Nonoscillation of Mathieu equations with two frequencies

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Abstract

As is well known, Mathieu's equation is a representative of mathematical models describing parametric excitation phenomena. This paper deals with the oscillation problem for Mathieu's equation with two frequencies. The ratio of these two frequencies is not necessarily a rational number. When the ratio is an irrational number, the coefficient of Mathieu's equation is is quasi-periodic, but not periodic. For this reason, the basic knowledge for linear periodic systems such as Floquet theory is not useful. Whether all solutions of Mathieu's equation oscillate or not is determined by parameters and frequencies. Our results provide parametric conditions to guarantee that all solutions are nonoscillatory. The advantage of the obtained parametric conditions is that it can be easily checked. Parametric nonoscillation region is drawn to understand these results easily. Finally, several simulations are carried out to clarify the remaining problems.

Key words: Nonoscillation; Parametric excitation; Mathieu's equation; Frequencies; Quasi-periodic

1. Introduction

A phenomenon where amplitude is magnified by varying some parameters is called *parametric excitation*. The work of Mathieu [11] is a pioneer research on parametric excitation. He studied the vibration of the oval type drum film and derived the second-order differential equation

$$x'' + (-\alpha + \beta\cos(2t))x = 0$$

as its mathematical approximation model. Here, the parameters α and β are real numbers. The equation above is named Mathieu's equation after him. Mathieu's equation has been applied to many problems in physics and natural sciences (refer to the book [12]).

By a simple transformation, Mathieu's equation becomes another form,

$$x'' + (-\tilde{\alpha} + \tilde{\beta}\sin t)x = 0.$$

The oscillation problem for Mathieu's equation and its equivalent form were already studied by several researchers, for example, El-Sayed [4], Leighton [9] and Sun et al. [16] (see

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also [12, p. 29]). The present authors [6] recently examined the second-order differential equation

$$x'' + (-\alpha + \beta\cos(\omega t))x = 0, \tag{1.1}$$

where the frequency ω is a positive number, and presented simple parametric conditions for oscillation and nonoscillation, respectively. Their results are as follows.

Theorem A. *If*

$$\alpha > 0 \quad \textit{and} \quad |\beta| \ge \omega\sqrt{2\alpha} + \alpha,$$

then all non-trivial solutions of (1.1) *are oscillatory*.

Theorem B. *If*

$$\alpha \ge 0 \quad \textit{and} \quad |\beta| \le \frac{\omega\sqrt{2\alpha}}{2} + \alpha,$$

then all non-trivial solutions of (1.1) *are nonoscillatory*.

Theorems A and B extend previous results in [4, 9, 16]. Some ideas in [8, 14, 15] are used for the proofs of Theorems A and B. In equation (1.1), the coefficient $-\alpha+\beta\cos(\omega t)$ is periodic. However, Mathieu's equations with a non-periodic coefficient often appear as practical physical models. To be able to deal well with even such cases, we consider the second-order differential equation

$$x'' + \big(-\alpha + \beta\cos(\omega_1 t) + \gamma\cos(\omega_2 t)\big)x = 0. \tag{1.2}$$

Here, the new parameter γ is a real number and the two frequencies ω_1 and ω_2 are positive numbers. Equation (1.2) coincides with the basic form (1.1) when $\gamma = 0$. If ω_1/ω_2 is a rational number, then the coefficient $-\alpha + \beta\cos(\omega_1 t) + \gamma\cos(\omega_2 t)$ is periodic. On the other hand, if ω_1/ω_2 is an irrational number, then the coefficient of (1.2) is not periodic. In such a case, equation (1.2) is called a *quasi-periodic* Mathieu equation (see [1, 3, 13, 20]). Quasi-periodic Mathieu equations have been studied mainly in the stability theory. For example, Zounes and Rand [20] discussed the case that $\beta = \gamma$, $\omega_1 = 1$ and ω_2 is any irrational number. Yagoubi et al. [18, 19] studied the stability of the free surface of a liquid layer by applying the method of harmonic balance to quasi-periodic equations of Mathieu type. Very recently, Kovacic et al. [7] systematically overviewed stability charts of a classical Mathieu's equation and its generalizations including quasi-periodic Mathieu equations. Note that the definition on their stability is not that of Lyapunov. They say that a quasi-periodic Mathieu equation is stable if all solutions are bounded and it is unstable if an unbounded solution exists. In contrast to the stability theory, the oscillation theory for quasi-periodic Mathieu equations has not been much reported so far.

The purpose of this paper is to give some sufficient conditions which guarantee that all non-trivial solutions of (1.2) are nonoscillatory.

To achieve our objective, it is worthwhile to explain fundamental results about nonoscillation (or oscillation) of the more general equation

$$x'' + c(t)x = 0, \tag{1.3}$$

which includes equations (1.1) and (1.2). Here, the coefficient c is a continuous function on $[0,\infty)$. A non-trivial solution of (1.3) is said to be *oscillatory* if it has an infinite number of zeros on $0 < t < \infty$. Otherwise, the solution is said to be *nonoscillatory*. Hence, a nonoscillatory solution of (1.3) is eventually positive or eventually negative. As can see immediately from this definition, whether a nontrivial solution of (1.3) is oscillatory or nonoscillatory is irrelevant to whether it is bounded or unbounded.

