + \alpha \int_{\theta_k}^{t} \sum_{i=1}^{j(s)} b_i (s-\theta_i)^{\alpha-1} ds \\
&= \alpha \int_{t_0}^{\theta_1} \sum_{i=1}^{0} b_i (s-\theta_i)^{\alpha-1} ds + \alpha \int_{\theta_1}^{\theta_2} \sum_{i=1}^{1} b_i (s-\theta_i)^{\alpha-1} ds \\
&\quad + \alpha \int_{\theta_2}^{\theta_3} \sum_{i=1}^{2} b_i (s-\theta_i)^{\alpha-1} ds + \cdots + \alpha \int_{\theta_k}^{t} \sum_{i=1}^{k} b_i (s-\theta_i)^{\alpha-1} ds \\
&= \alpha \int_{\theta_1}^{t} b_1 (s-\theta_1)^{\alpha-1} ds + \alpha \int_{\theta_2}^{t} b_2 (s-\theta_2)^{\alpha-1} ds \\
&\quad + \cdots + \alpha \int_{\theta_k}^{t} b_k (s-\theta_k)^{\alpha-1} ds \\
&= b_1 (t-\theta_1)^{\alpha} + b_2 (t-\theta_2)^{\alpha} + \cdots + b_k (t-\theta_k)^{\alpha} \\
&= \sum_{i=1}^{k} b_i (t-\theta_i)^{\alpha} = \sum_{i=1}^{j(t)} b_i (t-\theta_i)^{\alpha}
\end{aligned}$$

for $\theta_k < t \le \theta_{k+1}$.

Fubini's theorem gives

$$\int_{t_0}^{t}\int_{t_0}^{s} a(\tau)d\tau ds = \int_{t_0}^{t}\int_{\tau}^{t} a(\tau)dsd\tau = \int_{t_0}^{t}(t-s)a(s)ds$$

and

$$\int_{t_0}^{t}\int_{t_0}^{s}(t-s)^{\alpha-1}a(\tau)d\tau ds = \int_{t_0}^{t}\int_{\tau}^{t}(t-s)^{\alpha-1}a(\tau)dsd\tau = \int_{t_0}^{t}(t-s)^{\alpha}a(s)ds.$$

Hence, by combining with the equality (4.1), we see that conditions (1.6) and (3.1) are equivalent to

$$\lim_{t\to\infty}\frac{1}{t}\left\{\int_{t_0}^{t}(t-s)a(s)ds+\sum_{i=1}^{j(t)}b_i(t-\theta_i)\right\}=\infty$$

and

$$\limsup_{t\to\infty}\frac{1}{t^{\alpha}}\int_{t_0}^{t}\left\{\int_{t_0}^{s}(t-s)^{\alpha-1}a(\tau)d\tau+\alpha\sum_{i=1}^{j(s)}b_i(t-\theta_i)^{\alpha-1}\right\}ds=\infty,$$

respectively.

We can also use the equality (4.1) to get the following relationships:

Proposition 4. *Let $\alpha \ge 1$. Then*

$$\lim_{t\to\infty}\frac{1}{t^{\alpha}}\int_{t_0}^{t}\left\{\int_{t_0}^{s}(t-s)^{\alpha-1}a(\tau)d\tau+\alpha\sum_{i=1}^{j(s)}b_i(t-\theta_i)^{\alpha-1}\right\}ds=\infty \tag{4.2}$$

implies

$$\lim_{t\to\infty}\frac{1}{t^{\alpha+1}}\left\{\int_{t_0}^{t}(t-s)^{\alpha+1}a(s)ds+\frac{2}{\alpha+1}\sum_{i=1}^{j(t)}b_i(t-\theta_i)^{\alpha+1}\right\}=\infty. \tag{4.3}$$

Proof. For convenience, let

$$A(t)=\int_{t_0}^{t}(t-s)^{\alpha-1}a(s)ds.$$

It follows from (4.2) that for any $K>0$, there exists a $T^* > t_0$ such that

$$\int_{t_0}^{t}\left\{A(s)+\alpha\sum_{i=1}^{j(s)}b_i(t-\theta_i)^{\alpha-1}\right\}ds > (\alpha+1)Kt^{\alpha} \quad \text{for } t \ge T^*.$$

Hence, from the equality (4.1) it turns out that

$$\int_{t_0}^{t}A(s)ds+\sum_{i=1}^{j(t)}b_i(t-\theta_i)^{\alpha} > (\alpha+1)Kt^{\alpha} \quad \text{for } t \ge T^*. \tag{4.4}$$

Let

$$C = \int_{t_0}^{T^*} \left\{ \int_{t_0}^{s} A(\tau)d\tau + \sum_{i=1}^{j(s)} b_i(s-\theta_i)^\alpha \right\} ds.$$

