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A NOTE ON SOME WEAKLY MODULAR SEMIMODULATED LATTICES

by

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Introduction

D. Sachs [4] has introduced the notion of a modulated lattice which has enough modular elements and given a characterization of partition lattices. In the previous paper [1], we showed that in some non-atomic modulated lattices, modular elements play a role instead of points.

In the present paper, we introduce the notion of a semimodulated lattice (Definition (3. 7)) and give a characterization of some semimodulated Wilcox lattice (Theorem (3. 10)). And moreover we show that some modulated lattice L and $\mathfrak{M}$ which is the set of all modular elements in L have analogous properties (Theorem (4. 7)). By the above considerations, it seems that in some non-modular semimodulated lattice L, $\mathfrak{M}$ plays a role in the same way as a Wilcox lattice $L \equiv \Lambda - S$ does in Λ and that we obtain a generalization of modulated lattices.

§ 1. Preliminary statements.

In this section, we give some known definitions and lemmas which will be used without explicit mention throughout of this paper.

DEFINITION (1. 1). In a lattice L, $(a, b)M$ means $(c \vee a) \wedge b = c \vee (a \wedge b)$ for every $c \leqq b$ and $(a, b)M^*$ means $(c \wedge a) \vee b = c \wedge (a \vee b)$ for every $c \geqq b$. A lattice L is called an *M-symmetric* lattice when $(a, b)M$ implies $(b, a)M$. And a lattice L is called a *weakly modular* lattice when $a \wedge b \neq 0$ implies $(a, b)M$. Sometimes an M-symmetric lattice is called a semi-modular lattice. (Cf. [2], [4].)

DEFINITION (1. 2). Let L be a lattice with 0. When a covers b, we write $a \gtrdot b$. An element $p \in L$ is called an *atom* or a *point* when $p \gtrdot 0$. An

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element a is called a *modular* element when $(x, a)M$ for every $x \in L$. The elements 0, 1 and every points, if they exist, are modular elements. The set of all modular elements of L is denoted by $\mathfrak{M}$.

LEMMA (1. 3). *Let a, b and c be elements of a lattice L. If $(a, b)M$ and $(a \wedge b, c)M$, then $(a, b \wedge c)M$.*

PROOF. Cf. [3] p. 2.

LEMMA (1. 4). *Let a and b be modular elements of a lattice L, then $a \wedge b$ is a modular element of L.*

PROOF. Cf. [4] p. 326.

LEMMA (1. 5). *Let a be an element of a lattice L. Then $(a, x)M$ for every $x \in L$ if and only if $(a, x)M^*$ for every $x \in L$.*

PROOF. Cf. [3] p. 1.

When $a < b$ in a lattice L, then the interval $\{x \in L\,; a \leqq x \leqq b\}$ is denoted by $L[a, b]$.

LEMMA (1. 6). *If L is an M-symmetric lattice, $a \in \mathfrak{M}$ and $b \in L$, then the sublattices $L[a \wedge b, a]$ and $L[b, a \wedge b]$ are isomorphic by the following mutually inverse mappings: $x \to x \vee b$ and $y \to y \wedge a$.*

PROOF. Cf. [3] p. 2.

DEFINITION (1. 7). A lattice L is called a *relatively complemented* lattice when $a < x < b$ implies the existence y such that $x \vee y = b$, $x \wedge y = a$. Let L be a lattice with 0, then L is called a *left complemented* lattice when a, $b \in L$ implies the existence of b_1 such that $a \vee b_1 = a \vee b$, $b_1 \wedge a = 0$, $b_1 \leqq b$ and $(b_1, a)M$. (Cf. [6] p. 453.)

LEMMA (1. 8). *A left complemented lattice is a relatively complemented M-symmetric lattice.*

PROOF. Cf. [6] p. 454 and [3] p. 12.

DEFINITION (1. 9). Let $\{a_\delta\,; \delta \in D\}$ be an increasingly directed set of a complete lattice L. When $\vee (a_\delta\,; \delta \in D) = a$ implies $\vee (a_\delta \wedge b\,; \delta \in D) = a \wedge b$, L is called an *upper continuous* lattice.

LEMMA (1. 10). *Let $\{m_\delta\,; \delta \in D\}$ be an increasingly directed set of modular elements of an M-symmetric upper continuous lattice L, then $\vee (m_\delta\,; \delta \in D) = m$ is a modular element.*

PROOF. Cf. [4] p. 332.