Equation (1.3) has been widely studied in many books (for example, see [2, 17]). In case that c is a periodic function with period $T > 0$, equation (1.3) is called Hill's differential equation (refer to [5, 10, 12]). Equation (1.1) is an example of the Hill differential equation. When ω_1/ω_2 is a rational number, equation (1.2) is also a differential equation of Hill type.

It is well-known that if $c(t) \leq 0$ for all sufficiently large t, then all non-trivial solutions of (1.3) are nonoscillatory even if c is not a periodic function (see [17, p. 45]). Hence, if $\alpha \geq |\beta| + |\gamma|$, then all non-trivial solutions of (1.2) are nonoscillatory. Leighton-Wintner's oscillation criterion is also well-known: if

$$\int^{\infty} c(t)dt = \infty,$$

then all non-trivial solutions of (1.3) are oscillatory (see [17, p. 70]). Hence, $\alpha < 0$ implies that all non-trivial solutions of (1.2) are oscillatory. Thus, the unsettled case is that

$$0 \leq \alpha < |\beta| + |\gamma|. \tag{1.4}$$

In this paper, we report nonoscillation criteria that can also be applied in the case of (1.4). Our nonoscillation theorems are as follows.

Theorem 1.1. *If*

$$\alpha \geq |\gamma| \quad \textit{and} \quad |\beta| + |\gamma| \leq \frac{\omega_1\sqrt{2(\alpha - |\gamma|)}}{2} + \alpha, \tag{1.5}$$

then all non-trivial solutions of (1.2) *are nonoscillatory*.

Theorem 1.2. *If*

$$\alpha \geq |\beta| \quad \textit{and} \quad |\beta| + |\gamma| \leq \frac{\omega_2\sqrt{2(\alpha - |\beta|)}}{2} + \alpha, \tag{1.6}$$

then all non-trivial solutions of (1.2) *are nonoscillatory*.

These criteria (1.5) and (1.6) are expressed by the parameters α, β, γ and the frequencies ω_1 and ω_2. The proofs are given by using "phase plane analysis" that was proved in [14]. In the special case that $\gamma = 0$, the obtained results coincide with Theorem B.

2. Proof of nonoscillation theorems

Consider the second-order differential equation with damping,

$$y'' + a(t)y' + b(t)y = 0, \tag{2.1}$$

where a, $b \colon [0, \infty) \to \mathbb{R}$ are continuous functions. Using phase plane analysis, Sugie [14, Theorem 1.1] presented the following nonoscillation theorem.

Lemma 2.1. *Let S be a bounded, closed and convex set in the region*

$$R = \left\{(u,v) \colon u \geq 0 \ \text{ and } \ 0 \leq v \leq u^2/4\right\}.$$

Suppose that there exists a $T > 0$ such that

$$(a(t), b(t)) \in S \quad \text{for } \ t \geq T.$$

Then all non-trivial solutions of (2.1) *are nonoscillatory.*

His original result is applicable even to nonlinear differential equations including equation (2.1). By means of Lemma 2.1 and an equivalence transformation, we prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1. Recall that only case (1.4) is unresolved. From (1.4) and (1.5) it follows that

$$0 \leq \alpha < |\beta| + |\gamma| \leq \frac{\omega_1\sqrt{2(\alpha - |\gamma|)}}{2} + \alpha.$$

Hence, we see that $|\gamma| < \alpha < |\beta| + |\gamma|$. For simplicity, let

$$\begin{aligned} \rho &= \frac{|\beta| + |\gamma| - \alpha}{\omega_1} + \sqrt{\frac{(|\beta| + |\gamma| - \alpha)^2}{\omega_1^2} + 2\alpha}, \\ \sigma &= \frac{2(|\beta| + |\gamma| - \alpha)}{\omega_1}. \end{aligned} \tag{2.2}$$

Then, it turns out that $0 < \sigma < \rho$ and $\rho^2 = 2\alpha + \rho\sigma$. From now on, we will divide the argument into two cases: (i) $\beta \geq 0$ and (ii) $\beta < 0$.

Case (i): Consider equation (2.1) with

$$\begin{cases} a(t) = 2\rho - \sigma\sin(\omega_1 t), \\ b(t) = \alpha + \rho\sigma\big(1 - \sin(\omega_1 t)\big) + (\alpha - |\gamma|)\cos(\omega_1 t) + \gamma\cos(\omega_2 t) + \dfrac{1}{4}\sigma^2\sin^2(\omega_1 t). \end{cases} \tag{2.3}$$

Let η be any solution of (2.1) with (2.3). Define

$$\xi(t) = \eta(t)\exp\left(\frac{1}{2}\int_0^t a(s)ds\right).$$

Then, we have

$$\begin{aligned} \xi''(t) &= \left(\eta''(t) + a(t)\eta'(t) + \frac{1}{2}a'(t)\eta(t) + \frac{1}{4}a^2(t)\eta(t)\right)\exp\left(\frac{1}{2}\int_0^t a(s)ds\right) \\ &= \left(\frac{1}{4}a^2(t) + \frac{1}{2}a'(t) - b(t)\right)\eta(t)\exp\left(\frac{1}{2}\int_0^t a(s)ds\right). \end{aligned}$$