Since C is a constant, we can find a $T > T^*$ so that

$$-\frac{K}{4} < \frac{C}{t^{\alpha+1}} < \frac{K}{4} \quad \text{and} \quad -\frac{1}{4} < \left(\frac{T^*}{t}\right)^{\alpha+1} < \frac{1}{4}$$

for $t \geq T$. Using integration by parts twice, we obtain

$$\begin{aligned}
\int_{t_0}^{t} (t-s)^{\alpha+1} a(s) ds &= \int_{t_0}^{t} (t-2)^2 \left((t-s)^{\alpha-1} a(s) \right) ds \\
&= \left[(t-s)^2 A(s) \right]_{s=t_0}^{s=t} + 2\int_{t_0}^{t} (t-s)A(s)ds = 2\int_{t_0}^{t} (t-s)A(s)ds \\
&= 2\left[(t-s)\int_{t_0}^{s} A(\tau)d\tau \right]_{s=t_0}^{s=t} + 2\int_{t_0}^{t}\int_{t_0}^{s} A(\tau)d\tau = 2\int_{t_0}^{t}\int_{t_0}^{s} A(\tau)d\tau.
\end{aligned}$$

Hence, from the equality (4.1) again, we have

$$\begin{aligned}
&\frac{1}{t^{\alpha+1}} \left\{ \int_{t_0}^{t} (t-s)^{\alpha+1} a(s)ds + \frac{2}{\alpha+1} \sum_{i=1}^{j(t)} b_i(t-\theta_i)^{\alpha+1} \right\} \\
&= \frac{2}{t^{\alpha+1}} \int_{t_0}^{t} \left\{ \int_{t_0}^{s} A(\tau)d\tau + \sum_{i=1}^{j(s)} b_i(s-\theta_i)^\alpha \right\} ds \\
&= \frac{2C}{t^{\alpha+1}} + \frac{2}{t^{\alpha+1}} \int_{T^*}^{t} \left\{ \int_{t_0}^{s} A(\tau)d\tau + \sum_{i=1}^{j(s)} b_i(s-\theta_i)^\alpha \right\} ds
\end{aligned}$$

for $t \geq T^*$. From (4.4) we see that

$$\begin{aligned}
\frac{2C}{t^{\alpha+1}} + \frac{2}{t^{\alpha+1}} \int_{T^*}^{t} \left\{ \int_{t_0}^{s} A(\tau)d\tau + \sum_{i=1}^{j(s)} b_i(s-\theta_i)^\alpha \right\} ds &> \frac{2C}{t^{\alpha+1}} + \frac{2}{t^{\alpha+1}} \int_{T^*}^{t} (\alpha+1)Ks^\alpha ds \\
&= \frac{2C}{t^{\alpha+1}} + 2K - 2K\left(\frac{T^*}{t}\right)^{\alpha+1} \\
&> -\frac{K}{2} + 2K - \frac{K}{2} = K
\end{aligned}$$

for $t \geq T$. This completes the proof of Proposition 4. □

When $\alpha = 1$, condition (4.2) coincides with condition (1.6) and condition (4.3) becomes

$$\lim_{t\to\infty} \frac{1}{t^2} \left\{ \int_{t_0}^{t} (t-s)^2 a(s)ds + \sum_{i=1}^{j(t)} b_i(t-\theta_i)^2 \right\} = \infty.$$

Hence, Corollary 2 completely contains Theorem D. The following example is applied to Corollary 2, but cannot be applied to Theorem D.

Example 2. Let $t_0 = 3\pi/2$. Suppose that $\theta_k = (k+1)\pi$ and

$$b_k = \begin{cases} 1 & \text{if } k=1, \\ -1 & \text{if } k=2, \\ -1 & \text{if } k=3, \\ 1 & \text{if } k=4, \end{cases}$$

$b_{k+4} = b_k$ for $k \in \mathbb{N}$, and

$$a(t) = 2\sin t + 4t\cos t - t^2 \sin t + 2 \quad \text{for } t \geq t_0 = 3\pi/2.$$

Then equation (1.1) is oscillatory.