§ 2. Modular elements of some weakly modular lattices.

DEFINITION. (2. 1). Let L be a lattice with partially ordered by a relation $a \leqq b$ and having the operations $a \vee b$, $a \wedge b$. Let $\mathfrak{M}$ be the set of all modular elements of L. If $\mathfrak{M}$ is a lattice with partially ordered $a \leqq b$, then it is a lattice with operations $a \cup b$, $a \cap b$ such that $a \cup b \geqq a \vee b$, $a \cap b = a \vee q$. And the dual of $\mathfrak{M}$ is denoted by $\overline{\mathfrak{M}}$.

LEMMA (2. 2). *Let L be a lattice and $\mathfrak{M}$ be the set of all modular elements of L. If $\mathfrak{M}$ is a lattice and $a, b \in \mathfrak{M}$ implies $(a, b)M^*$ in $\mathfrak{M}$.*

PROOF. Suppose $a, b \in \mathfrak{M}$ and $a \vee b \in \mathfrak{M}$, then $a \cup b = a \vee b$. Let $c \geqq b$ and $c \in \mathfrak{M}$, then $c \cap (a \cup b) = c \wedge (a \vee b) = (c \wedge a) \vee b \leqq (c \cap a) \cup b$. The reverse inequality is obvious, and so $c \cap (a \cup b) = (c \cap a) \cup b$. Hence $(a, b)M^*$ in $\mathfrak{M}$.

DEFINITION (2. 3). Let L be a lattice with 0. L is called *semicomplemented* when for any element $a \in L$ (with $a \neq 1$ if 1 exists) there exists a non-zero element $b \in L$ such that $a \wedge b = 0$. (Cf. [3] p. 20.)

LEMMA (2. 4). *Let L be a weakly modular semicomplemented M-symmetric lattice. Then $a \in \mathfrak{M}$ and $a \neq 1$ imply $L[0, a] \subset \mathfrak{M}$.*

PROOF. Let $a \in \mathfrak{M}$ and $a \neq 1$. Since L is semicomplemented, there exists a non-zero element $b \in L$ such that $a \wedge b = 0$. Since L is M-symmetric, the intervals $L[0, a]$ and $L[b, a \vee b]$ are isomorphic by (1. 6). Since $b \neq 0$ and L is weakly modular, $L[b, a \vee b]$ is a modular lattice and hence $L[0, a]$ is a modular lattice. Let $a_1 \in L[0, a]$ and $x \in L$, then $(x, a)M$ and $(x \wedge a, a_1)M$ in L. By (1. 3) $(x, a \wedge a_1)M$ and hence $(x, a_1)M$.

REMARK (2. 5). By (2. 4) in a weakly modular semicomplemented M-symmetric lattice, modular elements are modular in the sense of [2].

THEOREM (2. 6). *Let L be a semicomplemented M-symmetric lattice. If L is a weakly modular complete lattice, then $\overline{\mathfrak{M}}$ is a weakly modular complete lattice.*

PROOF. We shall first show that if $a_\alpha \in \mathfrak{M}$ for every $\alpha \in I$, then the meet $a = \bigwedge (a_\alpha ; \alpha \in I)$ in L belongs to $\mathfrak{M}$. This is evident when I is empty or when $a_\alpha = 1$ for every $\alpha \in I$. When $a_\alpha \neq 1$ for some $\alpha \in I$, it follows from (2. 4) that $a = \bigwedge (a_\alpha ; \alpha \in I)$ belongs to $\mathfrak{M}$. Hence a is the meet $\bigcap (a_\alpha ; \alpha \in I)$ in $\mathfrak{M}$ and hence $\mathfrak{M}$ is complete and so is $\overline{\mathfrak{M}}$. Next we shall show that $\overline{\mathfrak{M}}$ is weakly modular. Let $a, b \in \overline{\mathfrak{M}}$ and $a \cap b \neq 0$ in $\overline{\mathfrak{M}}$. If $a \vee b \notin \mathfrak{M}$, then by (2. 4) $a \vee b \leqq x < 1$ implies $x \notin \mathfrak{M}$. Hence $a \cup b = 1$ in $\mathfrak{M}$ and hence $a \cap$

$b=0$ in $\overline{\mathfrak{M}}$. This contradicts $a \cap b \neq 0$. Hence by $a \vee b \in \mathfrak{M}$ and (2. 2), $(a, b)M^*$ in $\mathfrak{M}$ and hence $(a, b)M$ in $\overline{\mathfrak{M}}$.