Taking into account that $\rho^2 = 2\alpha + \rho\sigma$ and $\sigma\omega_1 = 2(\beta + |\gamma| - \alpha)$, we obtain

$$\begin{aligned} &\xi''(t) + \big(-\alpha + \beta\cos(\omega_1 t) + \gamma\cos(\omega_2 t)\big)\xi(t) \\ &= \left(\frac{1}{4}a^2(t) + \frac{1}{2}a'(t) - b(t) - \alpha + \beta\cos(\omega_1 t) + \gamma\cos(\omega_2 t)\right)\eta(t)\exp\left(\frac{1}{2}\int_0^t a(s)ds\right) \\ &= \left(\rho^2 - 2\alpha - \rho\sigma + \left(\beta + |\gamma| - \alpha - \frac{1}{2}\sigma\omega_1\right)\right)\eta(t)\exp\left(\frac{1}{2}\int_0^t a(s)ds\right) = 0. \end{aligned}$$

This means that the function ξ is a solution of (1.2). We therefore conclude that all nontrivial solutions of (1.2) are nonoscillatory if and only if those of (2.1) are nonoscillatory under the assumption (2.3).

Let $u = a(t)$ and $v = b(t)$. From (2.3), we see that $2\rho - \sigma \le u \le 2\rho + \sigma$,

$$\sin(\omega_1 t) = \frac{2\rho - u}{\sigma} \quad \text{and} \quad \cos(\omega_1 t) = \pm\sqrt{1 - \left(\frac{2\rho - u}{\sigma}\right)^2} \quad \text{for} \ \ t \ge 0.$$

Hence, we have

$$v = \alpha + \rho\sigma - \rho(2\rho - u) \pm (\alpha - |\gamma|)\sqrt{1 - \left(\frac{2\rho - u}{\sigma}\right)^2} + \gamma\cos(\omega_2 t) + \frac{1}{4}(2\rho - u)^2.$$

Let $f,\ g\colon [2\rho - \sigma, 2\rho + \sigma] \to \mathbb{R}$ be continuous functions defined by

$$f(u) = \alpha + \rho\sigma - \rho(2\rho - u) + (\alpha - |\gamma|)\sqrt{1 - \left(\frac{2\rho - u}{\sigma}\right)^2} + |\gamma| + \frac{1}{4}(2\rho - u)^2,$$

$$g(u) = \alpha + \rho\sigma - \rho(2\rho - u) - (\alpha - |\gamma|)\sqrt{1 - \left(\frac{2\rho - u}{\sigma}\right)^2} - |\gamma| + \frac{1}{4}(2\rho - u)^2,$$

respectively. Note that $g \le f$ in the whole interval $[2\rho - \sigma, 2\rho + \sigma]$. Define

$$S = \big\{(u, v)\colon 2\rho - \sigma \le u \le 2\rho + \sigma \ \text{ and } \ g(u) \le v \le f(u)\big\}. \tag{2.4}$$

The set S is bounded and closed. Since $|\cos(\omega_2 t)| \le 1$ for $t \ge 0$, we see that

$$(a(t), b(t)) \in S \quad \text{for} \ \ t \ge 0.$$

Using the fact that $\rho^2 = 2\alpha + \rho\sigma$ again, we can rewrite f and g as follows:

$$f(u) = \frac{1}{4}u^2 - (\alpha - |\gamma|)\left(1 - \sqrt{1 - \left(\frac{2\rho - u}{\sigma}\right)^2}\right);$$
$$g(u) = \frac{1}{4}u^2 - (\alpha + |\gamma|) - (\alpha - |\gamma|)\sqrt{1 - \left(\frac{2\rho - u}{\sigma}\right)^2}.$$

Since $0 < 2\rho - \sigma \le u \le 2\rho + \sigma$ and $\alpha > |\gamma|$, we see that

$$f(u) \le \frac{1}{4}u^2 \quad \text{and} \quad g(u) \ge \frac{1}{4}(2\rho - \sigma)^2 - (\alpha + |\gamma|) - (\alpha - |\gamma|) = \frac{1}{4}\sigma^2 > 0.$$

We therefore conclude that $S \subset R$. By a straightforward calculation, we have

$$\frac{d^2}{du^2}f(u) = \frac{1}{2} - \frac{\alpha - |\gamma|}{\sigma^2\left(1 - \left(\frac{2\rho - u}{\sigma}\right)^2\right)^{3/2}} < \frac{1}{2} - \frac{\alpha - |\gamma|}{\sigma^2}$$

and

$$\frac{d^2}{du^2}g(u) = \frac{1}{2} + \frac{\alpha - |\gamma|}{\sigma^2\left(1 - \left(\frac{2\rho - u}{\sigma}\right)^2\right)^{3/2}} > \frac{1}{2}$$

for $u \in [2\rho - \sigma, 2\rho + \sigma]$. From (1.5) and (2.2) it turns out that

$$\frac{d^2}{du^2}f(u) \le \frac{1}{2} - \frac{(\alpha - |\gamma|)\omega_1^2}{4(|\beta| + |\gamma| - \alpha)^2} \le \frac{1}{2} - \frac{(\alpha - |\gamma|)\omega_1^2}{2(\alpha - |\gamma|)\omega_1^2} = 0.$$

Hence, the upper boundary curve $v = f(u)$ is convex upward and the lower boundary curve $v = g(u)$ is convex downward. We therefore conclude that S is a convex set (see Fig. 1).

