Mem. Fac. Sci., Shimane Univ.,
12, pp. 15–18 Dec. 20, 1978

On Equivariant Line Bundles over Some Surfaces

Dedicated to Professor A. Komatu on his 70th birthday

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(Received September 6, 1978)

In § 1 it is proved that each member of the complex analytic family of K3-surfaces in [5]–I admits a canonical involution. The structure of the group of C^*-line bundles over some Hopf surfaces is analyzed in § 2, and some principal C-bundles over the projective line are treated.

§ 1. Some K3-surfaces

At first we refer some basic results from [5]–I. A K3-surface is a regular surface of which the first Chern class vanishes. By Noeter's formula and Hirzebruch signature theorem, the first Betti number is zero. Let C^{21} be the 21-dimensional complex number space. For each $\tau=(\tau_0, \tau_1,\dots, \tau_8; \sigma_1,\dots, \sigma_{12})\in C^{21}$, set

$$g(u, \tau)=\tau_0\prod_{v=1}^{8}(u-\tau_v), \quad h(u, \tau)=\prod_{v=1}^{12}(u-\sigma_v),$$

$$J_\tau=g(u, \tau)^3/(g(u, \tau)^3-27h(u, \tau)^2, \ u \text{ is a complex number},$$

$$C^{21}\supset N=\{\tau\in C^{21};\ 1)\ \tau_0\neq 0,\ \tau_0\neq 27,\ 2)\ g(\sigma_\lambda)=0 \Longrightarrow \sigma_v\neq\sigma_\lambda$$
$$\text{for} \quad v\neq\lambda,\ 3)\ J_\tau \text{ has no multiple pole}\}.$$

Let P^2 be a projective plane with homogeneous coordinates (x, y, z) and C be the complex number space. We define an identification in the union $W=P^2\times C_0\cup P^2\times C_1$ of two copies of the product space $\mathrm{P}^2\times C$ by

$$(x, y, z, u)\equiv(x_1, y_1, z_1, u_1)\longleftrightarrow uu_1=1,\ u^4x_1=x,\ u^6y_1=y,\ z_1=z.$$

Define a submanifold $\mathfrak{B}$ of $W\times N$ by

$$y^2z-4x^3+g(u, \tau)xz^2+h(u, \tau)z^3=0,$$

$$y_1^2z_1-4x_1^3+u_1^3g\left(\frac{1}{u_1}, \tau\right)x_1z_1^2+u_1^{12}h\left(\frac{1}{u_1}, \tau\right)z_1^3=0.$$

The restriction Ψ: $\mathfrak{B}\to N$ of the projection $W\times N\to N$ is a complex analytic family of algebraic elliptic K3-surfaces. By 9, [6], for any Kähler manifold with vanishing first Betti number, the group of automorphisms is discrete. Each member $B_\tau=\Psi^{-1}(\tau)$

admits a non trivial involution $g: y \to -y$, $y_1 \to -y_1$. The involution leaves invariant the global section defined by $x=z=x_1=z_1=0$.

PROPOSITION 1. *Any elliptic K3-surface whose singular fibres are of type* I_1 *or of type II admits a non trivial involution.*

PROOF. By Theorem 16, [5]–I, such a surface can be represented in the form $S = B^{h^*(s)}$, $s \in H^1(\Delta, \Omega(f))$, where Δ is the projective line, f is the normal bundle of Δ in B and $h^*: H^1(\Delta, \Omega(f)) \to H^1(\Delta, \Omega(B_0^\sharp)) \to 0$. ($B^\sharp$ and $B_0^\sharp$ are the ones in [4]–II.) The involution induces an automorphism of the sheaf $\Omega(B^\sharp)$. Referring to the construction in 14, [4]–II, let $\mu_j: V|E_j \to B|E_j$ be a biholomorphic fibre map. Then

$$L(\eta_{jk}): B|E_j \cap E_k \xrightarrow{\mu_k^{-1}} V|E_j \cap E_k \xrightarrow{\mu_j} B|E_j \cap E_k,$$

where $(\eta_{jk}) = \eta \in H^1(\Delta, \Omega(B^\sharp))$ and $V = B^\eta$. In the present case $E_{gj} = g^{-1}E_j = E_j$. We have