§ 3. Weakly modular semimodulated lattices.

DEFINITION (3. 1). Let L be a lattice and $\mathfrak{M}$ be the set of all modular elements of L. We introduce the following four conditions on L.

(α) If $a \in \mathfrak{M}$, $b \in L-\mathfrak{M}$ and $a < b$, then there exists a non-zero $c \in \mathfrak{M}$ such that $b \wedge c = a$ and $a < c < b$.

(β) If $a, b \in \mathfrak{M}$, $c \in L-\mathfrak{M}$ and $a < c < b$, then there exists $c' \in \mathfrak{M}$ such that $c \wedge c' = a$ and $a < c' < b$.

(γ) If $a, b \in \mathfrak{M}$, $c \in L-\mathfrak{M}$ and $a < c < b$, then there exists $c' \in \mathfrak{M}$ such that $c \vee c' = b$ and $c \wedge c' = a$.

(δ) In a lattice L with 1, if $a \in \mathfrak{M}$, $b \in L-\mathfrak{M}$ and $a < b$, then there exists $c \in \mathfrak{M}$ such that $b \vee c = 1$ and $b \wedge c = a$.

EXAMPLE. Let Λ be a relatively complemented modular lattice with 0 and the operations $\vee, \wedge$, of length $\geqq 3$ which contains a point p. We define

$$L \equiv \Lambda - \{p\}.$$

If L is partially ordered in the natural manner, then L is a weakly modular M-symmetric lattice with operations $\cup, \cap$ which satisfies the following conditions :

$$a \cup b = a \vee b,$$
$$a \cap b = a \wedge b \text{ if } a \wedge b \neq p,$$
$$a \cap b = 0 \text{ if } a \wedge b = p.$$

And for $a, b \in L$

$(a, b)M$ in L if and only if $a = 1$ or $a \not> p$.

It is easy to show that

(1) L satisfies (α) and if L has no unit 1, then (β) and (γ) are trivial propositions, and

(2) If L has unit 1, then it satisfies (δ). (Cf. [4] p. 327.)

LEMMA (3. 2). (i) *In any lattice*, $(\gamma) \Rightarrow (\beta)$. (ii) *In any lattice with* 1, $(\beta) \Rightarrow (\alpha)$ *and* $(\gamma) \Leftrightarrow (\delta)$.

PROOF. (i) It is evident. (ii) Let L be a lattice with 1. Since $1 \in \mathfrak{M}$,

$(\beta)\Rightarrow(\alpha)$ and $(\gamma)\Rightarrow(\delta)$ are evident. $(\delta)\Rightarrow(\gamma)$. Let $a, b\in\mathfrak{M}$, $c\in L-\mathfrak{M}$ and $a< c< b$. By (δ) there exists $c'\in\mathfrak{M}$ such that $c\wedge c'=a$ and $c\vee c'=1$. Let $d=c'\wedge b$, then $d\in\mathfrak{M}$ by (1. 4) and $c\wedge d=c\wedge(c'\wedge b)=(c\wedge c')\wedge b=a\wedge b=a$, $c\vee d=c\vee(c'\wedge b)=(c\vee c')\wedge b=1\wedge b=b$.

LEMMA (3. 3). *Let L be a lattice with (β) and $a, b\in\mathfrak{M}$. Then $a\lessdot b$ in $\mathfrak{M}$ if and only if $a\lessdot b$ in L.*

PROOF. Let $a, b\in\mathfrak{M}$. Assume $a\lessdot b$ in $\mathfrak{M}$ and there exists $x\in L-\mathfrak{M}$ such that $a<x<b$. By (β) there exists $x'\in\mathfrak{M}$ such that $a<x'<b$ which contradicts the hypothesis. Conversely if $a\lessdot b$ in L, then $a\lessdot b$ in $\mathfrak{M}$ by $\mathfrak{M}\subset L$.