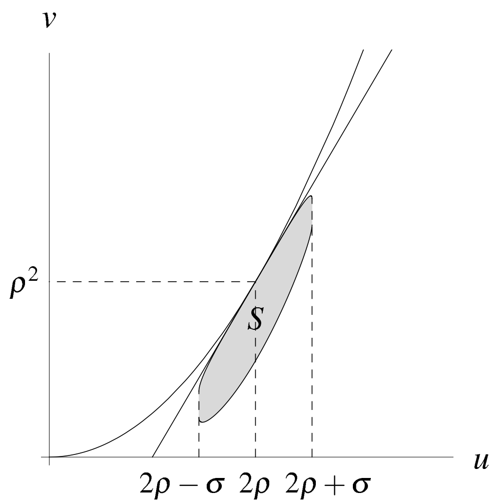


Figure 1: The shaded part is the set S that was given by (2.4). In this case, the functions f and g are given with $\alpha = 3$, $\gamma = 1$, $\rho = 1 + \sqrt{7}$ and $\sigma = 2$. The straight line $v = \rho u - \rho^2$ is the common tangent at the point $(2\rho, \rho^2)$.

As described above, we could check that all conditions of Lemma 2.1 were satisfied. By Lemma 2.1, all non-trivial solutions of (2.1) with (2.3) are nonoscillatory. Hence, all non-trivial solutions of (1.2) are also nonoscillatory.

Case (ii): Let

$$\begin{cases} a(t) = 2\rho + \sigma \sin(\omega_1 t), \\ b(t) = \alpha + \rho\sigma\big(1 + \sin(\omega_1 t)\big) - (\alpha - |\gamma|)\cos(\omega_1 t) + \gamma\cos(\omega_2 t) + \dfrac{1}{4}\sigma^2\sin^2(\omega_1 t) \end{cases} \tag{2.5}$$

instead of (2.3). Then, as in the proof of case (i), we can verify that equation (1.2) is equivalent to equation (2.1) under the assumption (2.5). Hence, we have only to show that all non-trivial solutions of (2.1) with (2.5) are nonoscillatory.

Let $u = a(t)$ and $v = b(t)$. Then, it follows from (2.5) that

$$2\rho - \sigma \le u \le 2\rho + \sigma$$

and

$$v = \alpha + \rho\sigma - \rho(2\rho - u) \pm (\alpha - |\gamma|)\sqrt{1 - \left(\frac{2\rho - u}{\sigma}\right)^2} + \gamma\cos(\omega_2 t) + \frac{1}{4}(2\rho - u)^2.$$

Using the same way as in case (i), we can check that the set S given by (2.4) is bounded, closed and convex, and it is contained in R. Hence, by Lemma 2.1, all non-trivial solutions of (2.1) with (2.5) are nonoscillatory.

The proof of Theorem 1.1 is now complete. □

Remark 2.1. The parabola $v = u^2/4$ and the upper boundary curve $v = f(u)$ have only one tangent point $(2\rho, \rho^2)$ and the equation of the common tangent is $v = \rho u - \rho^2$. The convex set S is located under the tangent line.

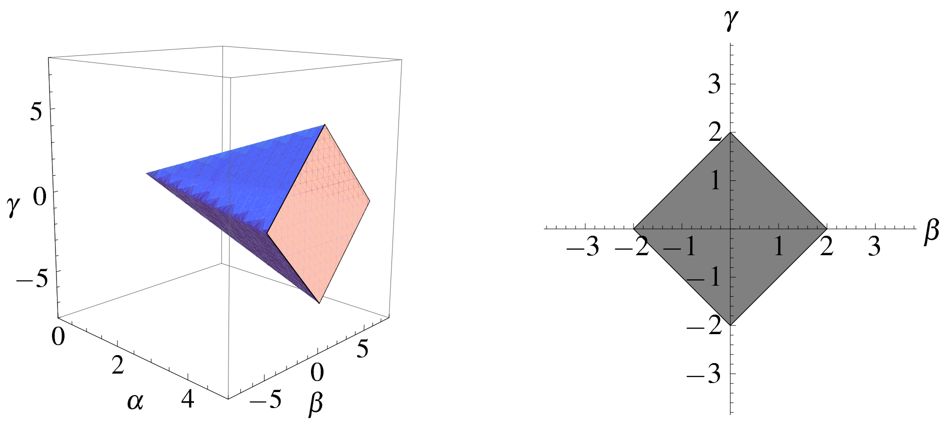
By replacing β, γ and ω_1 with γ, β and ω_2, respectively, and using the same method as the proof of Theorem 1.1, we can prove Theorem 1.2. We omit the detail.