We will confirm that condition (3.1) holds when $\alpha = 2$. We first calculate the integral term of (3.1). Since

$$\begin{aligned}(t-s)^2 a(s) = {} & t^2\big(2\sin s + 4s\cos s - s^2\sin s + 2\big) - 2t\big(2s\sin s + 4s^2\cos s - s^3\sin s + 2s\big) \\ & + 2s^2\sin s + 4s^3\cos - s^4\sin s + 2s^2,\end{aligned}$$

$$\begin{aligned}\int_{3\pi/2}^{t}\big(2\sin s + 4s\cos s - s^2\sin s + 2\big)ds &= \Big[2s\sin s + s^2\cos s + 2s\Big]_{3\pi/2}^{t} \\ &= 2t(\sin t + 1) + t^2\cos t,\end{aligned}$$

$$\begin{aligned}\int_{3\pi/2}^{t}\big(2s\sin s + 4s^2\cos s - s^3\sin s + 2s\big)ds &= \Big[s^2\sin s + s^3\cos s + s^2\Big]_{3\pi/2}^{t} \\ &= t^2(\sin t + 1) + t^3\cos t\end{aligned}$$

and

$$\begin{aligned}&\int_{3\pi/2}^{t}\big(2s^2\sin s + 4s^3\cos - s^4\sin s + 2s^2\big)ds \\ &\quad = \Big[4\cos s + 4s\sin s - 2s^2\cos s + s^4\cos s + \frac{2}{3}s^3\Big]_{3\pi/2}^{t} \\ &\quad = 6\pi - \frac{9}{4}\pi^3 + 4\cos t + 4t\sin t - 2t^2\cos t + \frac{2}{3}t^3 + t^4\cos t,\end{aligned}$$

we have

$$\int_{t_0}^{t}(t-s)^2 a(s)ds = 6\pi - \frac{9}{4}\pi^3 + 4\cos t + 4t\sin t - 2t^2\cos t + \frac{2}{3}t^3.$$

We next seek the impulsive term of (3.1). For $3\pi/2 \leq t \leq 2\pi$, we have

$$\sum_{i=1}^{j(t)} b_i(t-\theta_i)^2 = \sum_{i=1}^{0} b_i(t-\theta_i)^2 = 0.$$

We will show that

$$\sum_{i=1}^{j(t)} b_i(t-\theta_i)^2 = t^2 - 4(2n-1)\pi t + 4n(4n-3)\pi^2 \quad \text{for } (4n-2)\pi < t \le (4n-1)\pi \tag{4.5}$$

by using mathematical induction. Let $n = 1$. Then we have

$$\sum_{i=1}^{j(t)} b_i(t-\theta_i)^2 = \sum_{i=1}^{1} b_i(t-\theta_i)^2 = b_1(t-2\pi)^2 = t^2 - 4\pi t + 4\pi^2.$$

Hence, the equality (4.5) holds when $n = 1$. Assume that (4.5) holds when $n = m \in \mathbb{N}$; namely,

$$\sum_{i=1}^{j(t)} b_i(t-\theta_i)^2 = \sum_{i=1}^{4m-3} b_i(t-\theta_i)^2 = t^2 - 4(2m-1)\pi t + 4m(4m-3)\pi^2$$

for $(4m-2)\pi < t \le (4m-1)\pi$. We check that (4.5) also holds when $n = m+1$. For $(4m-1)\pi < t \le 4m\pi$, we have

$$\begin{aligned}\sum_{i=1}^{j(t)} b_i(t-\theta_i)^2 = \sum_{i=1}^{4m-2} b_i(t-\theta_i)^2 &= \sum_{i=1}^{4m-3} b_i(t-\theta_i)^2 + b_{4m-2}\big(t-(4m-1)\pi\big)^2 \\ &= t^2 - 4(2m-1)\pi t + 4m(4m-3)\pi^2 - \big(t-(4m-1)\pi\big)^2 \\ &= 2\pi t - (4m+1)\pi^2.\end{aligned}$$

For $4m\pi < t \le (4m+1)\pi$, we have

$$\begin{aligned}\sum_{i=1}^{j(t)} b_i(t-\theta_i)^2 = \sum_{i=1}^{4m-1} b_i(t-\theta_i)^2 &= \sum_{i=1}^{4m-2} b_i(t-\theta_i)^2 + b_{4m-1}(t-4m\pi)^2 \\ &= 2\pi t - (4m+1)\pi^2 - (t-4m\pi)^2 \\ &= -t^2 + 2(4m+1)\pi t - (16m^2+4m+1)\pi^2.\end{aligned}$$

For $(4m+1)\pi < t \le (4m+2)\pi$, we have

$$\begin{aligned}\sum_{i=1}^{j(t)} b_i(t-\theta_i)^2 = \sum_{i=1}^{4m} b_i(t-\theta_i)^2 &= \sum_{i=1}^{4m-1} b_i(t-\theta_i)^2 + b_{4m}\big(t-(4m+1)\pi\big)^2 \\ &= -t^2 + 2(4m+1)\pi t - (16m^2+4m+1)\pi^2 \\ &\quad + \big(t-(4m+1)\pi\big)^2 \\ &= 4m\pi^2.\end{aligned}$$