THEOREM (3. 4). *Let L be a lattice with (β) and $\mathfrak{M}$ be a lattice, then $(a, b)M^*$ in $\mathfrak{M}$ if and only if $a\vee b\in\mathfrak{M}$.*

PROOF. Suppose $a, b\in\mathfrak{M}$ and $(a, b)M^*$ in $\mathfrak{M}$. If $a\vee b\notin\mathfrak{M}$, then $b<a\vee b<a\cup b$ in L. By (β), there exists $c\in\mathfrak{M}$ such that $(a\vee b)\wedge c=b$, $b<c<a\cup b$. Then $c\cap a=c\wedge a=c\wedge((b\vee a)\wedge a)=(c\wedge(b\vee a))\wedge a=b\wedge a$, whence $(c\cap a)\cup b=(b\wedge a)\vee b=b<c=c\cap(a\cup b)$. Therefore $(a, b)\overline{M}^*$ ($\overline{M}^*$ being the negation of the relation M^*). This contradicts $(a, b)M^*$ in $\mathfrak{M}$. Sufficiency follows from (2. 2).

COROLLARY (3. 5). *If L is a lattice with (β) and $\mathfrak{M}$ is a lattice, then the following propositions hold.*

(i) *$\overline{\mathfrak{M}}$ is an M-symmetric lattice.*

(ii) *$\overline{\mathfrak{M}}$ is weakly modular if and only if $a, b\in\overline{\mathfrak{M}}$* and *$a\wedge b\notin\overline{\mathfrak{M}}$ imply $a\cap b=0$ in $\overline{\mathfrak{M}}$.*

THEOREM (3. 6). *Let L be a weakly modular semicomplemented M-symmetric lattice. If L is an upper continuous lattice, then (α), (β), (γ) and (δ) are equivalent.*

PROOF. Let $a\in\mathfrak{M}$, $b\in L-\mathfrak{M}$ and $a<b$. Define $S=\{c\in\mathfrak{M}; a<c$ and $b\wedge c=a\}$. By (α) $S\neq\phi$. Let X be a chain of S, then $c'=\bigvee(x; x\in X)$ is modular by (1. 10) and $a<c'$. Since L is upper continuous and the set $\{x\wedge b; x\in X\}$ is an increasingly directed set, $\bigvee(x\wedge b; x\in X)=c'\wedge b$. Therefore $c'\wedge b=a$ since $x\wedge b=a$ for every $x\in X$ and hence $c'\in S$. According to Zorn's lemma there exists a maximal element $c_0\in S$. If $c_0\vee b\neq 1$, then $c_0<c_0\vee b<1$. By (2. 4) $b\vee c_0\in L-\mathfrak{M}$ and by (α) there exists $c_1\in\mathfrak{M}$ such that $(b\vee c_0)\wedge c_1=c_0$ and $c_0<c_1$. Then $b\wedge c_1=b\wedge(b\vee c_0)\wedge c_1=b\wedge c_0=a$ and hence $c_1\in S$ which contradicts the definition of c_0. Therefore we have $b\vee c_0=1$ and hence $(\alpha)\Rightarrow(\delta)$. Consequently by (3. 2) and $(\alpha)\Rightarrow(\delta)$, in a

weakly modular semicomplemented M-symmetric lattice, all four conditions (α), (β), (γ) and (δ) are equivalent, if it is an upper continuous lattice.

DEFINITION (3. 7). Let L be an M-symmetric lattice with 1 and $\mathfrak{M}$. L is called a *semimodulated* lattice when it satisfies (δ). (Cf. [1] p. 112.)

THEOREM *Let Λ be a complemented modular lattice having the lattice operations $a \vee b, a \wedge b$. Let S be a fixed subset of $\Lambda - \{0, 1\}$ with the following two properties:*

(1) *$a \in S$ and $0 < b \leqq a$ imply $b \in S$.*

(2) *$a, b \in S$ implies $a \vee b \in S$.*

If in the set $L \equiv \Lambda - S$ we give the same order as Λ, then L is a weakly modular M-symmetric lattice where the lattice operations $a \cup b$ and $a \cap b$ satisfy the following conditions:

(3) $a \cup b = a \vee b$

$a \cap b = a \wedge b$ *if* $a \wedge b \in L$

$a \cap b = 0$ *if* $a \wedge b \in S$.

Moreover for $a, b \in L$

(4) *$(a, b)M$ if and only if $a \wedge b \in L$.*

(5) *$a \lessdot b$ in L if and only if $a \lessdot b$ in Λ.*

PROOF. Cf. [5] pp. 497-498.