3. Quasi-periodic Mathieu equation

As mentioned in Section 1, for any $\omega_1 > 0$ and $\omega_2 > 0$, all non-trivial solutions of (1.2) are nonoscillatory if

$$\alpha \geq |\beta| + |\gamma|. \tag{3.1}$$

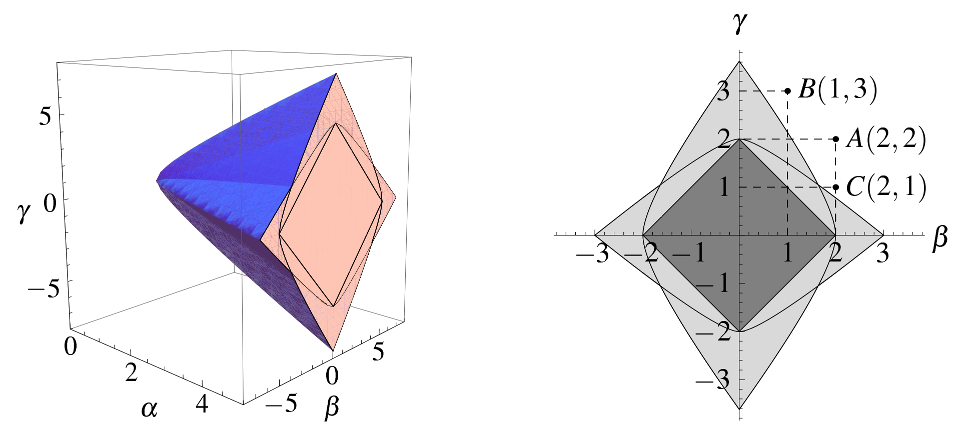
The set of three parameters (α, β, γ) satisfying the above inequality draws a regular quadrangular pyramid in the three dimensional space (see Figure 2(a)). Figure 2(b) gives the cross section of the regular quadrangular pyramid when $\alpha = 2$. Theorems 1.1 and 1.2 clarify how much the parametric nonoscillation region for equation (1.2) can be widened than the regular quadrangular pyramid. How much it spreads depends on the two frequencies ω_1 and ω_2.



(a) (b)

Figure 2: All non-trivial solutions of (1.2) are nonoscillatory if the set of (α, β, γ) is in the square pyramid given by (3.1). The dark shaded part is the cross section of the square pyramid when $\alpha = 2$.

In the case that ω_1/ω_2 is a rational number, the coefficient $-\alpha + \beta\cos(\omega_1 t) + \gamma\cos(\omega_2 t)$ is a periodic function and equation (1.2) is a differential equation of Hill type. In this case,



(a) (b)

Figure 3: All non-trivial solutions of (1.2) are nonoscillatory if the set of (α, β, γ) is in the three-dimensional region satisfying inequality (1.5) or (1.6). This parametric nonoscillation region contains the square pyramid given by (3.1). The dark shaded part is the same as that of Figure 2. The union of the light shaded part and the dark shaded part is the cross section of the parametric nonoscillation region when $\alpha = 2$.

Floquet theory might be useful for equation (1.2), but it is inapplicable if ω_1/ω_2 is an irrational number. For example, let

$$\omega_1 = 1 \quad \text{and} \quad \omega_2 = \frac{1+\sqrt{5}}{2} \stackrel{\text{def}}{=} \phi;$$

namely, the two frequencies are in the golden ratio. In Figure 3(a), we draw the parametric nonoscillation region satisfying inequality (1.5) or (1.6) to compare with the regular quadrangular pyramid. To make clearly understandable, we also give a cross section of the parametric nonoscillation region and the regular quadrangular pyramid in Figure 3(b). In Figure 3(b), two leaves seem to be overlapping each other.

Since each of inequalities (1.5) and (1.6) is not a necessary and sufficient condition, the real parametric nonoscillation region is wider than two leaves given in Figure 3(b). Here, a simple question arises. How much the real parametric nonoscillation region is different from the overlap of two leaves? To examine the situation, we give several numerical simulations.

Three points A, B and C are marked in Figure 3. For example, the point A means that both parameters β and γ are 2.

Simulation 3.1. Let $\alpha = \beta = \gamma = 2$. Then all non-trivial solutions of

$$x'' + \big(-\alpha + \beta\cos t + \gamma\cos(\phi t)\big)x = 0 \tag{3.2}$$

are oscillatory, where $\phi = (1+\sqrt{5})/2$.