For $(4m+2)\pi < t \le (4m+3)\pi$, we have

$$\begin{aligned}\sum_{i=1}^{j(t)} b_i(t-\theta_i)^2 = \sum_{i=1}^{4m+1} b_i(t-\theta_i)^2 &= \sum_{i=1}^{4m} b_i(t-\theta_i)^2 + b_{4m+1}\big(t-(4m+2)\pi\big)^2 \\ &= 4m\pi^2 + \big(t-(4m+2)\pi\big)^2 \\ &= t^2 - 4\big(2(m+1)-1\big)\pi t + 4(m+1)\big(4(m+1)-3\big)\pi^2.\end{aligned}$$

Hence, the equality (4.5) holds when $n=m+1$. By repeating the same calculation, we obtain

$$\sum_{i=1}^{j(t)} b_i(t-\theta_i)^2 = \begin{cases} 0 & \text{if } 3\pi/2 \le t \le 2\pi, \\ t^2-4(2n-1)\pi t+4n(4n-3)\pi^2 & \text{if } (4n-2)\pi < t \le (4n-1)\pi, \\ 2\pi t-(4n+1)\pi^2 & \text{if } (4n-1)\pi < t \le 4n\pi, \\ -t^2+2(4n+1)\pi t & \\ \quad -(16n^2+4n+1)\pi^2 & \text{if } 4n\pi < t \le (4n+1)\pi, \\ 4n\pi^2 & \text{if } (4n+1)\pi < t \le (4n+2)\pi. \end{cases}$$

Thus, it turns out from this formulation that the impulsive term $\sum_{i=1}^{j(t)} b_i(t-\theta_i)^2$ is a non-negative and increasing function on $[3\pi/2,\infty)$. We therefore conclude that

$$\frac{1}{t^2}\left\{\int_{t_0}^{t}(t-s)^{\alpha}a(s)ds+\sum_{i=1}^{j(t)} b_i(t-\theta_i)^2\right\} \ge \frac{6\pi}{t^2}-\frac{9\pi^3}{4t^2}+\frac{4\cos t}{t^2}+\frac{4\sin t}{t}-2\cos t+\frac{2}{3}t,$$

which diverges to infinity as t tends to ∞. This means that condition (3.1) holds when $\alpha=2$. Hence, by Corollary 2, equation (1.1) is oscillatory.

However, condition (1.6) does not hold. In fact, we have

$$\begin{aligned} \int_{3\pi/2}^{t}\int_{3\pi/2}^{s} a(\tau)d\tau ds &= \int_{3\pi/2}^{t}\int_{3\pi/2}^{s}\big(2\sin\tau+4\tau\cos\tau-\tau^2\sin\tau+2\big)d\tau ds \\ &= \int_{3\pi/2}^{t}\Big[2\tau\sin\tau+\tau^2\cos\tau+2\tau\Big]_{3\pi/2}^{s} ds \\ &= \int_{3\pi/2}^{t}\big(2s\sin s+s^2\cos s+2s\big)ds \\ &= \Big[s^2(\sin s+1)\Big]_{3\pi/2}^{t} = t^2(\sin t+1) \end{aligned}$$

and from a straightforward calculation we see that

$$\int_{3\pi/2}^{t}\sum_{i=1}^{j(s)} b_i\,ds = \begin{cases} 0 & \text{if } 3\pi/2 \le t \le 2\pi, \\ t-(4n-2)\pi^2 & \text{if } (4n-2)\pi < t \le (4n-1)\pi, \\ \pi & \text{if } (4n-1)\pi < t \le 4n\pi, \\ -t+(4n+1)\pi & \text{if } 4n\pi < t \le (4n+1)\pi, \\ 0 & \text{if } (4n+1)\pi < t \le (4n+2)\pi. \end{cases}$$

Let $t_n=(4n+3/2)\pi$ for $n\in\mathbb{N}$. Then

$$\int_{3\pi/2}^{t_n}\left\{\int_{3\pi/2}^{s} a(\tau)d\tau+\sum_{i=1}^{j(s)} b_i\right\}ds = t_n^2(\sin t_n+1)+0=0.$$

Thus, Theorem D is inapplicable to Example 2.

5. Conclusion

In this paper, we discussed the oscillation problem of the second-order linear differential equation (1.1) taking into account of impulsive effect. We gave some physical interpretation for equation (1.1) in introduction part. By using Riccati transformation and integral averaging technique, we presented Philos-type oscillation criteria showing how the impulsive term affects the oscillation behavior. The obtained results extend several previous studies. Based on these criteria, it became clear that there is a possibility that a mass point may oscillate due to the influence of the impulsive effect, even though the mass point did not oscillate in the model from which the impulsive effect was removed. What is more, our results also cover the linear differential equations without impulsive effect. Moreover, we chose the case of $H(t,s) = (t-s)^{\alpha}$ and $H(t,s) = \ln t / \ln s$ to give two useful corollaries. Finally, we gave an equivalent relationship between oscillation criteria using Fubini's theorem. In addition, we presented examples to illustrate our results.

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