$$g_V = \mu_j^{-1} L(\lambda_j(g)) g \mu_j \quad \text{on} \quad V|E_j.$$

Then g_V is an automorphism of V and it defines an automorphism of B by

$$g' = L(-\lambda_j(g_V)) \mu_j g_V \mu_j^{-1}.$$

Since $L^\sharp(-\lambda_j(g_V)) L^\sharp(\lambda_j(g))$ = the identity of $B^\sharp|E_j$, by theorem 9.2, [4]–II,

$$g' = L(-\lambda_j(g_V)) L(\lambda_j(g)) g = g.$$

Thus the involution g_V is non trivial.

COROLLARY. *The equivariant Picard number of a member* B_τ *is greater than one.*

PROOF. For any member B_τ, two divisors consisting of the global section $x=z=x_1=z_1=0$ and $C(\infty)$: $y_1^2 z_1 - 4x_1^3 + \tau_0 x_1 z_1^2 + z_1^3 = 0$ are g-invariant. Then they determine g-equivariant line bundles.

REMARK. We consider the local triviality of a family of line bundles over $\mathfrak{B}$. We have $L(G) = \Xi_\tau = \theta$, and since $h^{0,1} = q = 0$, $H^1(B_\tau, \Xi_\tau) = H^1(B_\tau, \theta) = 0$. Then by Theorem 7.3, [3], any family of line bundles over $\mathfrak{B}$ is locally trivial.

§2. On surfaces of class VII_0

Any surface of class VII_0 has numerical invariants $q = b_1 = 1$, $h^{1,0} = 0$ by Theorem 3, [5]–I. Let $\mathbf{O}(\mathbf{O}^*)$ be the sheaf of germs of (nowhere vanishing) holomorphic functions respectively. By (102), [5],

$$\mathfrak{P} = H^1(S, \mathbf{O})/H^1(S, \mathbf{Z}) \cong H^1(S, \mathbf{C})/H^1(S, \mathbf{Z}) \cong \mathbf{C}^* = \mathbf{C} - \{0\}.$$

Let $S^{[m]}$ be the surface defined by the union $\boldsymbol{C}^* \times \boldsymbol{C} \cup \boldsymbol{C}^* \times \boldsymbol{C}_1$, where $(w, u) \equiv (w_1, u_1) \leftrightarrow uu_1 = 1$ and $w_1 = u^m w$, (p. 75, [5]–III). We define a complex analytic automorphism

$$f': \Delta' \times S^{[m]} \longrightarrow \Delta' \times S^{[m]}$$

by

$$f'(t, w, u) = (t, (\gamma u + \delta)^m w, (\alpha u + \beta)/(\gamma u + \delta)),$$

where α, β, γ, δ are small and Δ' is the punctured unit disk $\{t \in C, 0 < |t| < 1\}$, (p. 75, [5]). The family $p: \Delta' \times S^{[m]}/\{f'\} \to \Delta'$ is a complex analytic family and each member $p^{-1}(t)$ is a Hopf surface, further it admits a $\boldsymbol{C}^*$-action. Since any $\boldsymbol{C}^*$-action on a surface induces the identity automorphism of $H^1(S, \boldsymbol{C})$, each element of $\mathfrak{B}$ is $\boldsymbol{C}^*$-invariant. Thus we have

Proposition 2. *For each member of the above family, any line bundle with degree zero is $\boldsymbol{C}^*$-equivariant.*

Next we consider principal torus bundles of class VII_0, (9, [5]–II). Let ω be a complex number with $\operatorname{Im}\omega > 0$, $\boldsymbol{Z}$ be the additive group of integers and $G = \boldsymbol{Z}[1] + \boldsymbol{Z}[\omega]$ generated by 1 and ω. Denote by C the torus $\boldsymbol{C}/G$. Let $\Omega(C)$ be the sheaf over the projective line Δ of germs of holomorphic functions with value in C. We have the exact sequence

$$0 \longrightarrow G \longrightarrow \boldsymbol{O} \longrightarrow \Omega(C) \longrightarrow 0.$$