Let $y = x'$. Then, equation (3.2) becomes the planar system

$$\begin{aligned} x' &= y, \\ y' &= \big(\alpha - \beta\cos t - \gamma\cos(\phi t)\big)x. \end{aligned} \tag{3.3}$$

Let $t_0 \geq 0$ and $(\xi, \eta) \in \mathbb{R}^2$. Because of the uniqueness of solutions to the initial conditions, there exists only one solution x of (3.2) satisfying that $(x(t_0), x'(t_0)) = (\xi, \eta)$. Let (x, y)

be the solution of (3.3) that corresponds to the solution x of (3.2). Then, the solution (x, y) satisfies that $(x(t_0), y(t_0)) = (\xi, \eta)$. The projection of the solution (x, y) of (3.3) onto the (x, y)-plane becomes a curve starting at the point (ξ, η). This curve is called a *solution curve*. It can be thought that the point $(x(t), y(t))$ moves on the solution curve as t increases.

A non-trivial solution x of (3.2) is oscillatory if and only if the point $(x(t), y(t))$ on the solution curve of (3.3) corresponding to the solution x moves from the right-half (resp., left-half) plane to the left-half (resp., right-half) plane, and then, it moves from the left-half (resp., right-half) plane to the right-half (resp., left-half) plane, and this movement is repeated infinitely many times. On the other hand, a non-trivial solution x of (3.2) is nonoscillatory if and only if the point $(x(t), y(t))$ on the solution curve of (3.3) corresponding to the solution x stays in either the right-half plane or the left-half plane.

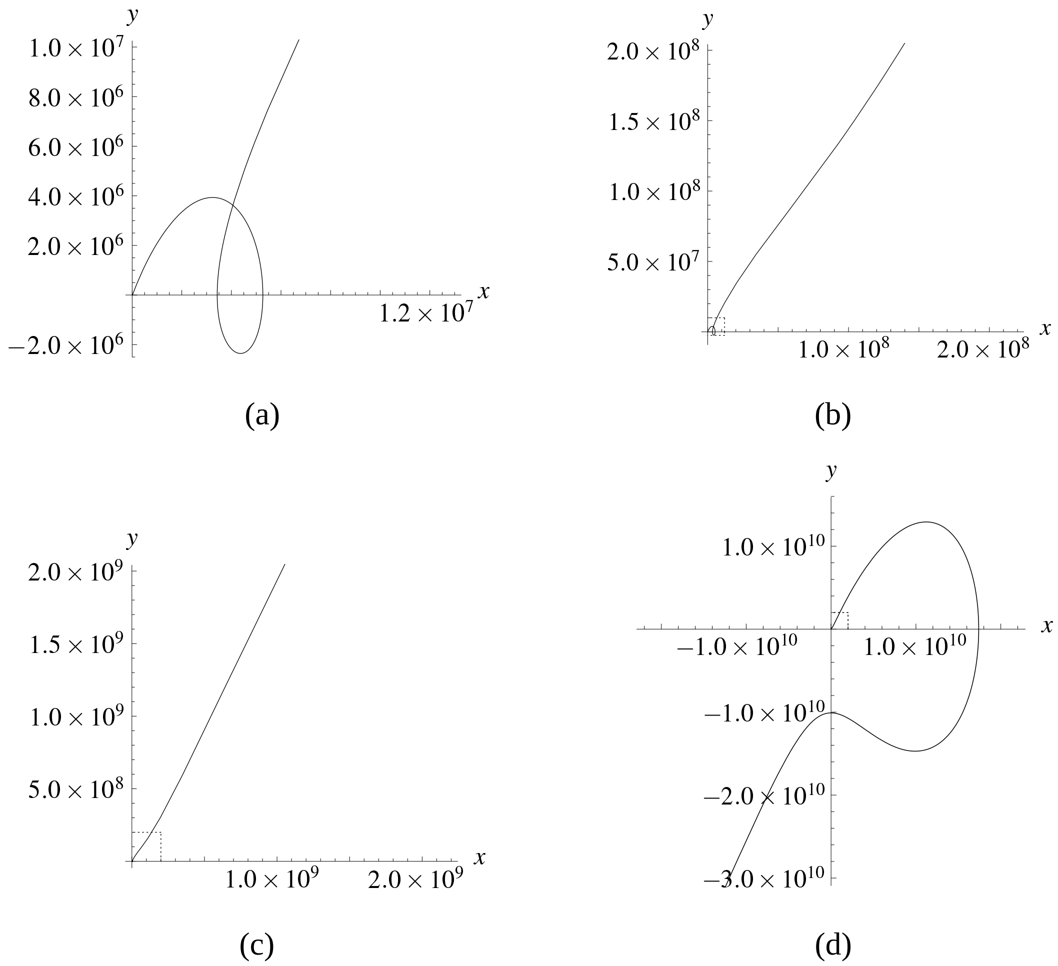


Figure 4: These four graphs represent the solution curve of (3.3) starting at the point $(0, 1)$ in the case that $\alpha = \beta = \gamma = 2$.

The drawing range of subfigure (a) is $[0,\ 1.2 \times 10^7] \times [-2.0 \times 10^6,\ 1.0 \times 10^7]$. In this range, the solution curve makes a loop. This range corresponds to the rectangle surrounded by dotted lines in subfigure (b). The drawing range of subfigure (b) corresponds to the rectangle surrounded by dotted lines in subfigure (c). In the drawing range of subfigure (c), the solution curve is like a straight line, there is no loop. The solution curve changes dramatically in the drawing range of subfigure (d) and moves from the right-half plane to the left-half plane. Such a behavior of the solution curve repeats infinitely many times. Thus, we see that all non-trivial solutions of (3.2) are oscillatory in this case.