DEFINITION (3. 8). When a weakly modular M-symmetric lattice L arises from a complemented modular lattice Λ in the manner describes the above theorem, L is called a *Wilcox* lattice. An element of S is called an *imagynary* element for L, and when S has a greatest element i it is called the *imagynary unit* for L. A non-zero element a of L is called a *regular* element when $a \wedge u = 0$ for all $u \in S$. (Cf. [3] pp. 12-14.)

LEMMA (3. 9). *Let $L \equiv \Lambda - S$ be a Wilcox lattice. Any regular element of L is modular and any modular element m of L with $0 < m < 1$ is regular if L is semicomplemented.*

PROOF. Cf. [3] p. 11.

THEOREM (3. 10). *Let $L \equiv \Lambda - S$ be a semicomplemented Wilcox lattice with imagynary unit i. L is semimodulated if and only if S is a set consisting of a point.*

PROOF. Suppose $L \equiv \Lambda - S$ be a semimodulated lattice. Let a be a complement of $i \in S$, then $a \vee i = 1$, $a \wedge i = 0$ in Λ. Since i is the greatest element of S, $a \wedge u = 0$ for every $u \in S$ and hence a is a regular element. Then by (3. 9) a is a modular element of L. Let b be a modular element of L such that $a \leqq b < 1$. Then b is a regular element by (3. 9). Hence $b \wedge i$

$= 0$. Let $\lambda \in \Lambda$ be a complement of a in $\Lambda[0, b]$, then $b = a \vee \lambda$, $a \wedge \lambda = 0$. Since L is a modular lattice, $\lambda \wedge a = 0$ and $(\lambda \vee a) \wedge i = 0$ implies $(a \vee i) \wedge \lambda = 0$. Since $a \vee i = 1$, $\lambda = 0$ and hence $a = b$. Thus $a \lessdot 1$ in $\mathfrak{M}$. By (3.3) $a \lessdot 1$ in L and hence $a \lessdot 1$ in Λ. Therefore $i \gtrdot 0$ in Λ and hence i is a point of Λ. Sufficiency is evident.** (Cf. [4] p. 327.)

§ 4. Modulated lattices.

DEFINITION (4. 1). Let L be a lattice with $\mathfrak{M}$. We introduce the following three conditions :

(α^*) If $a \in \mathfrak{M}$, $b \in L$ (with $b \neq 1$ if 1 exists) and $a \leqq b$, then there exists a non-zero $c \in \mathfrak{M}$ such that $b \wedge c = a$ and $a < c$.

(γ^*) If $a, b \in \mathfrak{M}$, $c \in L$ and $a < c < b$, then there exists $c' \in \mathfrak{M}$ such that $c \vee c' = b$ and $c \wedge c' = a$.

(δ^*) In a lattice L with 1, if $a \in \mathfrak{M}$, $b \in L$ and $a \leqq b$, then there exists $c \in \mathfrak{M}$ such that $b \vee c = 1$ and $b \wedge c = a$.

REMARK (4. 2). If a lattice with 0 satisfies (α^*), then it is semicomplemented and if a lattice with 0, 1 satisfies (δ^*), then it is a complemented lattice. In the example in § 3, it is easy to show that if every intervalsublattice of Λ is irreducible, then L satisfies (α^*) and moreover if L has 1, then it satisfies (δ^*).

LEMMA (4. 3). (i) *In any lattice with* 1, $(\gamma^*) \Rightarrow (\alpha^*)$ *and* $(\gamma^*) \Leftrightarrow (\delta^*)$.

(ii) *In an M symmetric upper continuous lattice,* $(\alpha^*) \Rightarrow (\delta^*)$. *And therefore the three conditions* (α^*), (γ^*) and (δ^*) are equivalent.

PROOF. (i) $(\gamma^*) \Rightarrow (\alpha^*)$ and $(\gamma^*) \Rightarrow (\delta^*)$ are evident since $1 \in \mathfrak{M}$. $(\delta^*) \Rightarrow (\gamma^*)$. It is similar to the proof of $(\delta) \Rightarrow (\gamma)$ in (3. 2). (ii) It is similar to the proof of (3. 6).

DEFINITION (4. 4). An M-symmetric lattice L with 0 and 1 is called a *modulated* lattice when it satisfies (δ^*). (Cf. [4] p. 326.)

REMARK (4. 5). A modulated lattice L is a complemented semimodulated lattice.