Since $H^1(\Delta, \boldsymbol{O}) = 0$, we have an isomorphism $H^1(\Delta, \Omega(C)) \xrightarrow{\delta^*} H^2(\Delta, G) \cong G$. For a point $a \in \Delta$, $a \neq \infty$, take a unit disk E_a: $|u - a| < \varepsilon$. Define a surface S by

$$L_a(\Delta \times C) = E_a \times C \cup (\Delta - a) \times C, \quad \text{where}$$

$$E_a \times C \ni (u, [\zeta_a]) \equiv (u, [\zeta]) \in (\Delta - a) \times C \longleftrightarrow [\zeta] = [\zeta_a + \eta_{12}(u)],\ \eta_{ij} \in H^1(\Delta, \Omega(C)).$$

The surface S is the total space of a principal C-bundle over Δ. By Theorem 1, [1], we have the extension of the tangent bundle $T(\Delta)$,

$$\boldsymbol{a}(S): 0 \longrightarrow \mathbf{1} \longrightarrow Q(S) \longrightarrow T(\Delta) \longrightarrow 0, \quad (\mathbf{1} \text{ is the trivial line bundle}),$$

where each point of $Q(S)$ is a field of tangent vectors to S along one of its fibres are invariant under C. The extension $\boldsymbol{a}(S)$ determines an element of $H^1(\Delta, T^*) = H^1(\Delta, \kappa) = \boldsymbol{C}$, where κ is the canonical bundle of Δ and T^* is the cotangent bundle of Δ. Since Δ is simply connected, any complex analytic connection is integrable. Thus we have

Proposition 3. $\boldsymbol{a}(S) = 0$ *if and only if* $S = \Delta \times C$.

$T(\Delta) = \det Q(S)$ and $c_1(T(\Delta)) = 2$. By Theorem 2.1, [2], $Q(S)$ is decomposable: $Q(S) = F_1 \oplus F_2$, and $c_1(F_1) + c_1(F_2) = 2$.

PROPOSITION 4. $Q(S)=F\oplus F$ and $c_1(F)=1$.

PROOF. The surface S admits another representation: $S=\boldsymbol{C}\times C\cup\boldsymbol{C}\times C$, where

$$(u_1, \zeta_1)\equiv(u_2, \zeta_2)\longleftrightarrow u_1u_2=1,\ \zeta_2=\zeta_1+\frac{\gamma}{2\pi i}\log u_1 \bmod G \text{ for } \gamma=\{\eta_{12}\}\in G.$$

Set $W=\boldsymbol{C}\times\boldsymbol{C}^*\cup\boldsymbol{C}^*\times\boldsymbol{C}=\boldsymbol{C}^2-\{0\}$ and define a holomorphic map $f\colon W\to S$ by

$$f(z_1, z_2)=\begin{cases}\left(z_1/z_2, \dfrac{\gamma}{2\pi i}\log z_2\right) & \text{for } z_2\neq 0,\\ \left(z_2/z_1, \dfrac{\gamma}{2\pi i}\log z_1\right) & \text{for } z_1\neq 0.\end{cases}$$

Then the map f induces a biholomorphic map $\hat{f}\colon W/H\to S$, if $\gamma\neq 0$, where H denotes an infinite cyclic group generated by $\exp 2\pi i\omega/\gamma$. We have equalities

$$W/H/\boldsymbol{C}^*/H=W/\boldsymbol{C}^*=\Delta$$

and

$$\boldsymbol{C}^*/H \xrightarrow{\sim} C \text{ by } z\longrightarrow\frac{\gamma}{2\pi i}\log z,$$

where the $\boldsymbol{C}^*/H$-action is diagonal. $Q(S)=W\times_{\boldsymbol{C}^*}\boldsymbol{C}^2$ is a plane bundle associated with the principal bundle $W\to\Delta$. Thus we have proved the proposition.

REMARK. Since the surface is non algebraic ([5]–I), $c_1^2=0$ and so $b_2=0$. Then any line bundle over S is of degree zero. Further the surface S admits C-action. Since $C=\boldsymbol{C}^*/\{t\}$, where $\omega=(1/2\pi i)\log t$, the action comes from the $\boldsymbol{C}^*$-action. Thus any line bundle over S is C-equivariant.

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