Although the point A is quite close to the light shaded part of Figure 3, this point does not enter in the real parametric nonoscillation region for equation (3.2), because an oscillatory solution exists as can be seen from Figure 4. Then, will not all non-trivial solutions of (3.2) oscillate if a point of the (β, γ)-plane is more closer to the light shaded part? The second simulation is an answer to this question.

Simulation 3.2. Let $\alpha = 2$, $\beta = 1$ and $\gamma = 3$. Then all non-trivial solutions of (3.2) are oscillatory.

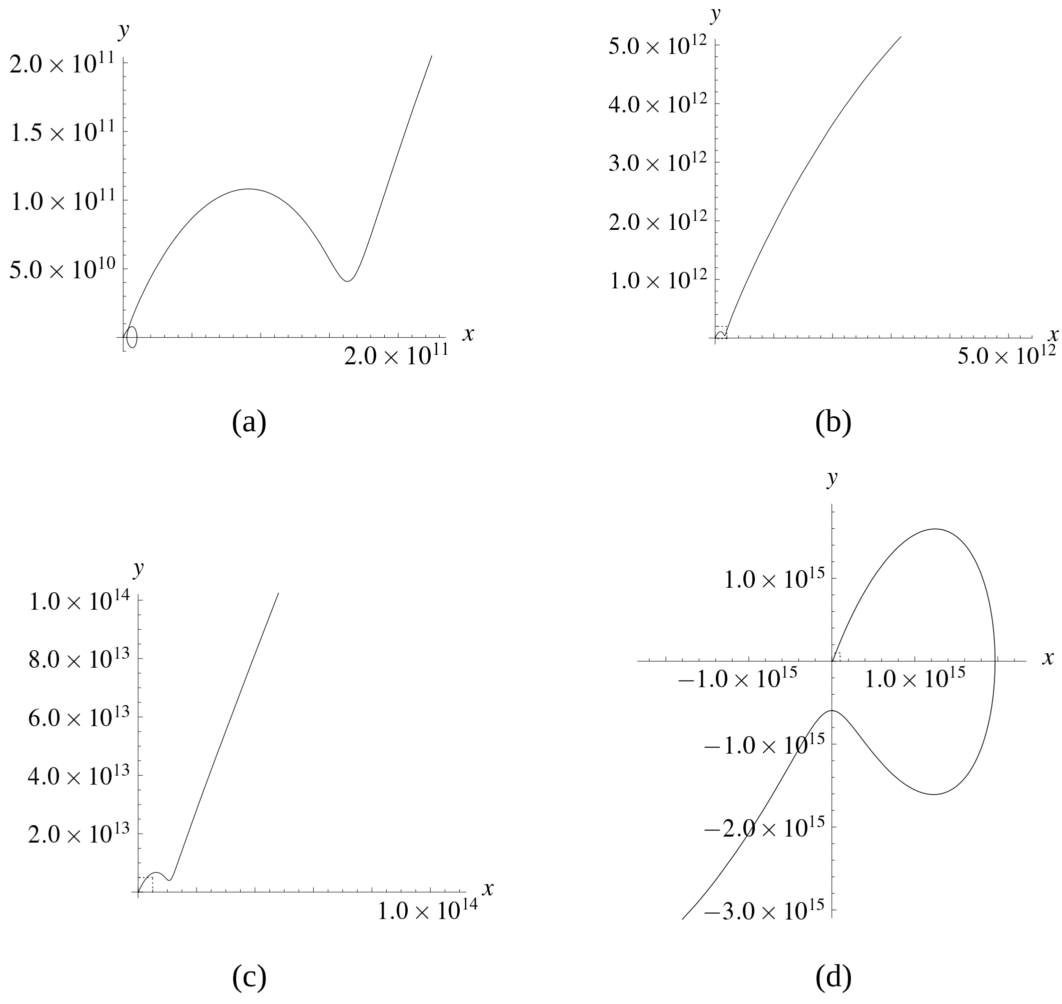


Figure 5: These four graphs represent the solution curve of (3.3) starting at the point $(0, 1)$ in the case that $\alpha = \beta = \gamma = 2$.

The style of Figure 5 is the same as that of Figure 4. The drawing range of subfigures (a), (b), (c) and (d) are enlarged in alphabetical order. In the drawing range of subfigure (a), the solution curve makes a loop. As shown in subfigures (b) and (c), the solution curve once bends but then flows like a straight line. After that, the solution curve moves from the right-half plane to the left-half plane in the drawing range of subfigure (d). Such a behavior of the solution curve repeats infinitely many times. Thus, we see that all non-trivial solutions of (3.2) are oscillatory in this case.

Figure 5 shows that the point B also does not enter in the real parametric nonoscillation region for equation (3.2). As shown in Figure 3, the union of the light shaded part and the dark shaded part is not convex. However, it seems to be reasonable to think that the real parametric nonoscillation region is convex.