LEMMA (4. 6). *If L is a semimodulated lattice and $\mathfrak{M}$ is a lattice, then the following conditions are equivalent.*

(i) *L is modulated.*

(ii) *$\mathfrak{M}$ is relatively complemented.*

(iii) *$\mathfrak{M}$ is left complemented.*

PROOF. (i) $\Rightarrow$ (ii). This follows from (4. 3) (i) and (γ^*). (ii) $\Rightarrow$ (iii). Let $a, b \in \mathfrak{M}$,

** This proof is indebted to Dr. S. Maeda for help.

then $a \vee b \geqq b$ in L. Since L satisfies (δ) and $\mathfrak{M}$ is relatively complemented, it is easy to show that there exists $b' \in \mathfrak{M}$ such that $(a \vee b) \vee b' = 1$ and $(a \vee b) \wedge b') = b$. Hence $b' \geqq b$ and $a \cap b' = a \wedge b' = a \wedge (a \vee b) \wedge b' = a \wedge b = a \cap b$, $a \cup b' \geqq a \vee b' = a \vee b \vee b' = 1$. Hence $a \cup b' = a \vee b' = 1$ and hence $(a, b)M^*$ in $\mathfrak{M}$ by (2. 2). Consequentry $\overline{\mathfrak{M}}$ is left complemented.

(iii) $\Rightarrow$ (i). Since L is semimodulated, it is sufficient to show that if $a, b \in \mathfrak{M}$ and $a \leqq b$, then there exists $c \in \mathfrak{M}$ such that $b \vee c = 1$ and $b \wedge c = a$. Assume that $\overline{\mathfrak{M}}$ is left complemented and $a, b \in \mathfrak{M}$ and $a \leqq b$ in L. Since $\overline{\mathfrak{M}}$ is left complemented, there exists $c \in \mathfrak{M}$ such that $b \cap c = a$, $b \cup c = 1$ and (b, c)M* in $\mathfrak{M}$. By (3. 4) $b \vee c = b \cup c = 1$ and $b \wedge c = b \cap c = a$.

THEOREM (4. 7). *Let L be a weakly modular modulated lattice. L is complete if and only if $\mathfrak{M}$ is complete.*

PROOF. (I) If L is complete, then so is $\mathfrak{M}$ by (2. 6). (II) Assume that $\mathfrak{M}$ is complete. (i) Let $m_\alpha \in \mathfrak{M}$ for every $\alpha \in$ I. It is easy to show that the meet $\cap(\mathrm{m}_\alpha; \alpha \in I)$ in $\mathfrak{M}$ is the meet $\wedge(m_\alpha; \alpha \in I)$ in L. (ii) Let $u_\alpha \in L - \mathfrak{M}$ for every $\alpha \in I$. When $\{u_\alpha\}$ has no lower bound except 0, we have $\wedge_\alpha u_\alpha = 0$ in L. When $\{u_\alpha\}$ has lower bound h with $h > 0$, we can take $m \in \mathfrak{M}$ such that $h \vee m = 1$ and $h \wedge m = 0$ in L since L is modulated. $u_\alpha \wedge m \in \mathfrak{M}$ for every $\alpha \in I$ by (2. 4) and hence there exists a meet $b = \cap (u_\alpha \wedge m; \alpha \in I)$ in $\mathfrak{M}$. By (1. 5) $(m, x)M^*$ for every $x \in L$ whence $(u_\alpha \wedge m) \vee h = u_\alpha \wedge (m \vee h) = u_\alpha$ and hence $u_\alpha \geqq b \vee h$ for every $\alpha \in I$. Therefore $b \vee h$ is an lower bound of $\{u_\alpha\}$ in L. If $\bar{h}$ is an arbitrary lower bound of $\{u_\alpha\}$ in L, then putting $c = (\bar{h} \vee h) \wedge m$, we have $c \in \mathfrak{M}$ and $c \leqq u_\alpha \wedge m$ for every $\alpha \in I$, whence $b \geqq c$. By (1. 5) $(m, x)M^*$ for every $x \in L$ whence

$b \vee h \geqq c \vee h = ((\bar{h} \vee h) \wedge m)) \vee h = (\bar{h} \vee h) \wedge (m \vee h) = \bar{h} \vee h \geqq \bar{h}$. Therefore $b \vee h$ is the meet $\wedge(u_\alpha; \alpha \in I)$ in L. (iii) By (i) and (ii), it is easy to show that any subset of L has its meet in L. Hence L is complete. (Cf. [3] p. 93.)

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