Consider the boundary curve of the light shaded part in Figure 3. This curve intersects the β-axis at $\beta = 3$ and $\beta = -3$. On the other hand, the line intersects the γ-axis at $\gamma = (5+\sqrt{5})/2$ and $\gamma = -(5+\sqrt{5})/2$. We can make the parallelogram

$$\frac{|\beta|}{3} + \frac{2|\gamma|}{5+\sqrt{5}} \leq 1$$

by connecting those four intersection points with a straight line. Note that the point C lies within this parallelogram.

Simulation 3.3. Let $\alpha = \beta = 2$ and $\gamma = 1$. Then all non-trivial solutions of (3.2) are nonoscillatory.

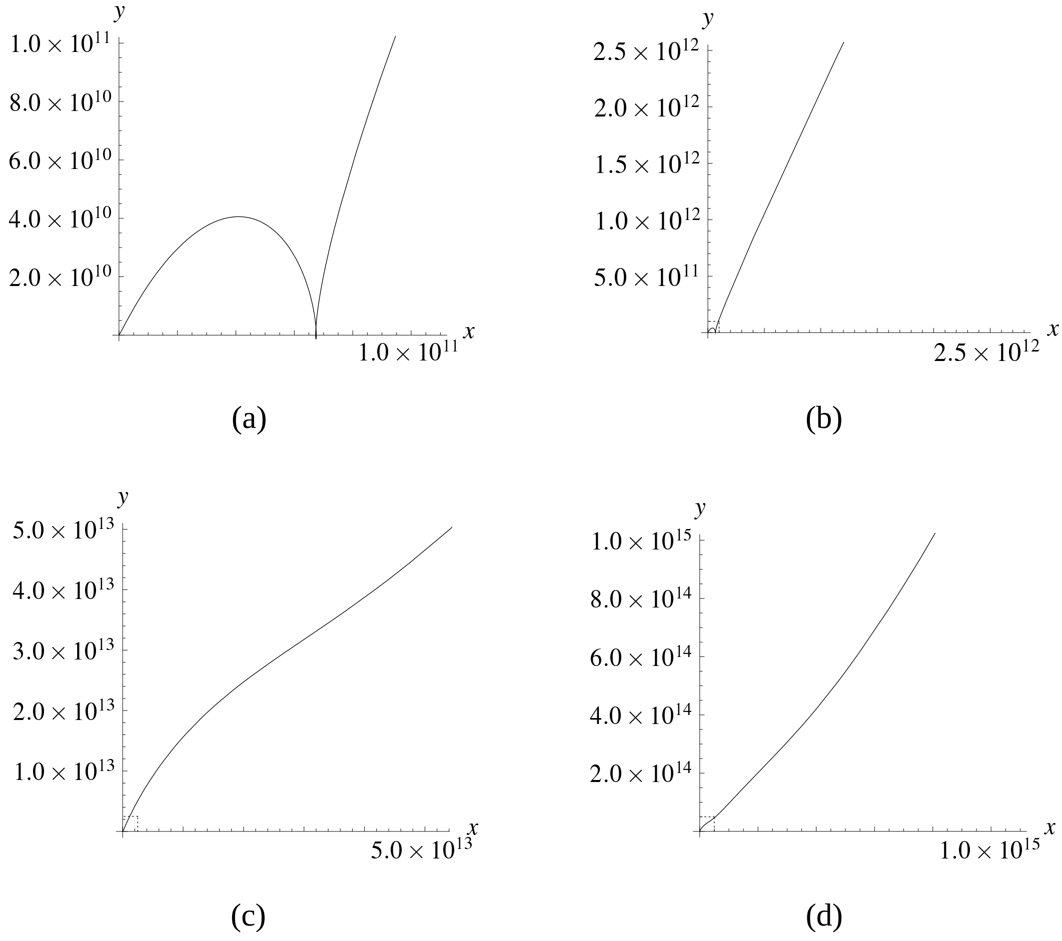


Figure 6: These four graphs represent the solution curve of (3.3) starting at the point $(0, 1)$ in the case that $\alpha = \beta = 2$ and $\gamma = 1$.

The style of Figure 6 is the same as those of Figures 4 and 5. We can find a small loop near the point $(6.75 \times 10^{10}, 0)$ in the drawing range of subfigure (a). The loop is too small to be identified in the drawing range of subfigure (b). As shown in subfigures (c) and (d), the solution curve flows in one direction without a loop. The same situation continues in a wider drawing range that of subfigure (d). Thus, we see that all non-trivial solutions of (3.2) are nonoscillatory in this case.

From Figure 6, we see that the point C enters in the real parametric nonoscillation region for equation (3.2). Judging from this fact, it is permissible to make the following conjecture.

Conjecture 3.1. *If*

$$\alpha > 0 \quad \textit{and} \quad \frac{|\beta|}{\omega_1\sqrt{\alpha/2}+\alpha} + \frac{|\gamma|}{\omega_2\sqrt{\alpha/2}+\alpha} \le 1,$$

then all non-trivial solutions of (1.2) *are nonoscillatory*.

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